A Simple Approximation for the Symbol Error Rate of Triangular Quadrature Amplitude Modulation

Tran Trung DUY†(a), Nonmember and Hyung Yun KONG†∗, Member

SUMMARY In this paper, we consider the error performance of the regular triangular quadrature amplitude modulation (TQAM). In particular, using an accurate exponential bound of the complementary error function, we derive a simple approximation for the average symbol error rate (SER) of TQAM over Additive White Gaussian Noise (AWGN) and fading channels. The accuracy of our approach is verified by some simulation results.

key words: Triangular QAM, complementary error function, AWGN channel, fading channels, symbol error rate

1. Introduction

The well-known square quadrature amplitude modulation (SQAM) was discovered by Campopiano and Glazer [1] and nowadays, it continues to gain interest in practical applications. The SQAM is a useful technique to achieve high rate transmission without increasing the bandwidth. Although this technique is easy to detect a received signal and obtains a quite good performance, it is not optimum in power efficiency.

Recently, the triangular QAM (TQAM) was proposed and analyzed in [2]–[5]. In this scheme, signal points are placed at the vertexes of the contiguous triangles and because the TQAM is more compact than of the SQAM, so it is more efficient in utilization of power resource. In [4], [5], a simple approximation for bit error rate (BER) and symbol error rate (SER) of the TQAM over AWGN channel was proposed. However up to now, the exact general expressions for these error performances have not been derived. In this paper, we employ an accurate exponential bound of the complementary error function to derive a simple and accurate approximation of SER for the regular \( M \)-ary TQAMs over AWGN and fading channels.

2. AWGN Channel

2.1 Power Gain

Figure 1 describes the signal constellation of the 64-ary regular TQAM, in which distance between adjacent signal points is equal to \( 2d \). Hence, the average energy per symbol for \( M \)-ary TQAM can be calculated as

\[
E_{E_{M \text{-TQAM}}} = \frac{7M^2 - 4}{12}d^2 \tag{1}
\]

For \( M \)-ary SQAM, the average energy per symbol is

\[
E_{E_{M \text{-SQAM}}} = \frac{2(M - 1)}{3}d^2 \tag{2}
\]

From (1) and (2), the power gain of the \( M \)-ary TQAM against the \( M \)-ary SQAM becomes

\[
PG = 10\log_{10}\left( \frac{8M^2 - 8}{7M^2 - 4} \right) \tag{3}
\]

2.2 Symbol Error Rate

We consider a demodulated symbol \( Z = x + jy \), where \( j = \sqrt{-1} \), \( x \) and \( y \) are the real part and imaginary part of \( Z \), respectively. In the \( M \)-ary TQAM, the symbol \( Z \) can be in one of \( 2M + 1 \) regions and the region \( i \) is determined according to the value of the real part \( x \) as

\[
i = \begin{cases} 
1 & \text{if } x < -(1 + 2M)d/2 \\
2 & \text{if } -(1 + 2M)d/2 \leq x < -(2M - 3)d/2 \\
i & \text{if } (2i - 3 - 2M)d/2 \leq x < (2i - 1 - 2M)d/2 \\
2M & \text{if } (2M - 3)d/2 < x < (1 + 2M)d/2 \\
2M + 1 & \text{if } x \geq (1 + 2M)d/2
\end{cases} \tag{4}
\]

In Fig. 1, the symbol \( Z \) falls in the region 11 \((R_{11})\) because the real part \( x \) of \( Z \) is larger than \( 3d/2 \) and less than \( 5d/2 \). Then, the demodulated symbol \( Z \) will be detected, relying on its imaginary part \( y \). For example, with 64-ary TQAM as shown in Fig. 1, the symbol \( Z \) can be detected as one of eight
signal points \((p_5, p_{14}, p_{21}, p_{30}, p_{37}, p_{46}, p_{53}, p_{62})\) by comparing the imaginary part \(y\) with the imaginary values of the seven boundary points \(D_j (1 \leq j \leq 7)\). Generally, in the \(M^2\)-ary TQAM, there are \(M - 1\) boundary points \(D_j\) in each region \(R_i (2 \leq i \leq 2M)\) and the imaginary part of these points can be easily found as follows:

\[
y_{D_j|R_i} = \left(\frac{M}{2} - j\right) \sqrt{\frac{3}{d}} + (-1)^j k_{R_i}
\]  
(5)

where, \(k_{R_i}\) depends on the real value \(x_{D_j}\) of \(D_j\) as

\[
k_{R_i} = (-1)^j \frac{x_{D_j} + (M + 1 - i) d}{\sqrt{3}}
\]  
(6)

In Fig. 1, if \(Z\) is demodulated symbol of the transmitted symbol \(p_{30}\), the symbol \(p_{30}\) is detected correctly if \(Z\) falls in the region \(R_{11}\) and the imaginary value of \(Z\) is less than \(y_{D_{11}|R_{11}}\) but larger than \(y_{D_{12}|R_{11}}\) or \(Z\) falls in the region \(R_{12}\) while the imaginary value of \(Z\) is less than \(y_{D_{12}|R_{12}}\) but larger than \(y_{D_{11}|R_{12}}\). Hence, the probability of this event is calculated by

\[
P_{e=30} = \frac{1}{\pi N} \int_{3d/2}^{5d/2} e^{-\frac{(x - \frac{y_{11}}{\sqrt{3}})^2}{2}} dx \int_{y_{11}}^{y_{12}} e^{-\frac{(y - \frac{y_{12}}{\sqrt{3}})^2}{2}} dy
\]

\[
+ \frac{1}{\pi N} \int_{5d/2}^{7d/2} e^{-\frac{(x - \frac{y_{12}}{\sqrt{3}})^2}{2}} dx \int_{y_{12}}^{y_{11}} e^{-\frac{(y - \frac{y_{11}}{\sqrt{3}})^2}{2}} dy
\]  
(7)

where, \(N/2\) is variance of the Gaussian probability distribution function, \(y_{D_{11}|R_{11}}, y_{D_{12}|R_{11}}, y_{D_{11}|R_{12}}\) and \(y_{D_{12}|R_{12}}\) are determined according to (5), (6).

After some manipulations, we can calculate the probability that the symbol \(p_{30}\) is detected incorrectly as

\[
P_{e=30} = \text{erfc} (\tau) + 2I_1
\]  
(8)

where, \(\tau = \frac{d}{\sqrt{3-directed}}\text{erfc}(\tau) = \frac{2}{\sqrt{e}} \int_{\tau}^{\infty} e^{-t^2} dt\) and \(I_1 = \int_{2\tau}^{2\tau} \left(1 - e^{-(t-2\tau)^2}\right) \text{erfc}\left(\frac{t}{\sqrt{3}}\right) dt\).

It should be noted that the symbols having the same detection area will have same correctly detected probability. In Fig. 1, there are 35 signal points that have same the detection area with the symbol \(p_{30}\), hence, the probability that these points are detected incorrectly equals to \(P_{e=1}\) in (8).

Also, it can be observed that there are 11 signal points have the same erroneous detection probability with \(p_{6}\) and this probability can be easily found by

\[
P_{e=2} = \text{erfc} (\tau) + I_1
\]  
(9)

Similarly, we can calculate the probability that the remaining symbols are detected incorrectly as follows:

With the points \(p_9\) and \(p_{57},\)

\[
P_{e=3} = \frac{1}{2} \text{erfc} (\tau) + \frac{1}{4} \text{erfc} (\tau) \text{erfc}\left(\frac{\tau}{\sqrt{3}}\right) + \frac{1}{2} I_2 + \frac{1}{2} I_2
\]  
(10)

where, \(I_2 = \int_{2\tau}^{3\tau} \frac{1}{\sqrt{3}} e^{-(t-2\tau)^2} \text{erfc}\left(\frac{t}{\sqrt{3}}\right) dt\).

With the points \(p_9\) and \(p_{56},\)

\[
P_{e=4} = P_{e=3} + \frac{1}{2} (I_1 + I_2 + I_3)
\]  
(11)

where, \(I_1 = \int_{3\tau}^{\infty} e^{-(t-2\tau)^2} \text{erfc}\left(\frac{t}{\sqrt{3}}\right) dt\).

With the points \(p_{1}\) and \(p_{64},\)

\[
P_{e=5} = \frac{1}{2} \text{erfc} (\tau) + \frac{1}{2} \text{erfc} (2\tau) + \frac{1}{2} I_4 + \frac{1}{2} I_4 - \frac{1}{2} I_5
\]  
(12)

where, \(I_4 = \frac{1}{\sqrt{3}} e^{-(t-2\tau)^2} \text{erfc}\left(\frac{t}{\sqrt{3}}\right) dt\) and \(I_5 = \frac{1}{\sqrt{3}} e^{-(t+2\tau)^2} \text{erfc}\left(\frac{t}{\sqrt{3}}\right) dt\).

With the points \(p_{16}, p_{17}, p_{32}, p_{33}, p_{48}\) and \(p_{49},\)

\[
P_{e=6} = \frac{1}{2} \text{erfc} (\tau) + \frac{1}{2} \text{erfc} (2\tau) + I_1 + I_4
\]  
(13)

Finally, with the points \(p_{24}, p_{25}, p_{40}\) and \(p_{41},\)

\[
P_{e=7} = \frac{1}{2} \text{erfc} (\tau) + \frac{1}{2} \text{erfc} (\sqrt{3}\tau) + I_1 + I_2
\]  
(14)

Hence, the average symbol error rate (SER) of the 64-ary TQAM is calculated by

\[
P_{e=64} = \frac{1}{64} \left(36P_{e=1} + 12P_{e=2} + 2P_{e=3} + 2P_{e=4} + 6P_{e=5} + 4P_{e=6}\right)
\]

\[
+ 96I_1 + 7I_2 + I_3 + 7I_4 - I_5 + 56 \text{erfc} (\tau) + 4 \text{erfc} (2\tau) + 3 \text{erfc} (\tau) \text{erfc} (\sqrt{3}\tau)
\]  
(15)

Furthermore, we have a general formula of SER for the \(M^2\)-ary TQAM as follows:

\[
P_{e=M^2} = \frac{1}{M^2} \left(2 (M-1) I_1 + (M-1) I_2 + I_3 + (M-1) I_4 - I_5 + (M^2 - M) \text{erfc} (\tau) + \text{erfc} (2\tau) + \frac{M^2}{2} \text{erfc} (\tau) \text{erfc} (\sqrt{3}\tau)\right)
\]  
(16)

Because the integrals \(I_1, I_2, I_3, I_4\) and \(I_5\) are not conventional integral forms, thus we must find some methods to approximate these integrals. In [6], a simple tight bound for the complementary error function was proposed as

\[
\text{erfc}(x) \approx \frac{1}{6} e^{-x^2} + \frac{1}{2} e^{-4x^2/3}
\]  
(17)

By applying (17) into the integral \(I_1\), we have

\[
I_1 \approx \int_{\tau}^{2\tau} \frac{1}{\sqrt{3}} e^{-(t-2\tau)^2} \left(\frac{1}{6} e^{-t^2} + \frac{1}{2} e^{-t^2/3}\right) dt
\]

\[
\approx 2C_1 \text{erf} \left(\frac{\tau}{\sqrt{3}}\right) + C_2 \left(\text{erf} \left(\frac{8\tau}{3\sqrt{15}}\right) + \text{erf} \left(\frac{5\tau}{3\sqrt{15}}\right)\right)
\]  
(18)

where, \(C_1 = \frac{\sqrt{5}}{12} e^{-\tau^2}\), \(C_2 = \frac{\sqrt{10}}{4\sqrt{15}} e^{-\tau^2/3}\) and \(\text{erf}(x) = 1 - \text{erfc}(x)\).

Similarly, we can approximate the integrals \(I_2, I_3, I_4\) and \(I_5\) as follows:

\[
I_2 \approx C_1 \left(\text{erf} \left(\sqrt{3}\tau\right) - \text{erf} \left(\frac{\tau}{\sqrt{3}}\right)\right) + C_2 \left(\text{erf} \left(\frac{7\tau}{3\sqrt{15}}\right) - \text{erf} \left(\frac{8\tau}{3\sqrt{15}}\right)\right)
\]

\[
I_3 \approx C_1 \text{erf} \left(\sqrt{3}\tau\right) + C_2 \text{erf} \left(\frac{7\tau}{3\sqrt{15}}\right)
\]
$$I_4 \approx C_1 \left( \text{erf}(\sqrt{3}r) + \text{erf}(\frac{\sqrt{3}r}{3}) \right) + C_2 \left( \text{erf}(\frac{3r}{5}) + \text{erf}(\frac{6r}{5}) \right)$$

$$I_5 \approx C_1 \text{erfc}(\sqrt{3}r) + C_2 \text{erfc}(\frac{6r}{5})$$

(19)

3. Rayleigh Fading Channel

Assume that the maximal ratio combining (MRC) technique is employed at the receiver, the average SER of the $M^2$-ary TQAM over Rayleigh fading channel is calculated by

$$p^{N}_{\text{Ray}} = \int_{0}^{\infty} P_{e-M^2}(\gamma) f_{N-\text{Ray}}(\gamma) \, d\gamma$$

(20)

where, $N$ is the number of diversity receiver branches and $f_{N-\text{Ray}}(\gamma)$ is probability density function (pdf) of the instantaneous signal to noise ratio (SNR) $\gamma$ and is expressed as

$$f_{N-\text{Ray}}(\gamma) = \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!}$$

(21)

Applying (17) to (16), (18), (19) and (20), we can approximate $p^{N}_{\text{Ray}}$ by

$$p^{N}_{\text{Ray}} \approx \frac{1}{M^2} \left[ 2(\text{M} - 1)^2 I_{2}^{\text{Ray}} + (\text{M} - 1) I_{1}^{\text{Ray}} \right]$$

(22)

where,

$$I_{1}^{\text{Ray}} = \int_{0}^{\infty} \left( -\frac{1}{(1+\alpha+\beta)^{2}} - \frac{1}{(1+\alpha+2\beta)^{2}} \right) f_{N-\text{Ray}}(\gamma) \, d\gamma$$

$$I_{2}^{\text{Ray}} = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right) \, d\gamma$$

$$I_{3}^{\text{Ray}} = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right)^2 \, d\gamma$$

$$I_{4}^{\text{Ray}} = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right)^3 \, d\gamma$$

(23)

4. Rician Channel

The average SER of the $M^2$-ary TQAM over Rician channel is calculated by

$$p^{R_{\text{IC}}} = \int_{0}^{\infty} P_{e-M^2}(\gamma) f_{R_{\text{IC}}}(\gamma) \, d\gamma$$

(24)

where, $K$ is Rician factor.

Similarly to above, applying (17) to (16), (18), (19) and (23), we can approximate $p^{R_{\text{IC}}}$ by

$$p^{R_{\text{IC}}} \approx \frac{1}{M^2} \left[ 2(\text{M} - 1)^2 R_{1}^{\text{IC}} + (\text{M} - 1) R_{2}^{\text{IC}} \right]$$

(25)

where,

$$R_{1}^{\text{IC}}(\alpha, \beta) = \int_{0}^{\infty} e^{-\gamma} \left( \frac{\gamma}{N} e^{-\gamma} + \frac{\gamma}{N} e^{-\gamma} \right) f_{R_{\text{IC}}}(\gamma) \, d\gamma$$

$$R_{2}^{\text{IC}}(\alpha, \beta) = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right) \, d\gamma$$

(26)

$$R_{3}^{\text{IC}}(\alpha, \beta) = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right)^2 \, d\gamma$$

$$R_{4}^{\text{IC}}(\alpha, \beta) = \int_{0}^{\infty} \frac{\gamma^{N-1} e^{-\frac{\gamma}{N}}}{(N-1)!} \left( \frac{\gamma}{N} \right)^3 \, d\gamma$$

(27)

$$R_{1}^{\text{IC}} \approx \frac{\sqrt{\pi}}{12} R_{1}^{\text{IC}} \left( \frac{\sqrt{\pi}}{12} \right) + \frac{1}{4} \frac{\sqrt{\pi}}{3} \left( \frac{3}{12} \right)$$

$$R_{2}^{\text{IC}} \approx \frac{\sqrt{\pi}}{24} R_{2}^{\text{IC}} \left( \frac{\sqrt{\pi}}{12} \right) + \frac{1}{4} \frac{\sqrt{\pi}}{3} \left( \frac{3}{12} \right)$$

$$R_{3}^{\text{IC}} \approx \frac{\sqrt{\pi}}{24} R_{3}^{\text{IC}} \left( \frac{\sqrt{\pi}}{12} \right) + \frac{1}{4} \frac{\sqrt{\pi}}{3} \left( \frac{3}{12} \right)$$

$$R_{4}^{\text{IC}} \approx \frac{\sqrt{\pi}}{24} R_{4}^{\text{IC}} \left( \frac{\sqrt{\pi}}{12} \right) + \frac{1}{4} \frac{\sqrt{\pi}}{3} \left( \frac{3}{12} \right)$$

(28)
In the expressions of $R_1^K$, $R_2^K$, and $R_3^K$, $Q_1$ is the first order Marcum Q-function, $\beta_1 = \frac{\alpha}{1+K} + 1$, $\beta_2 = \frac{(\alpha+\beta)^2}{1+K} + 1$ and $\beta_3 = \frac{(3\alpha+4\beta)^2}{3(1+K)} + 1$.

5. Simulation Results

In this section, we use the Monte-Carlo simulation to present several simulation results and verify the accuracy of the above approximate method. In Fig. 2, we draw SER as a function of the average symbol energy to noise power ($Es/No$) over AWGN channel. As we can see, the simulation results and the approximate results match each other very well at high $Es/No$. Furthermore, the TQAM outperforms the SQAM because the minimum distance between signal points in the TQAM is larger than that in the SQAM under the constraint that the average energy per symbol is fixed for a pair comparison. Figure 3 describes the SER of 16-ary TQAM and 64-ary TQAM over Rayleigh fading channel. In addition, the error performance of the TQAM in case that the system uses MRC technique is also presented. Similarly to above, at high $Es/No$ regions, the simulation results match with the approximate results quite well. In Fig. 4, we investigate the effect of Rician $K$ factor on the performance of the 16-ary TQAM. It can be observed that when $K$ increases, the performance of the 16-ary TQAM increases. Again, the deviation between simulation curve and approximate curve is quite small; this verifies the accuracy of our approach.

6. Conclusion

In this paper, an accurate and simple approximate expression of the complementary error function is used to approximate the average SER of the TQAM over AWGN channel, Rayleigh fading channel and Rician channel. This approximate solution is simple and it can be easily applied for the general $M^2$-ary TQAMs. The simulation results show that the TQAM outperforms SQAM and the approximate results matches very well with the simulation results at high $Es/No$ value.

References