Three Hoppy path problems and ternary paths

Eva Y.P. Deng\textsuperscript{a}, Toufik Mansour\textsuperscript{b, c}

\textsuperscript{a}Department of Applied Mathematics, Dalian University of Technology, 116024 Dalian, PR China
\textsuperscript{b}Department of Mathematics, University of Haifa, 31905 Haifa, Israel
\textsuperscript{c}Center for Combinatorics, LPMC, Nankai University, 300071 Tianjin, PR China

Received 2 October 2006; received in revised form 31 July 2007; accepted 12 August 2007
Available online 21 September 2007

Abstract

In this paper we solve several recurrence relations with two (three) indices using combinatorial methods. Moreover, we present several recurrence relations with two (three) indices related to ternary paths and \( k \)-ary paths.

© 2007 Elsevier B.V. All rights reserved.

MSC: 10A35; 65Q05; 05.10

Keywords: \( k \)-Ary trees; Ternary paths; Lattice paths

1. Introduction

The aim of this paper is to study combinatorial methods to solve recurrence relations with two indices. Recently, the second author \cite{2} introduced a combinatorial problem, called Hoppy’s problem, to study different types of recurrence relations with two indices. This study leads us to express the general term of such recurrence relations as paths, called Hoppy paths, in the plane integer lattice \( \mathbb{Z}^2 \). Moreover, \cite{2} gives several recurrence relations with two indices related to different types of Hoppy paths which are equivalent combinatorially to Dyck paths and Schröder paths. In this paper, we generalize Hoppy’s problem to study other types of recurrence relations with two indices for which combinatorial methods provide complete solutions. More precisely, in this paper we present three different generalizations of Hoppy’s problem (see Problems 1–3) to study three different kind of recurrence relations with two indices. These three problems lead us to \( k \)-ary paths (for \( k = 2 \) we get Dyck paths as shown in \cite{2}, and for \( k = 3 \) we get ternary paths).

A \( k \)-ary path of length \( kn \) is a lattice path from \((0,0)\) to \((kn,0)\) consisting of \((k−1)n\) up steps \( U = (1,1) \) and \( n \) down steps \( D = (1,−(k−1)) \) such that the path never goes below the \( x \)-axis. The number of \( k \)-ary paths of length \( kn \) is

\[
\frac{1}{(k−1)n+1} \binom{kn}{n}.
\]

When \( k = 2 \) it is called Dyck path, and when \( k = 3 \) it is called ternary path, see \cite[sequence A001764]{3}.

In this paper we study the following three combinatorial problems.

\textbf{Problem 1.} Fix \( k \geq 2 \). Let \( \mathcal{A}_k = \{(m,n) \mid 0 \leq m \leq (k−1)n\} \) be a subset of the plane integer lattice \( \mathbb{Z}^2 \). There is a rabbit, called \textit{Hoppy}, at the origin \( O = (0,0) \in \mathcal{A}_k \), and his \((k−1)n+1\) bunnies are located at the points \((i,n) \in \mathcal{A}_k \) for \( i = 0,1,\ldots,(k−1)n \). Hoppy can jump from the point \((i,j) \in \mathcal{A}_k \) to the point \((i′,j+1) \in \mathcal{A}_k \) with \( i \leq i′ \), see Fig. 1. For convenience, we call the step \((i,j) (i′, j+1)\) a \textit{north step} if \( i = i′ \) and a \textit{skew step} otherwise. The problem

\begin{itemize}
\item E-mail addresses: ypdeng@dlut.edu.cn (E.Y.P. Deng), toufik@math.haifa.ac.il (T. Mansour).
\end{itemize}

0166-218X/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2007.08.015
Fig. 1. The Hoppy path and Problem 1 with \( k = 3 \).

Fig. 2. The Hoppy path and Problem 2.

is to find \( a_k(i, n) \), the number of ways Hoppy can reach the bunny located at \( (i, n) \in \mathcal{A}_k \). It is not hard to see that \( a_k(i, n) \) satisfies the recurrence relation

\[
a_k(i, n) = a_k(0, n - 1) + \cdots + a_k(i, n - 1), \quad i = 0, 1, \ldots, (k - 1)n, \tag{1.1}
\]

with the initial conditions \( a_k(0, 0) = 1 \) and \( a_k(i, n) = 0 \) for \( i > (k - 1)n \).

**Problem 2.** Let \( \mathcal{B} = \{(m, n) \mid 0 \leq m \leq n\} \) be the second octant of the plane integer lattice \( \mathbb{Z}^2 \). The rabbit Hoppy is at the origin \( O = (0, 0) \in \mathcal{B} \), and his \( n + 1 \) bunnies are located at points \( (i, n) \in \mathcal{B} \) for \( i = 0, 1, \ldots, n \). Hoppy can jump from the point \( (i, j) \in \mathcal{B} \) to the point \( (i', j + 1) \in \mathcal{B} \) with \( i \leq i' \) or from the point \( (i, j) \in \mathcal{B} \) to the point \( (i + 1, j) \in \mathcal{B} \), see Fig. 2. We call the step \( (i, j)(i, j + 1) \) a north step, \( (i, j)(i + 1, j) \) an east step and \( (i, j)(i', j + 1) \), where \( i < i' \), a skew step. Our problem is to find \( b(i, n) \), the number of ways Hoppy can reach the bunny located at \( (i, n) \in \mathcal{B} \). \( b(i, n) \) satisfies the recurrence relation

\[
b(i, n) = b(0, n - 1) + \cdots + b(i, n - 1) + b(i - 1, n), \quad i = 0, 1, \ldots, n, \tag{1.2}
\]

with the initial conditions \( b(0, 0) = 1 \) and \( b(i, n) = 0 \) for \( i > n \).

**Problem 3.** Let \( \mathcal{C} = \{(i, j; n) \mid 0 \leq i \leq j \leq n\} \) be a subset of the integer lattice \( \mathbb{Z}^3 \). The rabbit Hoppy stays at the origin \( O = (0, 0; 0) \in \mathcal{C} \), and his \( \binom{n + 2}{2} \) bunnies are located at points \( (i, j; n) \in \mathcal{C} \) for \( 0 \leq i \leq j \leq n \). Hoppy can jump from the point \( (i, j; d) \in \mathcal{C} \) to the point \( (i + i', j + j'; d + 1) \in \mathcal{C} \) with \( 0 \leq i' \leq j' \). Our problem is to find \( c(i, j; n) \), the number of ways Hoppy can reach the bunny located at \( (i, j; n) \in \mathcal{C} \). \( c(i, j; n) \) satisfies the recurrence relation

\[
c(i, j; n) = \sum_{i' \leq i} \sum_{j' \leq j} c(i', j'; n - 1), \quad 0 \leq i \leq j \leq n, \tag{1.3}
\]

with the initial conditions \( c(0, 0; 0) = 1 \) and \( c(i, j; n) = 0 \) for either \( j > n \) or \( i > j \).

It is well known that there is no general procedure for solving multivariable recurrences, which is why it is an art. In this paper we present a combinatorial method for solving the above three recurrence relations, namely (1.1)–(1.3). More precisely, in Section 2 we use the kernel method technique and combinatorial argument to obtain an explicit formula for \( a_k(i, n) \). In Sections 3 and 4 we use combinatorial arguments to present numbers of Hoppy paths as described in Problems 2 and 3, respectively.
2. Problem 1 and $k$-ary paths

Here, we present two different methods for finding an explicit formula for the general term of the sequence $a_k(i,n)$.

2.1. The kernel method

The first of these methods can be described as follows. First, define $A_{k,n}(v) = \sum_{i=0}^{(k-1)n} a_k(i,n) v^i$ and $A_k(v,x) = \sum_{n \geq 0} A_{k,n}(v) x^n$. Multiplying (1.1) by $v^i$ and summing over all $i = 0, 1, \ldots, (k - 1)n$, we arrive at

$$A_{k,n}(v) = \sum_{i=0}^{(k-1)(n-1)} \frac{v^i - v^{(k-1)(n-1)+1}}{1-v} a_k(i,n-1) + \sum_{i=(k-1)(n-1)+1}^{(k-1)n} v^i A_{k,n-1}(1)$$

$$= \frac{1}{1-v} (A_{k,n-1}(v) - v^{(k-1)(n-1)+1} A_{k,n-1}(1)) + \frac{v^{(k-1)(n-1)+1} - v^{(k-1)n+1}}{1-v} A_{k,n-1}(1)$$

$$= \frac{1}{1-v} (A_{k,n-1}(v) - v^{(k-1)n+1} A_{k,n-1}(1)).$$

Again, multiplying the above recurrence relation by $x^n$, summing over all $n \geq 1$ while using the initial condition $A_{k,0}(v) = 1$ we obtain the functional equation

$$A_k(v,x) - 1 = \frac{x}{1-v} (A_k(v,x) - v^k A_k(1, xv^{k-1})),$$

which is equivalent to

$$\left(1 - \frac{x}{v^{k-1}(1-v)}\right) A_k(v,x/v^{k-1}) = 1 - \frac{xv}{1-v} A_k(1,x).$$

This type of functional equation can be solved systematically using the kernel method (see [1]). In this case, if we assume that $v = v_0$ and $g = g(x) = 1/v_0$, where $v_0 = v_0(x)$ satisfies the equation $v_0 = 1 - \frac{x}{v_0}$, then $A_k(1,x) = \frac{x}{1-xv_0} = \frac{1}{v_0}$. Hence, $A_k(1,x) = \frac{g(x)}{x} - 1$, where $g(x) = 1 + xg^{k}(x)$. In order to obtain an explicit formula for the coefficient of $x^n$ in the generating function $A_k(1,x)$ we need the Lagrange inversion formula (see [4, Theorem 5.4.2, 5]):

($\ast$) Let $G(x)$ be any function with $G(0) \neq 0$, and let $u(x) = x G(u(x))$ be any formal power series. Then $n! \sum_{n \geq 0} u(x)^n = k[x^n] G(x)^n$, for all $k, n \geq 0$, where $[z^n] F(z)$ is the coefficient of $z^n$ in the formal power series $F(z)$.

Thus $g(x) = \sum_{n \geq 0} \frac{1}{(k-1)n+1} \binom{kn}{n} x^n$, which implies that

$$A_k(1,x) = \frac{g(x)-1}{x} = \sum_{n \geq 0} \frac{1}{(k-1)(n+1)+1} \binom{k(n+1)}{n+1} x^n.$$

Denote the number of $k$-ary paths of length $k(n+1)$ by $f_k(n)$, that is, $f_k(n) = \frac{1}{(k-1)(n+1)+1} \binom{k(n+1)}{n+1}$. Therefore, we have

**Theorem 2.1.** The number of Hoppy paths on the lattice $\mathcal{A}_k$ to get from the origin to his $(k - 1)n + 1$ bunnies is given by

$$f_k(n) = \sum_{i=0}^{(k-1)n} a_k(i,n) = \frac{1}{(k-1)(n+1)+1} \binom{k(n+1)}{n+1}.$$
Theorem 2.3. The number of Hoppy paths on the lattice $\mathcal{A}_k$ to get from the origin to the $(i + 1)\text{st}$ bunny of his $(k - 1)n + 1$ bunnies is given by

$$a_k(i, n) = \binom{n + i - 1}{n - 1} - \sum_{j=0}^{n-1} \frac{1}{(k-1)(n-j)+1} \binom{k(n-j)}{n-j} \binom{j+i-1-(k-1)(n-j)}{j},$$

where $\binom{a}{b} = 0$ if $a < b$.

2.2. Combinatorial method

In this subsection we use a combinatorial method to solve the number of Hoppy paths $a_k(i, n)$. We will construct a bijection $\phi$ between the set of Hoppy paths on the lattice $\mathcal{A}_k$ to get from the origin to his $(k - 1)n + 1$ bunnies and the set of $k$-ary paths of length $k(n + 1)$.

Let $H = (0, 0)(x_1, 1) \ldots (x_n, n)$ be a Hoppy path from the origin to one of his $(k - 1)n + 1$ bunnies. For convenience, set $x_0 = 0$. We read $H$ from right to left and successively generate a $k$-ary path $\phi(H) = P$ as follows. First, when $x_a$ is read, the path $P$ starts with $(k - 1)(n + 1) - x_a$ up steps followed by a down step. When $x_j, j = n - 1, n - 2, \ldots, 0$, is read, then in the path $P$ we adjoin $x_{j+1} - x_j$ up steps, followed by a down step. We can see that the path $P$ never goes below the $x$-axis since there are $(k - 1)(n + 1) - x_j$ up steps and $n + 1 - i$ down steps after $x_j$ is read; and $P$ finally reaches $(k(n + 1), 0)$ since in total, there are $(k - 1)(n + 1)$ up steps and $n + 1$ down steps. So $P$ is a $k$-ary path.

The inverse of $\phi$ is defined as follows. Given a $k$-ary path $P$ of length $k(n + 1)$, assume the $x$-coordinate of the starting point of the down steps of $P$ are $y_0, y_1, \ldots, y_n$. Then the corresponding Hoppy path is $\phi^{-1}(P) = (0, 0)(x_1, 1) \cdots (x_n, n)$ with $x_i = k(n + 1) - y_{n-i} - (i + 1)$. For example, for the Hoppy path $(0, 0)(1, 1)(4, 2)(4, 3)(5, 4)(9, 5)(9, 6)$ on the lattice $\mathcal{A}_3$, the corresponding ternary path is $UUUUUDUUUUDUUUDUUUD$, see Fig. 3.

Therefore, we have

Corollary 2.2. The number of Hoppy paths on the lattice $\mathcal{A}_k$ to get from the origin to the $(i + 1)\text{st}$ bunny of his $(k - 1)n + 1$ bunnies is given by

$$a_k(i, n) = \binom{n + i - 1}{n - 1} - \sum_{j=0}^{n-1} \frac{1}{(k-1)(n-j)+1} \binom{k(n-j)}{n-j} \binom{j+i-1-(k-1)(n-j)}{j},$$

where $\binom{a}{b} = 0$ if $a < b$. 

Fig. 3. The Hoppy path and the corresponding $k$-ary path.
From the construction of bijection \( \phi(H) = P \), we can see that if the Hoppy path \( H \) ends at point \((x_n, n)\), then the corresponding \( k \)-ary path \( P \) starts with \((k - 1)(n + 1) - x_n\) up steps. When \( x_j, x_j \neq x_{j+1} \) or \( j = n \), is read, the adjoining up steps and down step bring a peak in the path \( P \). Therefore, we have

**Corollary 2.4.** (1) The number of Hoppy paths on the lattice \( \mathcal{A}_k \) from the origin to the bunny located at \((i, n)\) is equal to the number of \( k \)-ary paths of length \( k(n + 1) \) starting with \((k - 1)(n + 1) - i\) up steps.

(2) The number of Hoppy paths on the lattice \( \mathcal{A}_k \) from the origin to his \((k - 1)n + 1\) bunnies with \( m \) skew steps is equal to the number of \( k \)-ary paths of length \( k(n + 1) \) with \( m + 1 \) peaks (a peak is a segment of the two successive steps UD).

The above corollary leads to an explicit formula for the number of Hoppy paths from \((0, 0)\) to \((i, n)\). To do that we let \( F(x, q) \) be the ordinary generating function for the number of \( k \)-ary paths of length \( kn \) starting with \( m \) up steps, that is, \( F(x, q) = \sum_{n \geq 0} x^n \sum q^m \), where the internal sum is over all \( k \)-ary paths of length \( kn \) starting with \( m \) up steps.

An equation for the generating function \( F(x, q) \) is obtained from the “first return decomposition” of a \( k \)-ary path \( P : P = U P_1 U P_2 \cdots U P_{k-1} D P_k \), where \( P_1, \ldots, P_{k-1}, P_k \) are \( k \)-ary paths. Let us write an equation for \( F(x, q) \). If \( P_1, \ldots, P_{d-1} \) are empty paths and \( P_d \) is not empty, then the contribution of this case gives \( xq^d(F(x, q) - 1)F^{k-d}(x, 1) \) if \( d \leq k - 1 \), and \( xq^{k-1}F(x, 1) \) if \( d = k \). Hence,

\[
F(x, q) = 1 + xq^{k-1}F(x, 1) + \sum_{d=1}^{k-1} xq^d F^{k-d}(x, 1) - \frac{F(x, 1)}{q} \frac{1 - \frac{F(x, 1)}{q}}{1 - \frac{F(x, 1)}{q}}.
\]

which implies that

\[
F(x, q) = 1 + \frac{xq^{k-1}F(x, 1)}{1 - xq^k} - \frac{F(x, 1)}{1 - \frac{F(x, 1)}{q}} = 1 + \frac{xq^{k-1}F(x, 1)(1 - F(x, 1)/q)}{1 - \frac{F(x, 1)}{q} - xq^{k-1}F(x, 1) + xF^k(x, 1)}.
\]

Using the fact that \( xF^k(x, 1) = F(x, 1) - 1 \) we obtain

\[
F(x, q) = 1 + \frac{xq^{k-1}(1 - F(x, 1)/q)}{1 - \frac{F(x, 1)}{q} - xq^{k-1}} = 1 - q + xq^{k-1}F(x, 1).
\]

Since \( F(x, 1) = \sum_{n \geq 0} \frac{1}{(k-1)n+1} {kn \choose n} x^n \) is the ordinary generating function for the number of \( k \)-ary paths of length \( kn \), we obtain

\[
F(x, q) = \frac{1 - q}{1 - q + xq^k} + \frac{xq^{k-1}}{1 - q + xq^k} \sum_{d \geq 0} \frac{1}{(k-1)d+1} {kd \choose d} x^d.
\]

Finding the coefficient of \( x^{n+1}q^n \) in the generating function \( F(x, q) \), we obtain that the number of \( k \)-ary paths of length \( k(n + 1) \) starting with \( m \) up steps is

\[
\left( \begin{array}{c} n - 2 + (k - 1)(n + 1) - m \\ n - 1 \end{array} \right) - \sum_{j=0}^{n-1} \frac{1}{(k-1)(n-j) + 1} {k(n-j) \choose j} - \frac{1}{(k-1)(n-j) + 1} {k(n-j) \choose j}.
\]

where \( {a \choose b} = 0 \) if \( a < b \) or \( a, b \) are non-positive integer numbers. Thus, Corollary 2.4 with \( m = (k - 1)(n + 1) - (i + 1) \) yields Corollary 2.2 again.
In addition, Corollary 2.4 gives an explicit formula for the number of Hoppy paths on the lattice \( \mathcal{A}_k \) from the origin to his \( (k - 1)n + 1 \) bunnies with \( m \) skew steps. Let \( F'(x, q) = \sum_{n \geq 0} \sum x^n q^m \), where the internal sum runs over all \( k \)-ary paths of length \( kn \) with \( m \) peaks. Again, an equation for the generating function \( F'(x, q) \) is obtained from the “first return decomposition” of a ternary path \( P = UP_1UP_2 \cdots UP_{k-1}DQ \), where \( P_1, \ldots, P_{k-1}, Q \) are \( k \)-ary paths. Thus, the two possibilities of \( P_{k-1} \) either being an empty or non-empty \( k \)-ary path give contributions \( xqF^{k-1}(x, q) \) and \( xF^{k-1}(x, q)(F'(x, q) - 1) \). Therefore,

\[
F'(x, q) = 1 + xF^{k-1}(x, q)(q - 1 + F'(x, q)).
\]

Applying the Lagrange inversion formula, see (⋆), we find that the \( x^n q^m \) coefficient in \( F'(x, q) \) is given by

\[
\frac{1}{n} \left( \binom{k-1}{m-1} \right) \left( \binom{n}{m} \right).
\]

Thus, by Corollary 2.4, we can state the following result.

**Corollary 2.5.** The number of Hoppy paths on the lattice \( \mathcal{A}_k \) from the origin to his \( (k - 1)n + 1 \) bunnies with \( m \) skew steps is given by

\[
\frac{1}{n+1} \left( \binom{k-1}{n+1} \right) \left( \binom{n+1}{m+1} \right).
\]

3. Problem 2 and ternary paths

In this section, we will construct a bijection \( \varphi \) between the set of ternary paths and the set of Hoppy paths on the lattice \( \mathcal{B} \). Moreover, we discuss the number of ways Hoppy can reach his bunny located at \((i, n)\) and the number of Hoppy paths containing \( m \) skew steps.

Let \( \mathcal{T}_{n+1} \) be the set of all ternary paths of length \( 3(n+1) \). Let \( \mathcal{H}_{n+1} \) be the set of all Hoppy paths on the lattice \( \mathcal{B} \) to get from the origin to his \( n + 1 \) bunnies. We inductively construct a bijection \( \varphi \) from \( \mathcal{T}_{n+1} \) to \( \mathcal{H}_{n+1} \) which maps a ternary path \( P \) of length \( 3(n+1) \) with \( x \) points on the \( x \)-axis and \( \beta \) peaks into a Hoppy path \( H \) from \((0, 0)\) to \((n + 2 - x, n)\) with \( \beta \) skew steps.

We use induction on \( n \). First we set \( \varphi(UUD) = (0, 0) \). Assume we have gotten the correspondence for the ternary path of length less than \( 3(n+1) \). Now we consider the case of \( n + 1 \). Given a ternary path \( P \in \mathcal{T}_{n+1} \) which has \( x \) points on the \( x \)-axis with \( \beta \) peaks.

1. If \( x > 2 \). Assume \( P \) first touches the \( x \)-axis at \((3k+3, 0)\) after leaving the origin \((0, 0)\). We decompose \( P \), say \( P = P_1P_2 \), into two non-empty parts \( P_1 \) and \( P_2 \) by the point \((3k+3, 0)\). Suppose \( P_1 \) has \( \beta_1 \) peaks, then \( P_2 \) has \( \beta - \beta_1 \) peaks. By induction, there is a Hoppy path \( \varphi(P_1) \) from \((0, 0)\) to \((k, k)\) with \( k + 1 - \beta_1 \) skew steps corresponding to \( P_1 \), and a Hoppy path \( \varphi(P_2) \) from \((0, 0)\) to \((n - k + 2 - x, n - k - 1)\) with \( n - k - \beta + \beta_1 \) skew steps corresponding to \( P_2 \). Now we define \( \varphi(P) = \varphi(P_1)\varphi(P_2) \), where \( \varphi(P_2) \) denotes the Hoppy path to get from \((k, k+1)\) to \((n+2-x, n)\) by moving each point \((x, y)\) of \( \varphi(P_2) \) into the point \((x+k, y+k+1)\).

2. If \( x = 2 \). Then we decompose \( P \) into the following three cases:

   a. Let \( P = UP_1UD \), where \( P_1 \in \mathcal{T}_n \) and has \( \beta - 1 \) peaks. Suppose \( P_1 \) has \( z_1 \) points on the \( x \)-axis. By induction, there is a Hoppy path \( \varphi(P_1) \) from \((0, 0)\) to \((n + 1 - z_1, n - 1)\) with \( n + 2 - \beta \) skew steps corresponding to \( P_1 \). In this case we define

   \[
   \varphi(P) = \varphi(P_1)(n + 1 - z_1, n)(n + 2 - z_1, n) \cdots (n, n).
   \]

   b. Let \( P = UUP_1D \), where \( P_1 \in \mathcal{T}_n \) and has \( \beta \) peaks. Suppose \( P_1 \) has \( z_1 \) points on the \( x \)-axis. By induction, there is a Hoppy path \( \varphi(P_1) \) from \((0, 0)\) to \((n + 1 - z_1, n - 1)\) with \( n - \beta \) skew steps corresponding to \( P_1 \). In this case we define

   \[
   \varphi(P) = \varphi(P_1)(n + 2 - z_1, n)(n + 3 - z_1, n) \cdots (n, n).
   \]

   c. Let \( P = UP_1UP_2D \), where \( P_1 \in \mathcal{T}_k \) and \( P_2 \in \mathcal{T}_{n-k} \) with \( 0 < k < n \). Suppose \( P_1 \) has \( z_1 \) points on the \( x \)-axis and \( \beta_1 \) peaks. Then \( P_2 \) has \( z_1 \) points on the \( x \)-axis and \( \beta - \beta_1 \) peaks. By induction, there is a Hoppy path \( \varphi(P_1) \) from \((0, 0)\) to \((k + 1 - z_1, k - 1)\) with \( k - \beta_1 \) skew steps corresponding to \( P_1 \), and a Hoppy path \( \varphi(UUP_2D) \) from \((0, 0)\) to \((n - k, n - k)\) with \( n - k + 1 - \beta + \beta_1 \) skew steps corresponding to \( UUP_2D \). In this case we define \( \varphi(P) = \varphi(P_1)\varphi(UUP_2D) \), where \( \varphi(UUP_2D) \) denotes the Hoppy path from \((k + 1 - z_1, k)\) to \((n - k, n - k)\) with \( n - k + 1 - \beta + \beta_1 \) skew steps.
Theorem 3.1. The map $\varphi$ is a bijection between the set of ternary paths of length $3(n+1)$ and the set of Hoppy paths on the lattice $\mathcal{B}$ from the origin to his $n+1$ bunnies. Moreover,

- If $P$ is a ternary path of length $3(n+1)$ with $n+2-i$ points on $x$-axis, then $\rho(P)$ is a Hoppy path on the lattice $\mathcal{B}$ from the origin to his bunny located at $(i, n)$.
- If $P$ is a ternary path of length $3(n+1)$ with $n+1-m$ peaks, then $\rho(P)$ is a Hoppy path on the lattice $\mathcal{B}$ from the origin to his $n+1$ bunnies with $m$ skew steps.

Theorem 3.1 gives the following results.

![Fig. 4. The Hoppy path and the corresponding ternary path.](image)
Corollary 3.2. (1) The number of Hoppy paths on the lattice $B$ from the origin to the bunny located at $(i, n)$ is equal to the number of ternary paths of length $3(n+1)$ with $n+2-i$ points on the $x$-axis.

(2) The number of Hoppy paths on the lattice $B$ from the origin to his $n+1$ bunnies with $m$ skew steps is equal to the number of ternary paths of length $3(n+1)$ with $n+1-m$ peaks.

Corollary 3.2 leads to an explicit formula for the number of Hoppy paths from $(0,0)$ to $(i,n)$. To do this we let $G(x,q)$ be the ordinary generating function for the number of ternary paths of length $3n$ with $m$ points on the $x$-axis, that is,

$$G(x,q) = \sum_{n \geq 0} x^n \sum_{P \in \mathcal{F}_n} q^{\text{#points of } P \text{ on the } x \text{-axis}}.$$ 

An equation for the generating function $G(x,q)$ is obtained from the “first return decomposition” of a ternary path $P: P = UP_1UP_2DQ$, where $P_1, P_2, Q$ are ternary paths. Thus, the two possibilities of $Q$ either being an empty or non-empty ternary path give contributions $xq^2G^2(x,1)$ and $xqG^2(x,1)(G(x,q)-1)$, respectively. Hence, $G(x,q) = 1+xqG^2(x,1)(q-1+G(x,q))$, which is equivalent to

$$G(x,q) = 1 + \frac{xq^2G^2(x,1)}{1-xqG^2(x,1)}.$$

The above functional equation gives the ordinary generating function for the number of ternary paths of length $3n$ with exactly $m, m \geq 2$, points on the $x$-axis to be

$$x^{m-1}G^{2m-2}(x,1),$$

where $G(x,1) = 1 + xG^3(x,1)$. Applying the Lagrange inversion formula, see (⋆), we get that the $x^n$ coefficient in $x^{m-1}G^{2m-2}(x,1)$ is given by $\frac{2m-2}{3n-m+1} \binom{3n-m+1}{2m}$. Thus, Corollary 3.2 with $m=n+2-i$ provides the following result.

Corollary 3.3. The number of Hoppy paths on the lattice $B$ from the origin to his bunny located at $(i,n)$ is given by

$$b(i,n) = \frac{2n+2-2i}{2n+2+i} \binom{2n+2+i}{2n+2}.$$

Also, Corollary 3.2 leads to an explicit formula for the number of Hoppy paths from the origin to his $n$ bunnies with $m$ skew steps. To do this we let $G'(x,q)$ be the ordinary generating function for the number of ternary paths of length $3n$ with $m$ peaks, i.e.

$$G'(x,q) = \sum_{n \geq 0} x^n \sum_{P \in \mathcal{F}_n} q^{\text{#peaks in } P}.$$ 

An equation for the generating function $G'(x,q)$ is obtained from the “first return decomposition” of a ternary path $P: P = UP_1UP_2DQ$, where $P_1, P_2, Q$ are ternary paths. Thus, the two possibilities of $P_2$ either being an empty or non-empty ternary path give contributions $xqG^2(x,q)$ and $xG^2(x,q)(G'(x,q)-1)$, respectively. Hence,

$$G'(x,q) = 1 + xG^2(x,q)(q-1+G'(x,q)).$$

Applying the Lagrange inversion formula, we find the coefficient of $x^n$ in $G'(x,q)$ to be

$$\frac{1}{n} \sum_{i=0}^{n-1} \binom{2n}{i} \binom{n}{i+1} q^{i+1}.$$

Thus, Corollary 3.2 with $i = n-m-1$ yields the following result.

Corollary 3.4. The number of Hoppy paths on the lattice $B$ from the origin to his $n+1$ bunnies with $m$ skew steps is given by

$$\frac{1}{n+1} \binom{2(n+1)}{n-m} \binom{n+1}{n+1-m}.$$
4. A bijection between Problems 2 and 3

In this section, we construct a bijection ψ between Hoppy paths on the lattice ℂ and Hoppy paths on ℬ.

Let H1 be a Hoppy path on the lattice ℂ from the origin to his bunny located at the point (i, j; n). We read H1 one step at a time and successively generate a Hoppy path ψ(H1) = H2 on the lattice ℬ. Each step

(i, j; d)(i + i′, j + j′; d + 1)

of H1 corresponds to the steps

(j, d)(j + i′, d + 1)(j + i′ + 1, d + 1)⋯(j + j′, d + 1)

of H2.

On the other hand, let H2 be a Hoppy path on the lattice ℬ from the origin to one of his n bunnies. Suppose

H2 = (0, 0)(x10, 1)(x11, 1)(x20, 2)(x21, 2)(x22, 2)⋯(xn0, n)⋯(xn1, n).

(Usually, if xji = xji+1, then we omit (xji, j) since (xji, j)(xji+1, j) just is a point.) Then the corresponding Hoppy path ψ−1(H2) = H1 on the lattice ℂ is

H1 = (0, 0; 0)(x10, x11; 1)(x10 + (x20 − x11), x21; 2)⋯(x10 + (x20 − x11) + ⋅⋅⋅ + (xn0 − xn−11−1), xn1; n).

For example, for the Hoppy path (0, 0)(1, 1)(1, 2)(1, 3)(2, 3)(4, 4)(4, 5)(4, 6)(5, 6) on the lattice ℬ, the corresponding Hoppy path on the lattice ℂ is (0, 0; 0)(1, 1; 1)(1, 1; 2)(1, 2; 3)(3, 4; 4)(3, 4; 5)(3, 5; 6). From the bijection ψ, we obtain the following result.

**Theorem 4.1.** The map ψ is a bijection between the set of Hoppy paths from the origin to his \((n+\frac{1}{2})\) bunnies on the lattice ℂ and the set of Hoppy paths to get from the origin to his \(n+1\) bunnies on the lattice ℬ. Moreover, the number of Hoppy paths from the origin to his \((n+\frac{1}{2})\) bunnies on the lattice ℂ is given by \(\frac{1}{2n+1} \binom{3(n+1)}{n+1}\).

The construction of this bijection ψ gives rise to the following results.

**Corollary 4.2.** (1) The number of Hoppy paths \(\sum_i c(i, j; n)\) on the lattice ℂ is equal to the number of Hoppy paths on the lattice ℬ from the origin to the bunny located at (j, n).

(2) The number of Hoppy paths \(c(i, j; n)\) from the origin to his bunny located at \((i, j; n) \in ℂ\) is equal to the number of Hoppy paths on the lattice ℬ from the origin to the bunny located at \((j, n)\) with \(n + j - i\) steps.

We remark that our Problem 3 generalizes Hoppy problem in [2]. Indeed, from the definitions of Problem 3 together with [2, Theorem 2.2] we find the number of Hoppy paths from the origin to his bunny located at \((0, j; n) \in ℂ\) is the same as the number of Hoppy paths defined in [2, Theorem 2.2]. Thus, it follows that

\[ c(0, j; n) = \sum_{d=0}^{n-j} (-1)^d \binom{n-j-d}{d} C_{n-d}, \]

where \(C_m\) is the mth Catalan number and \(j = 0, 1, \ldots, n\).

**Acknowledgment**

The authors would like to thank Mark Dukes for reading a previous version of the present paper, and the referees for their very valuable comments.

**References**