COUNTING HUMPS AND PEAKS IN GENERALIZED DYCK PATHS

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Abstract

Let us call a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using $U = (1, k)$, $D = (1, -1)$, and $H = (a, 0)$ steps and never going below the $x$-axis a $(k, a)$-path (of order $n$). A super $(k, a)$-path is a $(k, a)$-path which is permitted to go below the $x$-axis. We relate the total number of humps in all of the $(k, a)$-paths of order $n$ to the number of super $(k, a)$-paths, where a hump is defined to be a sequence of steps of the form $UH_iD_i, i \geq 0$. This generalizes recent results concerning the cases when $k = 1$ and $a = 1$ or $a = \infty$. A similar relation may be given involving peaks (consecutive steps of the form $UD$).

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1. Introduction

A $(k, a)$-path of order $n$ is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$, and horizontal steps $(a, 0)$ (which we’ll denote by $U$, $D$, and $H$, respectively) and never going below the $x$-axis. We’ll use $\mathcal{P}_n(k, a)$ to denote the set of $(k, a)$-paths of order $n$. It is well known that $\mathcal{P}_n(k, \infty), \mathcal{P}_n(1, \infty), \mathcal{P}_n(1, 1),$ and $\mathcal{P}_n(1, 2)$ are, respectively, the sets of $k$-ary (see [2]), Dyck, Motzkin, and Schröder paths of length $n$. If we allow a $(k, a)$-path to go below the $x$-axis, then we get a super $(k, a)$-path. Let $\mathcal{SP}_n(k, a)$ denote the set of super $(k, a)$-paths of order $n$. A peak in a super $(k, a)$-path is any two consecutive steps $UD$. We denote the number of peaks in a path $P$ by $\#\text{Peaks}(P)$. A hump in a super $(k, a)$-path is an up step followed by zero or more horizontal steps followed by a down step. We denote the number of humps in a path $P$ by $\#\text{Humps}(P)$.

In 2010, Regev [4] noticed and proved that

\begin{equation}
|\mathcal{SP}_n(1, \infty)| = \binom{2n}{n} = 2 \sum_{P \in \mathcal{P}_n(1, \infty)} \#\text{Peaks}(P),
\end{equation}

\begin{equation}
|\mathcal{SP}_n(1, 1)| = \sum_{j \geq 0} \binom{n}{j} \binom{n-j}{j} = 2 \sum_{P \in \mathcal{P}_n(1, 1)} \#\text{Humps}(P) + 1.
\end{equation}
Let \( \delta_{a|n} \) be the number of humps, that is, using the first return decomposition, we obtain
\[
\sum_{P \in P_n(k,a)} \# \text{Humps}(P) = |SP_n(k,a)| - \delta_{a|n},
\]
where \( \delta_{a|n} \) equals 1 if \( a \) divides \( n \) and 0 otherwise.

As a corollary, the above theorem for \( k = 1 \) and \( a = \infty \) or \( a = 1 \) gives (1) and (2), respectively.

### 2. Proof of Theorem 1

Let \( P_{k,a}(x) \) and \( SP_{k,a}(x) \) denote, respectively, the generating functions for the number of \((k,a)\)-paths and super \((k,a)\)-paths of order \( n \), that is, \( P_{k,a}(x) = \sum_{n \geq 0} |P_n(k,a)|x^n \) and \( SP_{k,a}(x) = \sum_{n \geq 0} |SP_n(k,a)|x^n \). Using the first return decomposition, we obtain
\[
SP_{k,a}(x) = 1 + x^a SP_{k,a}(x) + (k+1)x^{2k}(P_{k,a}(x))^k SP_{k,a}(x).
\]
Let \( H_{k,a}(x,q) \) denote the generating function for the number of \((k,a)\)-paths of order \( n \) according to the number of humps, that is,
\[
H_{k,a}(x,q) = \sum_{n \geq 0} \left( \sum_{P \in P_n(k,a)} q^{\# \text{Humps}(P)} \right) x^n.
\]
Using the first return decomposition, we obtain
\[
H_{k,a}(x,q) = 1 + x^a H_{k,a}(x,q) + x^{2k} \left( H_{k,a}(x,q) - \frac{1}{1 - x^a} \right) H_{k,a}(x,q)^k + \frac{x^{2k}}{1 - x^a} (H_{k,a}(x,q))^k.
\]
Let
\[
H_{k,a}(x) = \frac{d}{dq} H_{k,a}(x,q) |_{q=1} = \sum_{n \geq 0} \left( \sum_{P \in P_n(k,a)} \# \text{Humps}(P) \right) x^n,
\]
which is the generating function for the number of humps in all of the \((k,a)\)-paths of order \( n \). By (4), we have
\[
H_{k,a}(x) = x^a H_{k,a}(x) + x^{2k} H_{k,a}(x) H_{k,a}(x,1)^k + kx^{2k} \left( H_{k,a}(x,1) - \frac{1}{1 - x^a} \right) H_{k,a}(x,1)^{k-1} H_{k,a}(x)
\]
\[
+ \frac{x^{2k}}{1 - x^a} (H_{k,a}(x,1))^k + \frac{kx^{2k}}{1 - x^a} (H_{k,a}(x,1))^{k-1} H_{k,a}(x).
\]
Note that \( H_{k,a}(x,1) = P_{k,a}(x) \), whence
\[
H_{k,a}(x) = \frac{x^{2k}(P_{k,a}(x))^k}{1 - x^a - (k+1)x^{2k} P_{k,a}(x)^k}.
\]
By (3), we then have
\[
(1 - x^a)H_{k,a}(x) = x^{2k}(P_{k,a}(x))^k SP_{k,a}(x) = \frac{1}{k+1} ((1 - x^a)SP_{k,a}(x) - 1),
\]
which implies

\[(k+1)H_{k,a}(x) = SP_{k,a}(x) - \frac{1}{1-x^a}.\]

Comparing the coefficients of \(x^n\) on both sides of the last equation gives Theorem 1.

**Remark 1:** If \(j \geq 0\), then let

\[a_j = \frac{1}{a}(n + (a - 1)(k + 1)j).\]

Then we have

\[|SP_n(k,a)| = \sum_j \left( \frac{a_j}{(k+1)j} \right) \binom{(k+1)j}{j}, \quad n \geq 1,
\]

where the sum is taken over all \(j\), \(0 \leq j \leq \left\lfloor \frac{n}{k+1} \right\rfloor\), such that \(a_j\) is integral, upon considering the number of up steps within a member of \(SP_n(k,a)\).

**Remark 2:** Modifying the combinatorial argument given by Ding and Du [1] for (1) and (2) proves Theorem 1 in the case when \(k = 1\) and \(a\) is general. We were unable to find a combinatorial proof of Theorem 1 for general \(k\), a seemingly more difficult problem which we will leave to the interested reader.

### 3. Peaks

Let \(PE_{k,a}(x,q)\) be the generating function for the number of \((k,a)\)-paths of order \(n\) according to the number of peaks, that is,

\[PE_{k,a}(x,q) = \sum_{n \geq 0} \left( \sum_{P \in \mathcal{P}_n(k,a)} q^{\#\text{Peaks}(P)} \right) x^n.
\]

Using the first return decomposition, we obtain

\[PE_{k,a}(x,q) = 1 + x^a PE_{k,a}(x,q) + x^{2k}(PE_{k,a}(x,q) - 1)(PE_{k,a}(x,q))^k + q x^{2k}(PE_{k,a}(x,q))^k.
\]

Thus, the generating function for the number of peaks in all of the \((k,a)\)-paths of order \(n\) is given by

\[PE_{k,a}(x) = \frac{d}{dq} PE_{k,a}(x,q) |_{q=1} = x^a PE_{k,a}(x) + x^{2k} PE_{k,a}(x)(PE_{k,a}(x,1))^k
\]

\[+ k x^{2k}(PE_{k,a}(x,1) - 1)(PE_{k,a}(x,1))^{k-1} PE_{k,a}(x)
\]

\[+ x^{2k}(PE_{k,a}(x,1))^{k} + k x^{2k}(PE_{k,a}(x,1))^{k-1} PE_{k,a}(x),
\]

which, since \(PE_{k,a}(x,1) = P_{k,a}(x)\), implies

\[PE_{k,a}(x) = \frac{x^{2k}(P_{k,a}(x))^k}{1 - x^a - (k+1)x^{2k}(P_{k,a}(x))^k}.
\]

By (3), we then have

\[(k+1)PE_{k,a}(x) = (k+1)x^{2k}(P_{k,a}(x))^k SP_{k,a}(x) = (1-x^a)SP_{k,a}(x) - 1.
\]

Thus, we can state the following result.
**Theorem 2.** For all $n \geq 1$,

$$(k + 1) \sum_{P \in \mathcal{P}_n(k,a)} \# \text{Peaks}(P) = |\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|,$$

where $\mathcal{SP}_j(k,a)$ is defined to be the empty set when $j < 0$.

**Bijective proof when $k = 1$:** One may give a bijective proof of Theorem 2 in the case when $k = 1$ and $a$ is arbitrary as follows. Suppose $\lambda \in \mathcal{SP}_n(1,a)$, not all $H$’s, has $U$ as its first non-horizontal step. Let us call a non-empty subpath of $\lambda$ a unit if it starts on the $x$-axis and is either of the form $U\alpha DH^i$ or $D\beta UH^i$, where $i \geq 0$ is maximal, $\alpha$ is a $(1,a)$-path, and $\beta$ is the reflection of some $(1,a)$-path in the $x$-axis. Note that $\lambda$ may be uniquely decomposed into a sequence of units, up to a (possibly empty) run of $H$ steps at the beginning. Let us call the unit of $\lambda$ directly preceding its left-most unit of the form $D\beta UH^i$ the right unit (if $\lambda$ never dips below the $x$-axis, then the right unit of $\lambda$ is defined to be its right-most one).

Let $\mathcal{SP}^*_n(1,a) \subseteq \mathcal{SP}_n(1,a)$ consist of those paths, not all $H$’s, whose first non-horizontal step is $U$ and in which the right-most hump in the right unit is a peak. Observe that

$$|\mathcal{SP}^*_n(1,a)| = \frac{|\mathcal{SP}_n(1,a)| - |\mathcal{SP}_{n-a}(1,a)|}{2}.$$

To see this, first note that there are $\frac{|\mathcal{SP}_n(1,a)| - \delta_{a,n}}{2}$ members of $\mathcal{SP}_n(1,a)$ whose first non-horizontal step is $U$, upon reflecting in the $x$-axis. Similarly, we see that there are $\frac{|\mathcal{SP}_{n-a}(1,a)| - \delta_{a,n-a}}{2}$ members of $\mathcal{SP}_n(1,a)$ whose first non-horizontal step is $U$ and whose right-most hump in the right unit is not a peak. The above expression for $|\mathcal{SP}^*_n(1,a)|$ then follows from subtraction. A simple modification of the argument used in [1] to show (1) now provides the needed bijection between $\mathcal{SP}^*_n(1,a)$ and the set of ordered pairs $(M,P)$, where $M \in \mathcal{P}_n(1,a)$ and $P$ is a peak of $M$. \[ \square \]

It would be interesting to find a bijective proof of Theorem 2 for all $k$.

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**References**