RECURSION RELATIONS FOR PATTERNS OF TYPE (2,1) IN FLATTENED PERMUTATIONS

TOUKIF MANSOUR, MARK SHATTUCK, AND DAVID G.L. WANG*

Abstract. We consider the problem of counting the occurrences of patterns of the form xy-z within flattened permutations of a given length. Using symmetric functions, we find recurrence relations satisfied by the distributions on $S_n$ for the patterns 12-3, 21-3, 23-1 and 32-1, and develop a unified approach to obtain explicit formulas. By these recurrences, we are able to determine simple closed form expressions for the number of permutations that, when flattened, avoid one of these patterns as well as expressions for the average number of occurrences. In particular, we find that the average number of 23-1 patterns and the average number of 32-1 patterns in Flatten($\pi$), taken over all permutations $\pi$ of the same length, are equal, as are the number of permutations avoiding either of these patterns. We also find that the average number of 21-3 patterns in Flatten($\pi$) over all $\pi$ is the same as it is for 31-2 patterns.

1. Introduction

The pattern counting problem for permutations has been studied extensively from various perspectives in both enumerative and algebraic combinatorics; see, e.g., [7]. The comparable problem has also been considered on other discrete structures such as $k$-ary words [2], compositions [11], and set partitions [10] (see also [6] and the references contained therein).

In his recent study of finite set partitions, Callan [3] introduced the notion of flattened partitions. In a previous paper [9], the authors considered permutations in the same sense and obtained formulas for the generating functions which count the flattened permutations of size $n$ according to the number of peaks and valleys. Here, we continue this work for some related statistics and derive recurrences for the distributions.

Let $[n] = \{1, 2, \ldots, n\}$ if $n \geq 1$, and $[0] = \emptyset$. Denote the set of permutations of $[n]$ by $S_n$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$. A pattern is any permutation of shorter length, and an occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ that is order-isomorphic to $\sigma$. If $r$ denotes the number of occurrences of a pattern $\sigma$ within a permutation $\pi$, then the case that has been studied most often previously is $r = 0$, i.e., the avoidance of $\sigma$ by $\pi$. Relatively little work has been done concerning the case when $r > 0$, and in what has been done, the patterns were usually of length three. For example, simple algebraic maps show that the six patterns of length three are divided into two classes with respect to pattern containment, a class with a representative pattern $\sigma = 123$ (see [14, 15]) and another with a representative pattern $\sigma = 132$ (see [12] and references contained therein).

By requiring that some of the letters within an occurrence of a pattern be adjacent, Babson and Steingr"ımsson [1] generalized the concept of pattern avoidance. Claesson and Mansour [4] considered the further notion of a pattern $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, said to be of type $(\ell_1, \ell_2, \ldots, \ell_k)$, wherein the subword $\sigma_i$ has length $\ell_i$ for each $i$ and the letters of each $\sigma_i$ are required to be adjacent within an occurrence of $\sigma$. In this notation, a classical pattern of length $k$ is of type $(1, 1, \ldots, 1)$. In particular,
the permutation $\pi$ is said to contain a pattern $\tau = xyz$ of type (2,1) if there exist indices $2 \leq i < j \leq n$ such that $\pi_{i-1}\pi_i\pi_j$ is order-isomorphic to $xyz$, where $xyz$ is some permutation of \{1,2,3\}. Otherwise, we say that $\pi$ avoids $\tau$.

Suppose $\pi \in S_n$ is represented in standard cycle form, that is, cycles arranged from left to right in ascending order according to the size of the smallest elements, where the smallest element is written first within each cycle. Let Flatten($\pi$) be the permutation of length $n$ obtained by erasing the parentheses enclosing the cycles of $\pi$ and considering the resulting word. For example, if $\pi = 71564328 \in S_8$, then the standard cycle form of $\pi$ is (172)(3546)(8) and Flatten($\pi$) = 17235468.

One can combine the ideas of the previous two paragraphs and say that a permutation $\pi$ contains a pattern $\tau$ in the flattened sense if and only if Flatten($\pi$) contains $\tau$ in the usual sense and avoids $\tau$ otherwise. Here, we will use this definition of pattern containment and consider the case when $\tau$ is a pattern of type (2,1). For example, the permutation $\pi = 71564328$ avoids 23-1 but has four occurrences of 31-2 in the flattened sense as Flatten($\pi$) = 17235468 avoids 23-1 but has four occurrences of 31-2.

Patterns where some of the elements are required to be adjacent are called generalized patterns in the literature. There is a trend in the combinatorics community to refer to such patterns now as vincular patterns. It is also becoming increasingly common to represent vincular patterns by overlining elements that are to be adjacent. In this way, the classical patterns would be denoted, for example, by 123, as they have been traditionally, instead of by 1-2-3. A pattern $xyz$ of type (2,1) would be represented by $\overline{xyz}$. In this paper, though, we will make use of the former notation (in accordance with [1]) when discussing vincular patterns not only for what we feel are aesthetic reasons but also to avoid possible confusion with the notion of barred patterns (see, for example, A201497 in [17]).

Let $st$ denote a statistic on $S_n$ which records the number of occurrences in the flattened sense of one of five patterns under consideration in this paper. In accordance with a previous paper [9], we will use the notation

$$g^s_t(a_1a_2\cdots a_k) = \sum_\pi q^s_t(Flatten(\pi)),$$

where $\pi$ ranges over all permutations of length $n$ such that Flatten($\pi$) starts with $a_1a_2\cdots a_k$. It is easy to see that $g^s_t(a_1a_2\cdots a_k) = 0$ if $a_1 \neq 1$. We will often denote $g^s_t(1)$ more simply by $g^s_t$.

In this paper, we use symmetric functions to develop recurrences for the generating functions $g^s_t$ in the cases when $st$ is the statistic recording the number of occurrences of $\tau$, where $\tau$ is any pattern of type (2,1) (except for 13-2). As a consequence, we obtain simple closed formulas for the number of permutations avoiding a pattern of type (2,1) in the flattened sense as well as for the average number of occurrences of a pattern taken over all permutations of a given size. We provide algebraic proofs of these results, as well as combinatorial proofs in all but one case.

We remark that our results can be expressed in terms of harmonic numbers, Stirling numbers, and two types of Bell numbers. Recall that $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the $n$-th harmonic number; see, e.g., Graham, Knuth and Patashnik [5]. Denote the Stirling numbers of the second kind by $S(n,k)$, the $n$th Bell number by $B_n$, and the $n$th complementary Bell number by $\hat{B}_n$ (also called the Rao Uppuluri-Carpenter number). Note that the sequence \{\hat{B}_n\} contains both positive and negative integers. More information about $\hat{B}_n$ may be found in [16] and in the entry A000587 of OEIS [17].

We summarize our results in Table 1 below and obtain as a consequence the following result.

**Theorem 1.1.** For any pattern $p$ of type (2,1), the average number $avr(n)$ of occurrences of $p$ in Flatten($\pi$) over all permutations $\pi$ of length $n$ satisfies

$$\lim_{n \to \infty} \frac{avr(n)}{n^2} = \frac{1}{12}. $$
Table 1. The number of avoiding and the average number of occurrences for patterns of type (2, 1) over flattened permutations of length n.

<table>
<thead>
<tr>
<th>pattern</th>
<th>number of avoiding</th>
<th>average number</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>13-2</td>
<td>$2^{n-1}$</td>
<td>$\frac{n^2 + 3n + 8}{12} - H_n$</td>
<td>[13]</td>
</tr>
<tr>
<td>31-2</td>
<td>$\binom{2n-2}{n-1}$</td>
<td>$\frac{n^3 - 3n^2 + 26n - 12}{12n}$</td>
<td>Corollary 2.2</td>
</tr>
<tr>
<td>21-3</td>
<td>$2 \sum_{k=1}^{n-1} k S(n - 1, k)$</td>
<td>$\frac{n^2 - 9n - 4}{12} + H_n$</td>
<td>Corollary 2.9</td>
</tr>
<tr>
<td>32-1</td>
<td>$\sum_{k=1}^{n-1} 2^k S(n - 1, k)$</td>
<td>$n^3 + 3n^2 - 40n + 24 \frac{12n}{12n}$</td>
<td>Corollary 2.13</td>
</tr>
<tr>
<td>23-1</td>
<td>$\sum_{k=1}^{n-1} 2^k S(n - 1, k)$</td>
<td>$\frac{n^2 - 9n - 4}{12} + H_n$</td>
<td>Corollary 2.5</td>
</tr>
<tr>
<td>12-3</td>
<td>$-2 \sum_{i=0}^{n-2} \binom{n-2}{i} (B_i + B_{i+1}) \tilde{B}_{n-i-3}$</td>
<td>$\frac{n^3 + 3n^2 - 40n + 24}{12n} + H_n$</td>
<td>Corollary 2.7</td>
</tr>
</tbody>
</table>

We will need the following notation. Let $X = \{x_1, x_2, \ldots, x_m\}$ be an ordered set. Define

$$
e_j(X) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq m} x_{i_1} x_{i_2} \cdots x_{i_j},$$

$$h_j(X) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq m} x_{i_1} x_{i_2} \cdots x_{i_j}.$$ 

In other words, $e_j(X)$ is the sum of products of any $j$ distinct elements of $X$ and $h_j(X)$ is the sum of products of any $j$ elements (non-distinct allowed) of $X$. For convenience, if $s_j$ represents either of these functions, let $s_0(X) = 1$, $s_j(X) = 0$ if $j < 0$, and $s_j(\emptyset) = \delta_{0,j}$, where the Kronecker’s delta notation $\delta_{i,j}$ is defined by

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

We will also follow the standard notation $[n] = \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}$ and $[m]! = \prod_{i=1}^{m}[i]$, with

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]![n-k]!}, \quad \text{if } 0 \leq k \leq n;$$

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = 0, \quad \text{otherwise},$$

where $q$ is an indeterminate. Note that $[n]_{q=0} = 1$ and $[n]_{q=1} = n$. Moreover, $[n]!_{q=1} = \binom{n}{2}$.

In what follows, we find that the recurrences satisfied by $g_{n}^{2,1}$ for the patterns 12-3, 21-3, 23-1 and 32-1 can be expressed in terms of various types of symmetric functions, while the enumeration of the pattern 31-2 can be done more or less explicitly, as it was in the case 13-2 [13]. In the final section, we provide a general approach to finding an explicit formula for the generating function for the number of permutations containing a fixed number of occurrences of a pattern of type (2,1) and illustrate it using the patterns 31-2 and 32-1.
2. Counting patterns of type \((2,1)\)

In this section, we find a recurrence satisfied by the generating function \(g_n^{st}\) for each of five patterns of type \((2,1)\). By using these recurrences, we are able to derive simple formulas for the average value of \(st\) over \(S_n\) in each case. We will suppress notation and omit the superscript \(st\) when discussing a particular pattern.

2.1. The pattern 31-2. Let \(\pi\) be a permutation of length \(n\). Suppose Flatten(\(\pi\)) starts with 1\(ij\). Let Flatten(\(\sigma\)) be the permutation obtained from Flatten(\(\pi\)) by removing the letter \(i\), and replacing each letter \(m\) \((m > i)\) by \(m - 1\). If \(j < i\), then each of the subsequences \(ij(i+1), ij(i+2), \ldots, ij(j-1)\) is an occurrence of 31-2 in Flatten(\(\pi\)). In this case, the number of occurrences of 31-2 in Flatten(\(\sigma\)) is \(i - j - 1\) less than the number of occurrences of 31-2 in Flatten(\(\pi\)). On the other hand, if \(j > i\), then Flatten(\(\sigma\)) has the same number of occurrences of 31-2 as Flatten(\(\pi\)). Therefore, for any \(3 \leq i \leq n\), we have

\[
g_n(1i) = \sum_{j<i} g_n(1ij) + \sum_{j>i} g_n(1ij) = \sum_{j<i} q^{i-j-1}g_{n-1}(j) + \sum_{j\geq i} g_{n-1}(j)
\]

\[
(2.1)
\]

The second-order difference transformation (in the index \(i\)) of the above formula gives

\[
g_n(1k) = (q + 1)g_n(1(k-1)) - q \cdot g_n(1(k-2)) + (q - 1)g_{n-1}(1(k-2)), \quad 5 \leq k \leq n.
\]

We will solve it with the initial values

\[
g_n(13) = g_{n-1}, \quad n \geq 3,
\]

\[
g_n(14) = g_{n-1} + 2(q - 1)g_{n-2}, \quad n \geq 4.
\]

Theorem 2.1. For any \(n \geq 2\), we have

\[
g_n = \sum_{j=1}^{\lfloor n/2 \rfloor} (q - 1)^{j-1}b_{n,j}g_{n-j},
\]

where

\[
b_{n,j} = \sum_{k=0}^{n-j-1} \frac{n-k}{j} \binom{n-j-1-k}{j-1} \binom{j-2+k}{j-2} q^k.
\]

Proof. For any \(k \geq 3\) and integer \(j\), define \(a_{k,j}\) by \(a_{3,j} = \delta_{j,1}, a_{4,j} = \delta_{j,1} + 2\delta_{j,2}\) and

\[
a_{k,j} = (q + 1)a_{k-1,j} - q \cdot a_{k-2,j} + a_{k-2,j-1}, \quad k \geq 5.
\]

Using (2.2) and (2.3), one can verify that

\[
g_n(1k) = \sum_{j \leq \lfloor k/2 \rfloor} a_{k,j}(q - 1)^{j-1}g_{n-j}, \quad 3 \leq k \leq n.
\]

Consequently, for any \(n \geq 2\), we have

\[
g_n = \sum_{k \geq 2} g_n(1k) = ng_{n-1} + \sum_{j=2}^{\lfloor n/2 \rfloor} (q - 1)^{j-1}b_{n,j}g_{n-j},
\]

where \(b_{n,j} = \sum_{k=3}^{n} a_{k,j}\). By (2.5), we have

\[
b_{n,j} = (q + 1)b_{n-1,j} - qb_{n-2,j} + b_{n-2,j-1} + 2\chi(j = 2 \text{ and } n \geq 4), \quad j \geq 2.
\]
It follows that
\[
B(x, y) = \sum_{n \geq 4} \sum_{j \geq 2} b_{n,j} x^{n-4} y^{j-2} = \frac{2-x}{(1-x)^2[(1-x)(1-qx) - x^2y]}.
\]

The desired expression for \( b_{n,j} \) can now be obtained by extracting the coefficient of \( x^{n-4} y^{j-2} \) in \( B(x, y) \).

The recurrence for \( g_n \) in the above theorem allows us to find the number of permutations \( \pi \) of length \( n \) such that Flatten(\( \pi \)) avoids 31-2. In fact, taking \( q = 0 \) in (2.4) gives
\[
g_n(0) = \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{j-1} \frac{n-j-1}{j} g_{n-j}(0), \quad n \geq 2,
\]
where \( g_n(0) \) denotes the generating function \( g_n \) evaluated at \( q = 0 \). By induction, one can show
\[
g_n(0) = \binom{2n-2}{n-1},
\]
for \( n \geq 1 \), as given in Table 1 above. We can also derive a formula for the average number of occurrences.

**Corollary 2.2.** If \( n \geq 1 \), then the average number of occurrences of 31-2 in Flatten(\( \pi \)) over \( \pi \in S_n \) is given by \( \frac{2^n - n^2 + 26n - 12}{12n} - H_n \).

**Proof.** It is easy to verify the formula for \( n = 1, 2, 3 \). Suppose \( n \geq 4 \). Differentiating both sides of the recurrence (2.4) with respect to \( q \), and setting \( q = 1 \), we get
\[
g_n'(1) = n g_{n-1}'(1) + (n-2) \frac{(n-2)(n-3)(n+2)}{6}.
\]
Solving this recurrence with the initial value \( g_3'(1) = 0 \), and noting that the average number of occurrences equals \( g_n'(1)/n! \), yields the desired formula. \[\square\]

2.2. The pattern 32-1. Let \( 3 \leq i \leq n \). By combinatorial reasoning similar to that used in the case 31-2 above, we have
\[
g_n(1i) = \sum_{j<i} q^{j+2} g_{n-1}(1j) + \sum_{j \geq i} q^n g_{n-1}(1j) = g_{n-1} + \sum_{j \geq i} (q^{j+2} - 1) g_{n-1}(1j).
\]
The first-order difference transformation applied to (2.6) gives the recurrence
\[
g_k(1k) = g_{n-1}(1(k-1)) + (q^{k+2} - 1) g_{n-1}(1(k-1)), \quad 4 \leq k \leq n,
\]
with the initial value \( g_n(13) = g_{n-1} \).

**Lemma 2.3.** Let \( k \geq 3 \). For all \( j \geq 1 \),
\[
e_j([1], [2], \ldots, [k-3]) = \frac{1}{(1-q)^2} \sum_{a=0}^{k-3} (-1)^a q^{(a+1)/2} [k-3] \binom{k-3-a}{k-3-j}.
\]
Proof. Let \( F(x_1, x_2, \ldots, x_k) = \sum_{j=0}^{k} e_j(x_1, x_2, \ldots, x_k) z^j \). By the definition of elementary symmetric functions, we have

\[
F([1], [2], \ldots, [k-3]) = \prod_{a=1}^{k-3} (1 + [a]z) = \left(1 + \frac{z}{1 - q} \right)^{k-3} \left( \frac{qz}{1 - q + z} \right)^{k-3} \]

\[
= \sum_{a=0}^{k-3} q^a(z) \left[ k - 3 \right] - a \frac{q(1 - q)^a z^{k-3-a}}{(1 - q)^{k-3}} \]

\[
= \sum_{a=0}^{k-3} \sum_{b=0}^{k-3-a} (-1)^a q^{\left( \frac{a+1}{2} \right)} \left( k - 3 - a \right) \left( k - 3 - a \right) \frac{z^{a+b}}{(1 - q)^{a+b}} .
\]

The desired formula now follows from comparing coefficients of \( z^j \) on both sides of the above identity. \( \square \)

We can now give an explicit recurrence for \( g_n \).

**Theorem 2.4.** For any \( n \geq 2 \), we have

\[
g_n = n g_{n-1} + \sum_{j=2}^{n-2} \left( \sum_{a=1}^{j} (-1)^j q^a(z) \sum_{k=0}^{n-2-j} \left( j - a + k \right) \left( j - 1 + k \right) \right) g_{n-j} .
\]

**Proof.** For any \( k \geq 3 \) and integer \( j \), define \( a_{k,j} = e_{j-1}([1], [2], \ldots, [k-3]) \). Using (2.7), one can verify that

\[
g_n(1k) = \sum_{j=1}^{k-2} a_{k,j}(q - 1)^{j-1} g_{n-j}, \quad 3 \leq k \leq n.
\]

Therefore,

\[
g_n = \sum_{k \geq 2} g_n(1k) = n g_{n-1} + \sum_{j=2}^{n-2} b_{n,j}(q - 1)^{j-1} g_{n-j},
\]

where \( b_{n,j} = \sum_{k=j+2}^{n} e_{j-1}([1], [2], \ldots, [k-3]) \). By Lemma 2.3, we have

\[
b_{n,j} = \frac{1}{(1 - q)^{j-1}} \sum_{k=j+2}^{n} \sum_{a=0}^{k-3} (-1)^a q^{\left( \frac{a+1}{2} \right)} \left( k - 3 - a \right) \left( k - 2 - j \right) ,
\]

which gives (2.8). \( \square \)

Setting \( q = 0 \) in (2.8) yields

\[
g_n(0) = n g_{n-1}(0) + \sum_{j=2}^{n-2} (-1)^j \left( n - 2 \right) g_{n-j}(0), \quad n \geq 2,
\]

with \( g_1(0) = 1 \). One can prove by induction that

\[
g_n(0) = \frac{1}{e^2} \sum_{k \geq 1} \frac{2^k k^{n-1}}{k!} = \sum_{k=1}^{n-1} 2^k S(n-1, k), \quad n \geq 2.
\]
We found that the sequence \( g_n(0) \) occurs as entry A001861 in OEIS [17]. There is also the following formula for the average number of occurrences of 32-1, which can be deduced similarly as it was in the case 31-2.

**Corollary 2.5.** If \( n \geq 1 \), then the average number of occurrences of 32-1 in Flatten(\( \pi \)) over \( \pi \in S_n \) is given by \( \frac{n^2 g_n(0)}{12} + H_n \).

### 2.3. The pattern 23-1. Let \( 3 \leq i \leq n \). We have

\[
(2.9) \quad g_n(1i) = \sum_{j<i} g_{n-1}(1j) + \sum_{j \geq i} q^{i-2} g_{n-1}(1j) = q^{i-2} g_{n-1} + (1 - q^{i-2}) \sum_{j \leq i-1} g_{n-1}(1j).
\]

The first-order difference transformation of the above formula gives the recurrence

\[
(2.10) \quad g_n(1k) = -\frac{q^{k-3} g_{n-1}}{k-3} + \left[ k - 2 \right] g_n(1(k-1)) + (1 - q)[k - 2]g_{n-1}(1(k-1)), \quad 4 \leq k \leq n,
\]

with \( g_n(13) = q \cdot g_{n-1} + 2(1 - q) g_{n-2} \).

**Theorem 2.6.** For all \( n \geq 2 \), we have

\[
(2.11) \quad g_n = (1 + [n - 1]) g_{n-1} + \sum_{j=2}^{n-1} b_{n,j}(1 - q)^{j-1} g_{n-j},
\]

where

\[
b_{n,2} = \sum_{k=1}^{n-2} \frac{[k]}{[k-1]} (1 + [k]) = \frac{(2 - n) q}{(q - 1)^2} + \frac{q^3 - 3q^2 + q + 4}{(q - 1)^3(q + 1)} + \frac{(q - 3)q^{n-1}}{(q - 1)^3} + \frac{q^{2n-2}}{(q - 1)^3(q + 1)},
\]

and for \( j \geq 3 \),

\[
b_{n,j} = \sum_{k=j+1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_{j-2} \leq k-3} (1 + [i_1])[i_1][i_2] \cdots [i_{j-2}][k-2].
\]

**Proof.** For any \( 3 \leq k \leq n \) and integer \( j \), define \( d_{n,k,j} \) by \( d_{n,3,j} = q \delta_{j,1} + 2(1 - q) \delta_{j,2} \) and

\[
d_{n,k,j} = -\frac{q^{k-3}}{k-3} \delta_{j,1} + \left[ k - 2 \right] d_{n,k-1,j} + (1 - q)[k - 2]d_{n-1,k-1,j-1}, \quad 4 \leq k \leq n.
\]

Using (2.10), one can verify that \( g_n(1k) = \sum_{j=1}^{k-1} d_{n,k,j} g_{n-j} \) for all \( 3 \leq k \leq n \). On the other hand, we may find \( d_{n,k,j} \) for \( k \geq 4 \) by iteration as follows:

\[
d_{n,k,1} = -[k - 2] \sum_{j=3}^{k-1} \frac{q^{k-j}}{[k-j][k-j + 1]} + [k - 2] d_{n,3,1} = q^{k-2},
\]

\[
d_{n,k,2} = [k - 2]d_{n,3,2} + (1 - q)[k - 2] \sum_{j=1}^{k-3} q^j = (1 - q)[k - 2](1 + [k - 2]),
\]

\[
d_{n,k,j} = -\frac{q^{k-3}}{k-3} \delta_{j,1} + \left[ k - 2 \right] d_{n,k-1,j} + (1 - q)[k - 2]d_{n-1,k-1,j-1}, \quad 4 \leq k \leq n, 3 \leq j \leq k.
\]
and for $j \geq 3$,
\[
d_{n,k,j} = (1 - q)[k - 2] \sum_{i=j}^{k-1} d_{n-1,i,j-1}
\]
\[
= (1 - q)^{j-2}[k - 2] \sum_{2<i_{j-2} < \cdots < i_2 < i_1 < k} [i_1 - 2][i_2 - 2] \cdots [i_{j-3} - 2] d_{n-j+2,i_{j-2},2}.
\]
\[
= (1 - q)^{j-1}[k - 2]a_{k,j},
\]
where
\[
a_{k,j} = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k - 3} (1 + [i_1])[i_1][i_2] \cdots [i_{j-2}], \quad j \geq 3.
\]

For all $3 \leq k \leq n$, we then have
\[
g_n(1k) = q^{k-2}g_{n-1} + [k - 2] \sum_{j=2}^{k-1} a_{k,j}(1 - q)^{j-1}g_{n-j},
\]
where $a_{k,2} = 1 + [k - 2]$. Formula (2.11) now follows from noting that $g_n = \sum_{k=2}^{n} g_n(1k)$. \qed

As in the case for 32-1 above, we may compute the number of permutations $\pi$ of length $n$ such that Flatten($\pi$) avoids 23-1 and find
\[
g_n(0) = \sum_{k=1}^{n-1} q^k S(n - 1, k), \quad n \geq 2.
\]

A similar proof as before may also be given for the average number of 23-1 occurrences.

**Corollary 2.7.** If $n \geq 1$, then the average number of occurrences of 23-1 in Flatten($\pi$) over $\pi \in S_n$ is given by $\frac{n^2 - n - 4}{12} + H_n$.

### 2.4. The pattern 21-3

#### Let $3 \leq i \leq n$. We have
\[
g_n(1i) = \sum_{j<i} q^{n-j}g_{n-1}(1j) + \sum_{j \geq i} g_{n-1}(1j) = g_{n-1} - (1 - q^{n-i}) \sum_{j \leq i-1} g_{n-1}(1j).
\]

In particular, since $g_n(12) = 2g_{n-1}$, we have
\[
g_n(13) = g_{n-1} - (1 - q^{n-3})g_{n-1}(12) = g_{n-1} + 2(q - 1)[n - 3]g_{n-2}, \quad n \geq 3.
\]

The first-order difference transformation of (2.12) gives
\[
g_n(1k) = \frac{q^{n-k}g_{n-1}}{[n - k + 1]} + \frac{[n - k]g_n(1(k - 1))}{[n - k + 1]} + (q - 1)[n - k]g_{n-1}(1(k - 1)), \quad 4 \leq k \leq n.
\]

**Theorem 2.8.** For all $n \geq 2$, we have
\[
g_n = ng_{n-1} + \sum_{j=2}^{n-1} b_{n,j}(q - 1)^{j-1}g_{n-j},
\]
where
\[
b_{n,2} = \sum_{k=3}^{n} (k - 1)[n - k] = -\frac{n^2}{2(q - 1)} + \frac{(q - 3)n}{2(q - 1)^2} + \frac{q(2q^{n-2} - q^{n-3} + q - 2)}{(q - 1)^3},
\]
and for $j \geq 3$,
\[
b_{n,j} = \sum_{k=j+1}^{n} \sum_{n-k \leq i_1 \leq i_2 \leq \cdots \leq i_{j-2} \leq n-j-1} (n-j-i_{j-2}+1)[i_1][i_2] \cdots [i_{j-2}][n-k].
\]

Proof. For any $3 \leq k \leq n$ and integer $j$, define $d_{n,k,j}$ by $d_{n,3,j} = \delta_{j,1} + 2(q-1)(n-3)\delta_{j,2}$ and
\[
d_{n,k,j} = \frac{q^{n-k}}{|n-k+1|} \delta_{j,0} + \frac{n-k}{|n-k+1|} d_{n,k-1,j} + (q-1)(n-k)d_{n-1,k-1,j-1}, \quad 4 \leq k \leq n.
\]
Using (2.14), one can verify that $g_n(1k) = \sum_{j=1}^{k-1} d_{n,k,j}g_{n-j}$ for any $3 \leq k \leq n$. On the other hand, we may find $d_{n,k,j}$ by iteration as follows:
\[
d_{n,k,1} = \frac{n-k}{|n-3|} + \sum_{j=3}^{k} \frac{n-k}{|n-j|(|n+1-j|)} q^{n-j} = 1,
\]
\[
d_{n,k,2} = \frac{n-k}{|n-3|} d_{n,3,2} + (k-3)(q-1)(n-k) = (q-1)(k-1)[n-k],
\]
and for $j \geq 3$,
\[
d_{n,k,j} = (q-1)[n-k] \sum_{i=j+1}^{k-1} d_{n-1,i,j-1}
\]
\[= (q-1)^{j-1}[n-k] \sum_{3 \leq i_{j-2} < \cdots < i_1 \leq k-1} (i_{j-2} - 1) \prod_{\ell=1}^{j-2} [n-\ell-i_\ell].
\]
Combining these formulas, we obtain
\[
g_n(1k) = g_{n-1} + [n-k] \sum_{j=2}^{k-1} a_{n,k,j}(q-1)^{j-1}g_{n-j}, \quad 3 \leq k \leq n,
\]
where $a_{n,k,2} = k-1$ and
\[
a_{n,k,j} = \sum_{n-k \leq i_1 \leq i_2 \leq \cdots \leq i_{j-2} \leq n-j-1} (n-j-i_{j-2}+1)[i_1][i_2] \cdots [i_{j-2}], \quad j \geq 3.
\]
Formula (2.15) now follows from the fact that $g_n = \sum_{k=2}^{n} g_n(1k)$. \hfill \Box

We can find the number of permutations of length $n$ such that Flatten($\pi$) avoids the pattern 21-3. Let $n \geq 3$. Taking $q = 0$ in (2.15) gives
\[
g_n(0) = ng_{n-1}(0) - \frac{n(n-3)}{2} g_{n-2}(0) + \sum_{j=3}^{n-1} (-1)^{j-1} \binom{n-2}{j} \binom{n-3}{j-1} g_{n-j}(0),
\]
with $g_1(0) = 1$ and $g_2(0) = 2$. By a routine application of the generating function method, we deduce that
\[
\sum_{n \geq 0} g_{n+2}(0) \frac{x^n}{n!} = 2e^{e^x+2x-1},
\]
which gives
\[
g_n(0) = 2 \sum_{k=1}^{n-1} kS(n-1,k), \quad n \geq 2.
\]
Differentiating both sides of recurrence (2.15) with respect to \( q \) yields the following corollary, the details of the proof we leave to the interested reader.

**Corollary 2.9.** If \( n \geq 1 \), then the average number of occurrences of 21-3 in Flatten(\( \pi \)) over \( \pi \in S_n \) is given by
\[
\frac{n^3 - 3n^2 + 26n - 12}{12n} - H_n.
\]

2.5. **The pattern 12-3.** Let \( 3 \leq i \leq n \). We have
\[
g_n(1i) = \sum_{j<i} g_n(1ij) + \sum_{j>i} g_n(1ij) = \sum_{j<i} q^{j-i+1}g_{n-1}(1j) + q^{n-i} \sum_{j \geq i} g_{n-1}(1j)
\]
(2.18)
\[
= q^{n-i}g_{n-1} - \sum_{j \leq i-1} (q^{n-i} - q^{j-i+1})g_{n-1}(1j).
\]
The first-order difference transformation of the above formula gives the recurrence
(2.19)
\[
g_n(1k) = q^{-1}g_n(1(k-1)) + (1 - q^{-k})g_{n-1}(1(k-1)), \quad 4 \leq k \leq n,
\]
with the initial value
(2.20)
\[
g_n(13) = q^{n-3}g_{n-1} - 2q^{n-3}(q^n - 1)g_{n-2}, \quad n \geq 3.
\]

**Lemma 2.10.** Let \( n \geq 0, 0 \leq j \leq k-1 \) and \( X = \{[n+i]: 0 \leq i \leq k-j-1\} \). Then
\[
h_{j-1}(X) = \sum_{i=0}^{j-1} (-1)^i \left[ \begin{array}{c} k-j-1+i \\ i \\ \end{array} \right] \left( \begin{array}{c} k-2 \\ j-i \\ \end{array} \right) \frac{q^i}{(1-q)^{j-1}}.
\]

**Proof.** By the definition of \( h_j \), we have \( \sum_{j \geq 0} h_j(X)z^j = \prod_{x \in X} (1 - xz)^{-1} \) for any set \( X \). Taking \( X = \{[n+i]: 0 \leq i \leq k-j-1\} \), we obtain
\[
\sum_{j \geq 1} h_{j-1}(X)z^{j-1} = \left( 1 - \frac{z}{1-q} \right)^{j-k} \prod_{i=0}^{k-1} \left( 1 + \frac{zq^i}{1-q-z}q^i \right)^{-1}
\]
\[
= \sum_{a \geq 0} (-1)^a \left[ \begin{array}{c} k-j-1+a \\ a \\ \end{array} \right] \frac{q^a}{(1-q)^a} \left( \frac{1-z}{1-q} \right)^{j-k-a}
\]
\[
= \sum_{a,b \geq 0} (-1)^a \left[ \begin{array}{c} k-j-1+a \\ a \\ \end{array} \right] \left( \begin{array}{c} k-j+a+b-1 \\ b \\ \end{array} \right) \frac{q^a}{(1-q)^{a+b}}.
\]
The desired formula now follows from comparing coefficients of \( z^{j-1} \) on both sides of the above identity. \( \square \)

**Theorem 2.11.** For any \( n \geq 3 \), we have
(2.21)
\[
g_n = \sum_{j=1}^{n-1} c_{n,j}g_{n-j},
\]
where
\[
c_{n,j} = \sum_{i=0}^{j-1} \sum_{k=0}^{n-1-j} (-1)^i \left( 2 \left[ \begin{array}{c} k+i \\ i \\ \end{array} \right] \left[ \begin{array}{c} k+j-1 \\ j-i-1 \\ \end{array} \right] - \left[ \begin{array}{c} k+i-1 \\ i \\ \end{array} \right] \left[ \begin{array}{c} k+j-2 \\ j-i-1 \\ \end{array} \right] \right) q^{(i+1)(n-j-k-1)}.
\]
Proof. Using (2.19) and (2.20), one can verify that

\[ g_n(1k) = q^{n-k} \sum_{j=1}^{k-1} (1-q)^{j-1} a_{n,k,j} g_{n-j}, \quad 3 \leq k \leq n, \]

where \( a_{n,k,j} = 2h_{j-1}(\{[n-\ell]: 1 \leq \ell \leq k\}) - h_{j-1}(\{[n-\ell]: 2 \leq \ell \leq k\}) \). Therefore,

\[ g_n = \sum_{k=2}^{n} g_n(1k) = (2q^{n-2} + [n-2])g_{n-1} + \sum_{j=2}^{n-1} b_{n,j}(1-q)^{j-1} g_{n-j}, \]

where

\[ b_{n,j} = \sum_{k=j+1}^{n} \left( 2h_{j-1}(\{[n-\ell]: 1 \leq \ell \leq k\}) - h_{j-1}(\{[n-\ell]: 2 \leq \ell \leq k\}) \right) q^{n-k}. \]

Recurrence (2.21) now follows from Lemma 2.10. □

By using recurrence (2.21), one can obtain the number of permutations of length \( n \) avoiding the pattern 12-3. Recall that the Bell numbers \( B_n \) satisfy

\[ e^{e^x-1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}, \quad e^{e^x+x-1} = \sum_{n \geq 0} B_{n+1} \frac{x^n}{n!}, \]

and that the complementary Bell numbers \( \tilde{B}_n \) satisfy

\[ \int_0^x e^{1-e^t} dt = \sum_{n \geq 1} \tilde{B}_n \frac{x^n}{n!} dt = \sum_{n \geq 1} \tilde{B}_{n-1} \frac{x^n}{n!}. \]

We define \( \tilde{B}_{-1} = -1 \).

Corollary 2.12. If \( n \geq 2 \), then the number of permutations \( \pi \) of length \( n \) with \( \text{Flatten}(\pi) \) avoiding 12-3 is \(-2 \sum_{i=0}^{n-2} \binom{n-2}{i} (B_i + B_{i+1}) \tilde{B}_{n-i-3}\).

Proof. Letting \( q = 0 \) in (2.21) gives

\[ g_n(0) = \sum_{j=1}^{n-2} \left( \binom{n-3}{j-1} + \binom{n-4}{j-2} \right) g_{n-j}(0), \quad n \geq 4, \]

with \( g_2(0) = g_3(0) = 2 \). Let \( G(x) = \sum_{n \geq 2} g_n(0) \frac{x^n}{(n-2)!} \). Then the above recurrence translates to

\[ G''(x) = (e^x G(x))' + e^x G(x). \]

Solving this differential equation with the initial conditions \( G(0) = G'(0) = 2 \) gives

\[ G(x) = 2(e^x + 1)e^{e^x-1} \left( 1 - \int_0^x e^{1-e^t} dt \right) - 2. \]

Extracting the coefficient of \( x^{n-2} \) in this formula for \( G(x) \), and using (2.22) and (2.23), yields the requested result. □

Corollary 2.13. If \( n \geq 1 \), then the average number of occurrences of 12-3 in \( \text{Flatten}(\pi) \) over \( \pi \in S_n \) is given by

\[ \frac{n^3 + 3n^2 - 40n + 24}{12n} + H_n. \]
Proof. Differentiating both sides of (2.21) with respect to \( q \), and letting \( q = 1 \), yields
\[
g'_n(1) = \left( 2(n - 2)q^{n-3} + \frac{[n - 2]'}{q} \right) \bigg|_{q=1} \sum_{\sigma} g_{n-1}(1) + ng'_{n-1}(1) - b_{n,2}\bigg|_{q=1} = \left( 2(n - 2) + \frac{n - 2}{2} \right)(n-1)! + ng'_{n-1}(1) - \frac{(n+2)(n-2)(n-3)}{3} (n-2)!, \quad n \geq 2,
\]
with \( g'_n(1) = 0 \). The desired result now follows from solving this recurrence and noting that the average number of occurrences is given by \( g'_n(1)/n! \).

3. Combinatorial proofs

In this section, we provide combinatorial proofs of all entries in Table 1 above except for the one concerning avoidance of the pattern 12-3. Given positive integers \( m \) and \( n \), let \([m, n] = \{m, m+1, \ldots, n\}\) if \( n \geq m \), with \([m, n] = \emptyset \) if \( m > n \). We first prove the statements concerning the avoidance.

3.1. The avoidance numbers.

3.1.1. The avoidance numbers for 23-1 and 32-1. We first treat the case 23-1. Recall that the Stirling number of the second kind \( S(m, k) \) counts the partitions of an \( m \)-element set into exactly \( k \) blocks. Then the sum \( \sum_{k=1}^{n-1} 2^k S(n-1, k) \) counts the partitions of the set \([2, n]\) having any number of blocks in which some subset of the blocks are marked. We will denote the set of such partitions by \( \pi(n)^* \). Let \( \pi = B_1/B_2/\cdots/B_k \in \pi(n)^* \), \( 1 \leq k \leq n-1 \), where the \( B_i \)'s are arranged in ascending order of smallest elements and some subset of the \( B_i \)'s are marked. Furthermore, we assume within each block \( B_i \) that the elements are written in descending order. Finally, let \( m_i \) denote the smallest element of block \( B_i \), \( 1 \leq i \leq k \).

We now transform \( \pi \) into a permutation of size \( n \) whose flattened form avoids the pattern 23-1. We start by writing the element 1 in a cycle by itself as \((1\cdots)\). We then consider the block \( B_1 \). If \( B_1 \) is not marked, then write the elements of \( B_1 \) in descending order after 1 within its cycle to obtain the longer cycle \((1B_1\cdots)\). If \( B_1 \) is marked, then write all elements of \( B_1 \) except for the last in the cycle with 1 and start a new cycle with \( m_1 = 2 \); in this case, one would have two cycles \((1\tilde{B}_1\cdots)\) and \((2\cdots)\), where \( \tilde{B}_1 = B_1 - \{m_1\} \).

Continue in this fashion, inductively, as follows. If \( i \geq 2 \) and block \( B_i \) is not marked, then write all of the elements of \( B_i \) at the end of the last current cycle, while if \( B_i \) is marked, write all of the elements of \( B_i \) except \( m_i \) at the end of the last current cycle and then write the element \( m_i \) in a new cycle by itself. Doing this for each of the \( k \) blocks of \( \pi \) yields a permutation \( \sigma \) of length \( n \) such that Flatten(\( \sigma \)) avoids 23-1 and has exactly \( k \) ascents. The above procedure is seen to be reversible, and hence is a bijection, upon considering whether or not the smaller number in an ascent within Flatten(\( \sigma \)) starts a new cycle of \( \sigma \).

For example, if \( n = 10 \) and \( \pi = \{6,5,2\}, \{10,7,3\}, \{4\}, \{9,8\} \in \pi(10)^* \), with the second and third blocks marked, then the corresponding permutation in standard cycle form would be \( \sigma = (1,6,5,2,10,7), (3), (4,9,8) \).

For the case 32-1, we now define a bijection between permutations avoiding 23-1 and those avoiding 32-1 in the flattened sense. To do so, given \( \sigma \) avoiding 23-1, let \( t_1 = 1 < t_2 < \cdots < t_\ell \) denote the set of numbers consisting of the first (i.e., left) letters of the ascents of Flatten(\( \sigma \)), going from left to right. Note that the \( t_i \) are increasing since there is no occurrence of 23-1 in Flatten(\( \sigma \)). Given \( 1 < i \leq \ell \), let \( \alpha_i \) denote the sequence (possibly empty) of numbers occurring between \( t_{i-1} \) and \( t_i \) in Flatten(\( \sigma \)). Note that the letters of each \( \alpha_i \) must all belong to the same cycle of \( \sigma \), by definition of the \( t_i \), since \( \sigma \) is assumed to be in standard cycle form. Write the sequence \( \alpha_i \) in reverse order for each \( i \), where the
letters remain in the same cycle of $\sigma$. If $\sigma'$ denotes the resulting permutation, then it may be verified that the mapping $\sigma \mapsto \sigma'$ is the requested bijection. For example, if $\sigma$ is as above, then the order of the letters between 1 and 2 and between 2 and 3 is reversed, which gives $\sigma' = (1, 5, 6, 2, 7, 10), (3, 4, 9, 8)$. Given a permutation $\rho = \rho_1\rho_2 \cdots \rho_n$ which avoids 32-1, the above mapping is reversed by considering the subsequence $\rho_i$ of Flatten$(\rho)$, where $\rho_1 = 1$ and $\rho_i$ is the smallest letter to the right of $\rho_{i-1}$ if $r > 1$, and changing the order of the letters between $\rho_{i-1}$ and $\rho_i$ for each $r$.

3.1.2. The avoidance numbers for 31-2. We will show that a permutation avoids 31-2 in the flattened sense if and only if it avoids 3-1-2, whence the result will follow from Theorem 2.4 in [8] where a combinatorial proof was given. Clearly, a permutation avoiding 3-1-2 also avoids 31-2. So suppose a permutation $\sigma$ contains an occurrence of 3-1-2 in the flattened sense. We will show that it must contain an occurrence of 31-2. Let Flatten$(\sigma) = \sigma_1\sigma_2 \cdots \sigma_n$, which we will denote by $\sigma'$.

First suppose that there is at least one ascent to the right of $n$ in $\sigma'$. Let $j$ denote the index of the left-most such ascent. That is, there exist indices $i$ and $j$ with $i < j < n$ such that $\sigma_i = n$, $\sigma_i > \sigma_{i+1} > \cdots > \sigma_j$ and $\sigma_{j+1} > \sigma_j$. Since $\sigma_j < \sigma_{j+1} < \sigma_i$, there exists an index $\ell$ with $i < \ell \leq j$ such that $\sigma_i < \sigma_{j+1} < \sigma_{\ell-1}$. Then the subsequence $\sigma_{i-1} \sigma_i \sigma_{j+1}$ would be an occurrence of 31-2 in $\sigma'$, which completes this case.

On the other hand, suppose there is no ascent in $\sigma'$ to the right of the letter $n$. If $\sigma_i = n$, then $\sigma_i > \sigma_{i+1} > \cdots > \sigma_n$. Apply now the reasoning of the previous paragraph to the subpermutation $\sigma_1\sigma_2 \cdots \sigma_{i-1}$, considering instead of $n$, the largest element among the first $i - 1$ positions of $\sigma'$. If an occurrence of 31-2 arises as before, then we are done. Otherwise, continue with still a smaller subpermutation. If no occurrence of 31-2 arises before all of the positions of $\sigma'$ are exhausted, then it must be the case that there is the following decomposition of $\sigma'$:

$$\sigma' = T_r T_{r-1} \cdots T_1,$$

for some $r \geq 1$, where $T_i$ starts with the letter $n$ and is decreasing and $T_i$, $1 < i \leq r$, starts with the largest letter to the left of $T_{i-1}T_{i-2} \cdots T_1$ followed by a possibly empty decreasing sequence.

Now suppose $\sigma'$ contains an occurrence of 3-1-2 consisting of the letters x, y, and z, respectively. If $m_k = \max(T_k)$, $1 \leq k \leq r$, then $m_1 > m_2 > \cdots > m_r$, which implies that we may assume that $x$ and $y$ belong to the same block $T_i$ for some $i$, with $x = m_i$, and that $z$ belongs to a block $T_j$ for some $j$ with $j < i$. (Note that if $x \in T_i$ and $y \in T_{i+1}$, then the occurrence of 31-2 would be in $\sigma'$.) Since the letters are decreasing between $x$ and $y$, inclusive, with $x > z > y$, there must exist an occurrence of 31-2 in $\sigma'$ where the 3 and 1 correspond to a pair of adjacent letters between $x$ and $y$ (and possibly including $x$ or $y$) and the 2 corresponds to the letter $z$. Thus, $\sigma'$ contains an occurrence of 31-2 in all cases, which completes the proof.

3.1.3. The avoidance numbers for 21-3. Suppose $\sigma \in S_n$ is such that $\sigma' = $ Flatten$(\sigma)$ avoids 21-3. Then $\sigma'$ must be of the form

$$\sigma' = 1L_1L_2 \cdots L_s,$$

for some $s \geq 1$, where $L_1$ is an increasing sequence whose last letter is $n$ and $L_i$ is an increasing sequence whose last letter is the largest occurring to the right of $1L_1L_2 \cdots L_{i-1}$ in $\sigma'$ if $i > 1$. Note that the $L_i$ may be identified as blocks of a partition of the set $[2, n]$, arranged left to right in descending order of largest elements, where the elements within a block are written in ascending order.

Observe that a letter corresponding to a right-to-left minimum in $\sigma'$ either may or may not start a cycle of $\sigma$, when expressed in standard cycle form, and for all such letters greater than one, both alternatives are possible. Note that no other letters in $\sigma'$ may start cycles. From this, we see that the
The average number of occurrences for 3.2.1. will denote the total number of occurrences of the pattern under consideration. The 2 corresponds to the actual letter i and an occurrence in which the 2 and 3 correspond to the letters i and j, respectively, as an (i, j)-occurrence. In the proofs that follow, tot(τ) will denote the total number of occurrences of the pattern τ under consideration.

3.2. The average numbers. We first introduce some notation which we will use in this subsection. Given a pattern τ of type (2, 1), we will refer to an occurrence of τ (in the flattened sense) in which the 2 corresponds to the actual letter i as an i-occurrence of τ and an occurrence in which the 2 and 3 correspond to the letters i and j, respectively, as an (i, j)-occurrence. In the proofs that follow, tot(τ) will denote the total number of occurrences of the pattern τ under consideration.

3.2.1. The average number of occurrences for 23-1 and 32-1. We first treat the case 32-1 and argue that the total number of i-occurrences of 32-1 (in the flattened sense) within all of the permutations of size n is given by

\[ (n - i) \binom{i-1}{2} \frac{(n-1)!}{i} \text{ for } i \in [3, n-1]. \]

Summing over i would then give the total number of occurrences of 32-1.
Note that within an \(i\)-occurrence of 32-1, the letter \(i\) cannot start a cycle since there is a letter to the right of it in the flattened form which is smaller. Note further that the position of \(j\) is determined by that of \(i\)'s within an \((i, j)\)-occurrence of 32-1, where \(i + 1 \leq j \leq n\). Given \(i\) and \(j\), we count the permutations of size \(n\) for which there are exactly \(r\) \((i, j)\)-occurrences of 32-1, respectively, where \(1 \leq r \leq i - 2\). Note that the position of \(i\) is determined within such permutations once the positions of the elements of \([i - 1]\) have been, which also determines the position of \(j\) (note that \(i\) must be placed within a current cycle so that there are exactly \(r\) members of \([i - 1]\) to its right within the flattened form).

Thus, there are \(\frac{n!}{i!}\) such permutations for each \(r\), which implies that the total number of \((i, j)\)-occurrences of 32-1 is given by

\[
\sum_{r=1}^{i-2} r \frac{(n-1)!}{i!} = \binom{i-1}{2} \frac{(n-1)!}{i!}.
\]

Since there are \(n - i\) choices for \(j\), given \(i\), with each choice yielding the same number of \((i, j)\)-occurrences of 32-1, it follows that there are \(\binom{n-1}{2} \frac{(n-1)!}{i!}\) \(i\)-occurrences of 32-1, as desired. Summing over \(3 \leq i \leq n - 1\), and simplifying, then gives

\[
tot(32-1) = \sum_{i=3}^{n-1} (n-i) \binom{i-1}{2} \frac{(n-1)!}{i!} = n! \sum_{i=3}^{n-1} \left( i - \frac{3}{2} + \frac{1}{i} \right) - (n-1)! \binom{n-1}{3}
\]

\[
= \frac{n!}{2} \left( \binom{n-3}{2} + n! \left( H_{n-1} - \frac{3}{2} \right) \right) - (n-1)! \binom{n-1}{3}
\]

\[
= \frac{n!}{12} (n^2 - 9n - 4) + n! H_n.
\]

Dividing by \(n!\) yields the average value formula given in Corollary 2.5.

Writing the letter corresponding to 3 directly after (instead of directly before) the letter corresponding to 2 shows that the total number of \((i, j)\)-occurrences of 23-1 is the same as the total number of \((i, j)\)-occurrences of 32-1 for each \(i\) and \(j\), which yields the result for 23-1.

3.2.2. The average number of occurrences for 31-2. Similar reasoning as in the prior proof shows that the total number of \(i\)-occurrences of 31-2 in the flattened sense within all of the permutations of \([n]\) is given by \((n-i) \binom{i-1}{2} \frac{(n-1)!}{i!}\) for \(i \in [3, n-1]\). To see this, first note that there are \(n-i\) choices for the letter \(j\) to play the role of the 3, given \(i\), within an occurrence of 31-2. Let \(\sigma \in S_n\). Note that for each \(r\), \(1 \leq r \leq i - 3\), there are \(\frac{(n-1)!}{i!}\) \(r\) permutations that have an \((i, j)\)-occurrence of 31-2 in which the letter \(i\) comes somewhere between the \((r+1)\)-st and \((r+2)\)-nd members of \([i-1]\) from the left within Flatten(\(\sigma\)), and \(2\frac{(n-1)!}{i!}(i-2)\) permutations in which the letter \(i\) occurs to the right of all the members of \([i-1]\) within Flatten(\(\sigma\)). Observe that in the latter case, the letter \(i\) would either occur within a cycle whose first letter is a member of \([i-1]\) or as the first letter of a cycle. Note that in all cases, the possible positions for \(j\) are determined by the value of \(r\) and is independent of the value of \(i\). Summing over \(r\), the total number of \((i, j)\)-occurrences of 31-2 is thus given by

\[
(1 + 2 + \cdots + (i-3) + 2(i-2)) \frac{(n-1)!}{i} = \binom{i}{2} - 1 \frac{(n-1)!}{i}.
\]
Summing over $i$, and simplifying, then implies

$$\text{tot}(31-2) = \sum_{i=3}^{n-1} (n-i) \left( \binom{i}{2} - 1 \right) \frac{(n-1)!}{i}$$

$$= \frac{n!}{2} \sum_{i=2}^{n-1} (i - \frac{2}{i}) - (n-1)! \left( \binom{n}{3} - (n-2) \right)$$

$$= \frac{n!}{12n} (n^3 - 3n^2 + 26n - 12) - n! H_n.$$  

Dividing by $n!$ yields the average value formula given in Corollary 2.2.

**Remark:** A similar proof may also be given for the entry in Table 1 above for the average number of occurrences of 13-2.

### 3.2.3. The average number of occurrences for 12-3 and 21-3.

We first treat the case 21-3. To handle this case, we will simultaneously consider occurrences of the pattern 3-21. If $i \in [3, n-1]$, first note that there are $(i-2)(n-1)!$ permutations $\sigma$ of size $n$ in which the letter $i$ directly precedes a member of $[i-1]$ in Flatten($\sigma$). To show this, first insert $i$ directly before some member of $[2, i-1]$ within a permutation of $[i-1]$ expressed in standard cycle form. Upon treating $i$ and the letter directly thereafter as a single object, we see that there are $\prod_{s=i+1}^{n} (s - 1)$ choices for the positions of the elements of $[i+1, n]$ and thus the total number of such permutations is $(i-2)(n-1)! \prod_{s=i+1}^{n} (s - 1) = (i-2)(n-1)!$, as claimed.

Within each of these permutations $\sigma$, every letter of $[i+1, n]$ contributes either an $i$-occurrence of 3-21 or of 3-21 depending on whether the letter goes somewhere before or somewhere after $i$ within Flatten($\sigma$). This implies

$$\text{tot}(i\text{-occurrences of 21-3}) + \text{tot}(i\text{-occurrences of 3-21}) = (n-i)(i-2)(n-1)!, \quad 3 \leq i \leq n-1,$$

and summing over $i$ gives

$$\text{(3.2)} \quad \text{tot}(21-3) + \text{tot}(3-21) = (n-1)! \sum_{i=3}^{n-1} (n-i)(i-2).$$

We now count the total number of occurrences of 3-21 within permutations of size $n$, which is apparently easier. We first count the number of permutations having an $(i, j)$-occurrence of 3-21, where $3 \leq i < j \leq n$ are given. To do so, we first create permutations of the set $[i] \cup \{j\}$ by writing some permutation of $[i-1]$ in standard cycle form and then deciding on the positions of the letters $i$ and $j$. Either $i$ and $j$ can directly precede different members of $[2, i-1]$ or can precede the same member (in which case $j$ would come before $i$), whence there are $\binom{(i-2)}{2} + (i-2) = \binom{(i-1)}{2}$ choices regarding the placement of $i$ and $j$. Upon treating $i$ and the letter directly thereafter as a single object and adding the remaining members $r$ of $[i+1, n] - \{j\}$, we see that the number of permutations of length $n$ having an $(i, j)$-occurrence of 3-21 is

$$\binom{\binom{(i-1)}{2}}{(i-1)!} \prod_{r=i+1}^{j-1} r \prod_{r=j+1}^{n} (r-1) = \binom{\binom{(i-1)}{2}}{(i-1)!} \prod_{j=1}^{(i-1)!} r = \binom{\binom{(i-1)}{2}}{(i-1)!} \prod_{r=1}^{n} (r-1).$$

$$= \binom{\binom{(i-1)}{2}}{(i-1)!} \prod_{r=1}^{n} (r-1).$$
Since there are $n - i$ choices for $j$, given $i$, the total number of $i$-occurrence of 3-21 is then given by $(n - i)\binom{(n-1)!}{i}$. Summing over $3 \leq i \leq n - 1$ gives

$$\text{tot}(3-21) = (n - 1)! \sum_{i=3}^{n-1} \frac{n-i}{i} \binom{i-1}{2}.$$  

Subtracting (3.3) from (3.2) yields

$$\text{tot}(21-3) = (n - 1)! \sum_{i=3}^{n-1} \frac{n-i}{i} \binom{i-1}{2} - \frac{(n-1)!}{2} \sum_{i=2}^{n-1} (i^2 - i - 2)$$

$$= \frac{n!}{12n} (n^3 - 3n^2 + 26n - 12) - n!H_n,$$

which completes the proof in the case 21-3.

A similar proof may be given for the case 12-3. In fact, there are the comparable formulas

$$\text{tot}(12-3) + \text{tot}(3-12) = (n - 1)! \sum_{i=2}^{n-1} (n-i)i$$

and

$$\text{tot}(3-12) = (n - 1)! \sum_{i=2}^{n-1} \frac{n-i}{i} \binom{i-1}{2} - \frac{(n-1)!}{2} \sum_{i=2}^{n-1} (i^2 - i - 2).$$

Subtracting, simplifying, and dividing by $n!$ then gives the average value formula found in Corollary 2.13.

4. Further Results

In Section 2, we presented a recurrence in several cases for the generating function $g_n^{st}$ for the statistic $st$ recording the number of occurrences of a given pattern of type (2,1) in Flatten($\pi$), where $\pi$ is a permutation of length $n$. In this section, we consider the problem of trying to find an explicit formula for the coefficient of $q^r$ in $g_n^{st}$, which we will denote by $g_{n,r}^{st}$. Note that $g_{n,r}^{st}$ is the number of permutations $\pi$ of length $n$ such that Flatten($\pi$) contains exactly $r$ occurrences of the given pattern. We will denote the generating function for the sequence $\{g_{n,r}^{st}\}_{n \geq 3}$, where $r$ is fixed, by $G_r^{st}(x)$. In [13], it was shown that the generating function $G_{13}^{r}(x)$ is rational for all $r \geq 0$. For the five patterns discussed in this paper, however, the comparable problem is apparently more difficult.

Let $g_{n,r}^{st}(1i)$ denote the coefficient of $q^r$ in $g_n^{st}(1i)$. Note that $g_{n,r}^{st}(1i)$ is the number of permutations $\pi$ of length $n$ such that Flatten($\pi$) starts with the letters $1i$ and contains exactly $r$ occurrences of the associated pattern. Let

$$G_{n,r}^{st}(v) = \sum_{i \geq 2} g_{n,r}^{st}(1i)v^{i-2},$$

$$G_r^{st}(x, v) = \sum_{n \geq 3} G_{n,r}^{st}(v)x^n = \sum_{n \geq 3} \sum_{i \geq 2} g_{n,r}^{st}(1i)v^{i-2}x^n.$$  

(4.1)
It follows that \( G_{n,r}^{st}(1) = g_{n,r}^{st} \) if \( n \geq 2 \) and
\[
G_r^{st}(x) = G_r^{st}(x, 1) = \sum_{n \geq 3} g_{n,r}^{st} x^n, \quad r \geq 0.
\]

We now describe a general approach for finding the generating function \( G_r^{st}(x) \). First, we will need a recurrence for \( g_{n,r}^{st}(1i) \). For the patterns of type \((2,1)\) featured, we have already obtained such a recurrence in each case; see Section 2 above. We then extract the coefficient of \( q^r \) in these recurrences for \( g_{n,r}^{st}(1i) \) and obtain comparable recurrences for the numbers \( g_{n,r}^{st}(1i) \), which we convert into generating functional form. Using this form of the recurrence, we then derive explicit expressions for \( G_r^{st}(x) \) in a recursive manner, starting with \( r = 0 \). To do so, we make use of various generating function techniques, including the kernel method. We will illustrate how this approach can be applied to find \( G_r^{st}(x) \) in the cases 31-2 and 32-1. Similar arguments apply to the other three \((2,1)\) patterns discussed.

4.1. The pattern 31-2. We illustrate the approach described with the pattern 31-2. We often will omit the superscript 31-2 and write, for example, \( G_r(x, v) \) in place of \( G_r^{31-2}(x, v) \). We first obtain a recurrence for the generating function \( G_r(x, v) \).

**Theorem 4.1.** For any integer \( r \geq 0 \), we have
\[
(1 + \frac{v^2 x}{1-v}) G_r(x, v) = x \sum_{j=0}^{r-1} v^{r-j+1} G_j(x, v) + \frac{2-v}{1-v} x G_r(x, 1) + 2(2+v)x^3 \delta_{r,0} + H_r(x, v),
\]
where \( H_r(x, v) = -x \sum_{s=0}^{r-1} \sum_{n=3}^{n} g_{n,r}(1j) v^{j+r-s-1} x^n. \)

**Proof.** Let \( r \geq 0 \). Extracting the coefficient of \( q^r \) in (2.1) gives
\[
g_{n,r}(1i) - g_{n-1,r} + \sum_{j=2}^{i-2} g_{n-1,r}(1j) - \sum_{j=2}^{i-2} g_{n-1,r-i+j+1}(1j) = 0, \quad 3 \leq i \leq n,
\]
with \( g_{2,r} = g_{2,r}(12) = 2 \delta_{r,0} \). Multiplying each of the four terms on the left-hand side of (4.3) by \( v^{i-2} x^n \), and summing over \( n \geq 3 \) and \( 3 \leq i \leq n \), yields
\[
\sum_{n \geq 3} \sum_{i=3}^{n} g_{n,r}(1i) v^{i-2} x^n = G_r(x, v) - 2x G_r(x, 1) - 4x^3 \delta_{r,0},
\]
\[
\sum_{n \geq 3} \sum_{i=3}^{n} g_{n-1,r} v^{i-2} x^n = \frac{v x}{1-v} G_r(x, 1) - \frac{x}{1-v} G_r(v x, 1) + 2v x^3 \delta_{r,0},
\]
\[
\sum_{n \geq 3} \sum_{i=3}^{n} \sum_{j=2}^{i-2} g_{n-1,r}(1j) v^{i-2} x^n = \frac{x}{1-v} \left( v^2 G_r(x, v) - G_r(v x, 1) \right),
\]
\[
\sum_{n \geq 3} \sum_{i=3}^{n} \sum_{j=2}^{i-2} g_{n-1,r}(1j) v^{i-2} x^n = \frac{x}{1-v} \left( v^2 G_r(x, v) - G_r(v x, 1) \right),
\]
and
\[
\sum_{n \geq 3} \sum_{i=3}^{n} \sum_{j=i-1-r}^{i-2} g_{n-1,r-i+j+1}(1j)v^{i-2}x^n
= x \sum_{n \geq 3} \sum_{i=4}^{n+1} \sum_{j=i-1-r}^{i-2} g_{n,r-i+j+1}(1j)v^{i-2}x^n
= x \sum_{i \geq 4} \sum_{j=i-1-r}^{i-2} \sum_{n \geq i-1} g_{n,r-i+j+1}(1j)v^{i-2}x^n
= x \sum_{s \leq r-1} \sum_{j \geq 2} \sum_{n \geq r+s} g_{n,s}(1j)v^{r+j-s-1}x^n,
\]
which combine to give (4.2). \(\square\)

Taking \(r = 0\) in recurrence (4.2) gives
\[
\left(1 + \frac{v^2x}{1 - v}\right) G_0(x, v) = \frac{2 - v}{1 - v} xG_0(x) + 2(2 + v)x^3.
\]

To solve this equation, we use the kernel method and substitute \(v = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}\) to obtain
\[
G_0(x) = \frac{x}{\sqrt{1 - 4x}} - x - 2x^2
\]
and
\[
(4.4) \quad G_0(x, C(x)) = \lim_{v \to C(x)} G_0(x, v) = \frac{x^2(\sqrt{1 - 4x} + 8x - 1)}{1 - 4x}.
\]

Taking \(r = 1\) in (4.2) and using the fact that \(H_1(x, v) = -x \sum_{j \geq 3} g_{j,0}(1j)v^jx^j = -\frac{2x^4v^3}{1 - 2x}\), we obtain
\[
(4.5) \quad \left(1 + \frac{v^2x}{1 - v}\right) G_1(x, v) = xv^2G_0(x, v) + \frac{2 - v}{1 - v} xG_1(x) - 2 \frac{x^4v^3}{1 - xv}.
\]

Substituting \(v = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}\) into this equation, and using (4.4), yields
\[
G_1(x) = \frac{(3x - 1)(1 - 5x + 2x^2) + (1 - 6x + 7x^2)\sqrt{1 - 4x}}{x\sqrt{1 - 4x}^3},
\]
and thus
\[
G_1(x, C(x)) = \lim_{v \to C(x)} G_1(x, v)
= \frac{(1 - 4x)(5x^4 - x^3 - 16x^2 + 8x - 1) - (17x^4 + 19x^3 - 30x^2 + 10x - 1)\sqrt{1 - 4x}}{x\sqrt{1 - 4x}^3}.
\]

Continuing in this way for \(r = 2, 3\), we obtain the following result.

**Corollary 4.2.** For \(0 \leq r \leq 3\), we have
\[
G_r^{31-2}(x) = \frac{a_r(x) + b_r(x)\sqrt{1 - 4x}}{\sqrt{1 - 4x}^{2r+1}},
\]
where
(i) \(a_0(x) = x, b_0(x) = -x - 2x^2;\)
(ii) \( a_1(x) = \frac{1}{3}(3x - 1)(1 - 5x + 2x^2) \), \( b_1(x) = \frac{1}{2}(-1 - 6x + 7x^2) \);
(iii) \( a_2(x) = \frac{1}{3}(1 - 12x + 50x^2 - 76x^3 + 224x^4) \), \( b_2(x) = \frac{1}{2}(-1 + 10x - 32x^2 + 28x^3) \);
(iv) \( a_3(x) = \frac{1}{3}(2 - 37x + 270x^2 - 972x^3 + 1748x^4 - 1346x^5 + 220x^6) \), \( b_3(x) = \frac{1}{2}(-2 + 33x - 208x^2 + 614x^3 - 824x^4 + 368x^5) \).

4.2. The pattern 32-1. In this subsection, we provide a second example of our approach using the pattern 32-1. We often omit the superscript 32-1 when discussing the various generating functions. We first establish a recurrence for \( G_r(x, v) \).

**Theorem 4.3.** For any integer \( r \geq 0 \), we have

\[
(1 - v + vx)G_r(x, v) = x(2 - v + 2vx)G_r(x, 1) - 2vx^2G_r(vx, 1)
\]

\[
+ 2x^3(1 - v)(2 + v + 2vx + 2v^2x)\delta_{r,0} + H_r(x, v),
\]

where \( H_r(x, v) = \sum_{n \geq 3} \sum_{j=0}^{n} g_{n, r, j}(1) x^{n+1} (v^{j-1} - v^n) \).

**Proof.** Let \( r \geq 0 \). Extracting the coefficient of \( q^r \) in (2.6) gives

\[
g_{n, r}(1) - g_{n, r-1} + \sum_{j \leq i-1} g_{n, r-1}(1j) - \sum_{j \leq i-1} g_{n, r-j}(1j) = 0, \quad 3 \leq i \leq n,
\]

with \( g_{2, r} = g_{2, r}(12) = 2\delta_{r,0} \). Multiplying each of the four summands on the left-hand side of (4.7) by \( v^{j-2}x^n \), and summing over \( n \geq 3 \) and \( 3 \leq i \leq n \), yields

\[
\sum_{n \geq 3} \sum_{i=3}^{n} g_{n, r}(1i)v^{i-2}x^n = G_r(x, v) - 2xG_r(x, 1) - 4x^3\delta_{r,0}.
\]

\[
\sum_{n \geq 3} \sum_{i=3}^{n} g_{n-1, r}v^{i-2}x^n = \frac{vx}{1 - v}G_r(x, 1) - \frac{x}{1 - v}G_r(vx, 1) + 2vx^3\delta_{r,0},
\]

\[
\sum_{n \geq 3} \sum_{j=2}^{n} \sum_{i=3}^{n-j} g_{n, r-1}(1j)v^{i-2}x^n = \frac{vx}{1 - v}G_r(x, v) - \frac{x}{1 - v}G_r(vx, 1) + 2vx^3\delta_{r,0},
\]

and

\[
\sum_{n \geq 3} \sum_{j=2}^{n} \sum_{i=3}^{n-j} g_{n, r-j}(1j)v^{i-2}x^n
\]

\[
= x \sum_{n \geq 2} \sum_{i=3}^{n} g_{n, r-j}(1j)v^{i-2}x^n
\]

\[
= \frac{x}{1 - v} \sum_{n \geq 2} \sum_{j=2}^{n} g_{n, r-j}(1j)x^n (v^{j-1} - v^n)
\]

\[
= 2vx^3(1 + 2x + 2vx)\delta_{r,0} + \frac{2vx^2}{1 - v} (G_r(x, 1) - G_r(vx, 1)) + H_r(x, v),
\]

which combine to give (4.6). \( \square \)

Taking \( r = 0 \) in recurrence (4.6) gives

\[
(1 - v + vx)G_0(x, v) = x(2 - v + 2vx)G_0(x, 1) - 2vx^2G_0(vx) + 2x^3(1 - v)(2 + v + 2vx + 2v^2x).
\]
Substituting $v = 1/(1 - x)$, we obtain

\begin{equation}
G_0(x) = 2xG_0(x/(1 - x)) + \frac{2x^3(3 - x)}{(1 - x)^2}.
\end{equation}

Iterating this recurrence an infinite number of times yields

\begin{equation}
G_0^{32}(x) = \sum_{i \geq 1} \frac{2^i x^{i+2}(3 - (3i - 2)x)}{(1 - (i - 1)x)(1 - ix)\prod_{j=0}^{i-1}(1 - jx)}.
\end{equation}

Taking $r = 1$ in (4.6) gives

\begin{equation}
(1 - v + vx)G_1(x, v) = x(2 - v + 2vx)G_1(x) - 2vx^2G_1(vx) + H_1(x, v),
\end{equation}

where

\begin{equation}
H_1(x, v) = x \sum_{n \geq 3} g_{n,0}(13)x^{n+1}(v^2 - v^n) = x^2 \sum_{n \geq 3} g_{n-1,0}x^{n-1}(v^2 - v^n)
\end{equation}

\begin{equation}
= x^2v^2(G_0(x) + 2x^2) - x^2v(G_0(xv) + 2x^2v^2).
\end{equation}

Substituting $v = 1/(1 - x)$ and using (4.8), we obtain

\begin{equation}
G_1(x) = 2xG_1(x/(1 - x)) + \frac{1 - 3x}{2(1 - x)}G_0(x) - \frac{3x^3}{1 - x}.
\end{equation}

Iterating this recurrence yields

\begin{equation}
G_1^{32}(x) = \sum_{i \geq 0} \frac{2^{i-1}x^i}{\prod_{j=0}^{i+1}(1 - jx)} \left( (1 - (3 + i)x)(1 - ix)G_0^{32}(x) \left( \frac{x}{1 - ix} \right) - \frac{6x^3}{1 - ix} \right).
\end{equation}

4.3. Remarks. We have found explicit formulas for the generating functions $G_r^{31-2}(x)$ when $0 \leq r \leq 3$ as well as expressions for $G_0^{32-1}(x)$ when $r = 0, 1$. We see that the procedure discussed for finding these formulas may be extended to other small $r$. Our formulas for $G_r^{31-2}(x)$ lead one to ask the following question: For any $r \geq 0$, does the generating function $G_r^{31-2}(x)$ have the form

\begin{equation}
\frac{A_r(x) + B_r(x)\sqrt{1 - 4x}}{x'\sqrt{(1 - 4x)^{2r+1}}}
\end{equation}

where $A_r(x)$ and $B_r(x)$ are polynomials with integer coefficients?

For the other patterns $\tau \in \{12-3, 21-3, 23-1\}$ featured in this paper, one can apply similar arguments and obtain in each case comparable explicit formulas for the generating function $G_r^{\tau}(x)$ when $r$ is small, the details of which we leave to the interested reader.

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References


