Continued Fractions and Generalized Patterns

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Babson and Steingrìmssson (2000, Séminaire Lotharingien de Combinatoire, B44b, 18) introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

Let \( f_{\tau,r}(n) \) be the number of 1-3-2-avoiding permutations on \( n \) letters that contain exactly \( r \) occurrences of \( \tau \), where \( \tau \) is a generalized pattern on \( k \) letters. Let \( F_{\tau,r}(x) \) and \( F_{\tau}(x, y) \) be the generating functions defined by \( F_{\tau,r}(x) = \sum_{n \geq 0} f_{\tau,r}(n)x^n \) and \( F_{\tau}(x, y) = \sum_{n \geq 0} F_{\tau,r}(x)^{y^n} \). We find an explicit expression for \( F_{\tau}(x, y) \) in the form of a continued fraction for \( \tau \) as a generalized pattern: \( \tau = 123- \ldots -k \), \( \tau = 213- \ldots -k \), \( \tau = 123 \ldots -k \), or \( \tau = k \ldots 321 \). In particular, we find \( F_{\tau}(x, y) \) for any \( \tau \) generalized pattern of length 3. This allows us to express \( F_{\tau,r}(x) \) via Chebyshev polynomials of the second kind and continued fractions.

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1. INTRODUCTION

Let \([p] = \{1, \ldots, p\}\) denote a totally ordered alphabet on \( p \) letters, and let \( \pi = (\pi_1, \ldots, \pi_m) \in [p]^m \), \( \beta = (\beta_1, \ldots, \beta_m) \in [p]^m \). We say that \( \pi \) is order-isomorphic to \( \beta \) if for all \( 1 \leq i < j \leq m \) one has \( \pi_i < \pi_j \) if and only if \( \beta_i < \beta_j \). For two permutations \( \pi \in S_n \) and \( \tau \in S_k \), an occurrence of \( \tau \) in \( \pi \) is a subsequence \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that \( (\pi_{i_1}, \ldots, \pi_{i_k}) \) is order-isomorphic to \( \tau \); in such a context \( \tau \) is usually called the pattern (classical pattern). We say that \( \pi \) avoids \( \tau \), or is \( \tau \)-avoiding, if there is no occurrence of \( \tau \) in \( \pi \).

More generally, we say \( \pi \) containing \( \tau \) exactly \( r \) times, if there exists \( r \) different occurrences of \( \tau \) in \( \pi \).

The set of all \( \tau \)-avoiding permutations of all possible sizes including the empty permutation is denoted \( S(\tau) \). Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [8] to singularities of Schubert varieties [10]. A complete study of pattern avoidance for the case \( \tau \in S_3 \) is carried out in [16].

On the other hand, [1] introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The idea of [1] introducing these patterns was the study of Mahonian statistics.

We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1324, as 1-3-2-4, and if we write, say 24-1-3, then we mean that if this pattern occurs in permutation \( \pi \), then the letters in the permutation \( \pi \) that correspond to 2 and 4 are adjacent. For example, the permutation \( \pi = 35421 \) has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas \( \pi \) has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342 and 341.

Reference [3] presented a complete solution for the number of permutations avoiding any pattern of length three with exactly one adjacent pair of letters. Reference [4] presented a complete solution for the number of permutations avoiding any two patterns of length three with exactly one adjacent pair of letters. Reference [7] almost presented results avoiding two or more 3-patterns without internal dashes, that is, where the pattern corresponds to a contiguous subword in a permutation. Besides, [5] presented the following generating functions regarding the distribution of the number of occurrences of any generalized pattern of length 3:

\[
\sum_{\pi \in S} \gamma(123)\pi x^{\left\lfloor \frac{\left\lceil \frac{|\pi|}{3!} \right\rceil}{2} \right\rceil} = \frac{2f(y)e^{2(f(y)-y+1)x}}{f(y)+y+1+f(y)-y-1)e^{f(y)x}}.
\]

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The purpose of this paper is to point out an analogue of [15], and some interesting consequences of this analogue. Generalizations of this theorem have already been given in [6, 9, 12]. In the present paper we study the generating function for the number $1-3-2$-avoiding permutations in $S_n$ that contain a prescribed number of generalized pattern $\tau$. The study of the obtained continued fraction allows us to recover and to present an analogue of the results of [2, 6, 9, 12] that relates the number of $1-3-2$-avoiding permutations that contain no $12-3$-...-3 patterns to Chebyshev polynomials of the second kind.

Let $f_{\tau,r}(n)$ stand for the number of $1-3-2$-avoiding permutations in $S_n$ that contain exactly $r$ occurrences of $\tau$. We denote by $F_{\tau,r}(x)$ and $F_{\tau}(x,y)$ the generating function of the sequence $\{f_{\tau,r}(n)\}_{n \geq 0}$ and $\{f_{\tau,r}(n)\}_{n,r \geq 0}$, respectively, that is,

$$F_{\tau,r}(x) = \sum_{n \geq 0} f_{\tau,r}(n)x^n, \quad F_{\tau}(x,y) = \sum_{r \geq 0} F_{\tau,r}(x)y^r.$$ 

The paper is organized as follows. The cases $\tau = 12-3$-...-$k$, $\tau = 21-3$-...-$k$, $\tau = 123$...$k$, and $\tau = k$...321 are treated in Section 2. In Section 3, we present the cases $\tau = 123$, 213, 231, 312, and 321, that is, $\tau$ is a 3-letters generalized pattern without dashes. In Section 4, we treat the cases when $\tau$ is a 3-letters generalized pattern with one dash. Finally, in Section 5, we present examples of restricted more than one generalized pattern of 3-letters.

2. Four General Cases

In this section, we study the following four cases: $\tau = 12-3$-...-$k$, $\tau = 21-3$-...-$k$, $\tau = 123$...$k$, and $\tau = k$...21, by the following three subsections.

2.1. Pattern $12-3$-...-$k$. Our first result is a natural analogue of the main theorems of [9, 12, 15].

**Theorem 2.1.** The generating function $F_{12-3$-...-$k}(x,y)$ for $k \geq 2$ is given by the continued fraction

$$\frac{1}{1 - x + xy^{d_1} - \frac{x y^{d_2}}{1 - x + xy^{d_2} - \frac{x y^{d_3}}{1 - x + xy^{d_3} - \frac{x y^{d_4}}{1 - x + xy^{d_4} - \frac{x y^{d_5}}{\ddots}}}}}$$

where $d_i = \binom{i-1}{k-2}$, and $(\binom{a}{b})$ is assumed 0 whenever $a < b$ or $b < 0$.

**Proof.** Following [12] we define $\eta_j(\pi), j \geq 3$, as the number of occurrences of $12-3$-...-$j$ in $\pi$. Define $\eta_2(\pi)$ for any $\pi$, as the number of occurrences of $12$ in $\pi$, $\eta_1(\pi)$ as the number of letters of $\pi$, and $\eta_0(\pi) = 1$ for any $\pi$, which means that the empty pattern occurs exactly once in each permutation. The weight of a permutation $\pi$ is a monomial in $k$ independent variables $q_1, \ldots, q_k$ defined by

$$w_k(\pi) = \prod_{j=1}^{k} q_j^{\eta_j(\pi)}.$$ 

The total weight is a polynomial

$$W_k(q_1, \ldots, q_k) = \sum_{\pi \in S(1-3-2)} w_k(\pi).$$
The following proposition is implied immediately by the definitions.

**PROPOSITION 2.2.** $F_{12-3-\ldots-k}(x, y) = W_k(x, 1, \ldots, 1, y)$ for $k \geq 2$.

We now find a recurrence relation for the numbers $\eta_j(\pi)$. Let $\pi \in S_n$, so that $\pi = (\pi', n, \pi'')$.

**PROPOSITION 2.3.** For any nonempty $\pi \in S(1-3-2)$

$$\eta_j(\pi) = \eta_j(\pi') + \eta_j(\pi'') + \eta_{j-1}(\pi'),$$

where $j \neq 2$. Besides, if $\pi'$ is nonempty then

$$\eta_2(\pi) = \eta_2(\pi') + \eta_2(\pi'') + 1,$$

otherwise

$$\eta_2(\pi) = \eta_2(\pi'').$$

**PROOF.** Let $l = \pi^{-1}(n)$. Since $\pi$ avoids $1-3-2$, each number in $\pi'$ is greater than any of the numbers in $\pi''$. Therefore, $\pi'$ is a $1$-$3$-$2$-avoiding permutation of the numbers $\{n - l + 1, n - l + 2, \ldots, n - 1\}$, while $\pi''$ is a $1$-$3$-$2$-avoiding permutation of the numbers $\{1, 2, \ldots, n - l\}$. On the other hand, if $\pi'$ is an arbitrary $1$-$3$-$2$-avoiding permutation of the numbers $\{n - l + 1, n - l + 2, \ldots, n - 1\}$ and $\pi''$ is an arbitrary $1$-$3$-$2$-avoiding permutation of the numbers $\{1, 2, \ldots, n - l\}$, then $\pi = (\pi', n, \pi'')$ is $1$-$3$-$2$-avoiding. Finally, if $(i_1, \ldots, i_j)$ is an occurrence of $12-3-\ldots-j$ in $\pi$ then either $i_j < l$, and so it is also an occurrence of $12-3-\ldots-j$ in $\pi'$, or $i_j > l$, and so it is also an occurrence of $12-3-\ldots-j$ in $\pi''$, or $i_j = l$, and so $(i_1, \ldots, i_{j-1})$ is an occurrence of $12-3-\ldots-(j - 1)$ in $\pi'$, where $j \neq 2$. For $j = 2$ the proposition is trivial. The result follows.

Now we are able to find the recurrence relation for the total weight $W$. Indeed, by Proposition 2.3,

$$W_k(q_1, \ldots, q_k) = 1 + \sum_{\pi \neq \pi' \in S(1-3-2)} \prod_{j=1}^k q_j^{\eta_j(\pi)}$$

$$= 1 + \sum_{\pi \neq \pi' \in S(1-3-2)} \sum_{\pi'' \in S(1-3-2)} \prod_{j=1}^k q_j^{\eta_j(\pi'')}, q_1^{\eta_1(\pi')} + q_2, \prod_{j=2}^{k-1} (q_j q_{j+1})^{\eta_j(\pi')} \cdot q_k^{\eta_k(\pi')} + \sum_{\pi'' \in S(1-3-2)} q_1 \prod_{j=1}^k q_j^{\eta_j(\pi'')}.$$

Hence

$$W_k(q_1, \ldots, q_k) = 1 + q_1 W_k(q_1, \ldots, q_k) + q_1 q_2 W_k(q_1, \ldots, q_k) (W_k(q_1, q_2 q_3, \ldots, q_{k-1} q_k, q_k) - 1). \quad (1)$$

Following [12], for any $d \geq 0$ and $2 \leq m \leq k$ define

$$q^{d,m} = \prod_{j=2}^k q_j^{\binom{d-m}{j-d}},$$

recall that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. The following proposition is implied immediately by the well-known properties of binomial coefficients.
PROPOSITION 2.4. For any \( d \geq 0 \) and \( 2 \leq m \leq k \)

\[ q^{d,m} q^{d,m+1} = q^{d+1,m}. \]

Observe now that \( W_k(q_1, \ldots, q_k) = W_k(q_1, q_0^2, \ldots, q_0^k) \) and that by (1) and Proposition 2.4

\[ W_k(q_1, q_0^2, \ldots, q_0^k) = 1 + q_1 W_k(q_1, q_0^2, \ldots, q_0^k) + q_1 q_0^2 W_k(q_1, q_0^2, \ldots, q_0^k)(W_k(q_1, q_0^d, \ldots, q_0^k) - 1). \]

Therefore

\[ W_k(q_1, \ldots, q_k) = \frac{1}{1 - q_1 + q_1 q_0^2 - \frac{q_1 q_0^d}{1 - q_1 + q_1 q_0^d - \frac{q_1 q_0^k}{1 - q_1 + q_1 q_0^k - \frac{q_1 q_0^{k-1}}{\ldots}}}}. \]

To obtain the continued fraction representation for \( F(x, y; k) \) it is enough to use Proposition 2.2 and to observe that

\[ q_1 q_0^d \bigg|_{q_1=q_2=\ldots=q_{k-1}=1,q_k=y} = x y^{(d-2)} \]

COROLLARY 2.5.

\[ F(x, y; 2) = \frac{1 - x + x y - \sqrt{(1 - x)^2 - 2x(1 + x)y + x^2 y^2}}{2xy}, \]

in other words, for any \( r \geq 1 \)

\[ f_{12}(n) = \frac{r + 1}{n(n - r)} \left( \frac{n}{r + 1} \right)^2. \]

PROOF. For \( k = 2 \), Theorem 2.1 yields

\[ F_{12}(x, y) = \frac{1}{1 - x + xy - \frac{x y}{1 - x + xy - \frac{x y}{1 - x + xy - \frac{x y}{1 - x + xy - \ldots}}}}. \]

which means that

\[ F_{12}(x, y) = \frac{1}{1 - x + xy - xy F_{12}(x, y)}. \]

So the rest is easy to see.

\[ \square \]

Now, we find an explicit expression for \( F_{12,3,\ldots,k,r}(x) \) where \( 0 \leq r \leq k - 2 \). Following [12], consider a recurrence relation

\[ T_j = \frac{1}{1 - x T_{j-1}}, \quad j \geq 1. \]
The solution of (2) with the initial condition \( T_0 = 0 \) is denoted by \( R_j(x) \), and the solution of (2) with the initial condition

\[
T_0 = G_{12-3-\ldots-k}(x, y) = \frac{1}{1 - x + xy\left(\frac{k-2}{2}\right) - \frac{1}{1 - x + xy\left(\frac{k-1}{2}\right)} \frac{1}{1 - x + xy\left(\frac{k}{2}\right)} - \frac{1}{1 - x + xy\left(\frac{k+1}{2}\right)}}
\]

is denoted by \( S_j(x, y; k) \), or just \( S_j \) when the value of \( k \) is clear from the context. Our interest in (2) is stipulated by the following relation, which is an easy consequence of Theorem 2.1:

\[
F_{12-3-\ldots-k}(x, y) = S_{k-2}(x, y; k).
\]  

(3)

Following [12, Eqn. (4)], for all \( j \geq 1 \)

\[
R_j(x) = \frac{U_{j-1} \left( 1 - \frac{1}{2\sqrt{x}} \right)}{\sqrt{x} U_j \left( \frac{1}{2\sqrt{x}} \right)},
\]  

(4)

where \( U_j(\cos \theta) = \sin(j + 1)\theta \)/\( \sin \theta \) are the Chebyshev polynomials of the second kind. Next, we find an explicit expression for \( S_j \) in terms of \( G \) and \( R_j \)

**Lemma 2.6.** For any \( j \geq 2 \) and any \( k \geq 2 \)

\[
S_j(x, y; k) = R_j(x) \frac{1 - x R_{j-1}(x) G_{12-3-\ldots-k}(x, y)}{1 - x R_j(x) G_{12-3-\ldots-k}(x, y)}.
\]  

(5)

**Proof.** Indeed, from (2) and \( S_0 = G \) we get \( S_1 = 1/(1 - x G) \). On the other hand, \( R_0 = 0 \), \( R_1 = 1 \), so (5) holds for \( j = 1 \). Now let \( j > 1 \), then by induction

\[
S_j = \frac{1}{1 - x S_{j-1}} = \frac{1}{1 - x R_{j-1}} \frac{1 - x R_{j-1} G}{1 - x R_{j-1} G}
\]

Relation (2) for \( R_j \) and \( R_{j-1} \) yields \((1 - x R_{j-2}) R_{j-1} = (1 - x R_{j-1}) R_j = 1 \), which together with the above formula gives (5). \( \square \)

As a corollary from Lemma 2.6 and (3) we get the following expression for the generating function \( F_{12-3-\ldots-k}(x, y) \).

**Corollary 2.7.** For any \( k \geq 3 \)

\[
F_{12-3-\ldots-k}(x, y) = R_k(x) + \left( R_{k-2}(x) - R_{k-3}(x) \right) \sum_{m \geq 1} \left( x R_{k-2}(x) G_{12-3-\ldots-k}(x, y) \right)^m.
\]

Now we are ready to express the generating function \( F_{12-3-\ldots-k;r}(x) \) where \( 0 \leq r \leq k - 2 \), via Chebyshev polynomials.

**Theorem 2.8.** For any \( k \geq 3 \), \( F_{12-3-\ldots-k;r}(x) \) is a rational function given by

\[
F_{12-3-\ldots-k;r}(x) = \frac{x^{r-1} U_{k-2}^{-1} \left( \frac{1}{2\sqrt{x}} \right)}{(1 - x) x^{r+1} \left( \frac{1}{2\sqrt{x}} \right)}, \quad 1 \leq r \leq k - 2,
\]

\[
F_{12-3-\ldots-k;0}(x) = \frac{U_{k-1} \left( \frac{1}{2\sqrt{x}} \right)}{\sqrt{x} U_k \left( \frac{1}{2\sqrt{x}} \right)},
\]

where \( U_j \) is the \( j \)th Chebyshev polynomial of the second kind.
The total weight \( (\pi) \) where \( d \) is assumed \( 0 \) whenever \( a < b \) or \( b = 0 \).

**Proof.** Following [12] we define \( v_j(\pi) \), \( j \geq 3 \), as the number of occurrences of \( 21-3-\ldots-j \) in \( \pi \). Define \( v_2(\pi) \) for any \( \pi \), as the number of occurrences of \( 21 \) in \( \pi \), \( v_1(\pi) \) as the number of letters of \( \nu \), and \( v_0(\pi) = 1 \) for any \( \pi \), which means that the empty pattern occurs exactly once in each permutation. The weight of a permutation \( \pi \) is a monomial in \( k \) independent variables \( q_1, \ldots, q_k \) defined by

\[
v_k(\pi) = \prod_{j=1}^{k} q_j^{v_j(\pi)}.
\]

The total weight is a polynomial

\[
V_k(q_1, \ldots, q_k) = \sum_{\pi \in S(1\ldots k)} v_k(\pi).
\]

The following proposition is implied immediately by the definitions.

**Proposition 2.10.** \( F_{21-3-\ldots-k}(x, y) = V_k(x, 1, \ldots, 1, y) \) for \( k \geq 2 \).

We now find a recurrence relation for the numbers \( v_j(\pi) \). Let \( \pi \in S_n \), so that \( \pi = (\pi', n, \pi'') \).

**Proposition 2.11.** For any nonempty \( \pi \in S(1\ldots k) \)

\[
v_j(\pi) = v_j(\pi') + v_j(\pi'') + v_{j-1}(\pi'),
\]

where \( j \neq 2 \). Besides, if \( \pi'' = \pi' \) is nonempty then

\[
v_2(\pi) = v_2(\pi') + v_2(\pi'') + 1,
\]

otherwise

\[
v_2(\pi) = v_2(\pi').
\]
PROOF. Similar to Proposition 2.3 we get $\pi'$ avoids 1-3-2 if and only if $\pi'$ is a 1-3-2-avoiding permutation of the numbers $[n - l + 1, n - l + 2, \ldots, n - 1]$, while $\pi''$ is a 1-3-2-avoiding permutation of the numbers $[1, 2, \ldots, n - l]$. Finally, if $(i_1, \ldots, i_j)$ is an occurrence of 21-3-\ldots-j in $\pi$ then either $i_j < l$ and so it is also an occurrence of 21-3-\ldots-j in $\pi'$, or $i_1 > l$ and so it is also an occurrence of 21-3-\ldots-j in $\pi''$, or $i_j = l$ and so $(i_1, \ldots, i_{j-1})$ is an occurrence of 21-3-\ldots-(j-1) in $\pi$, where $j \neq 2$. For $j = 2$ the proposition is trivial. 

Now we are able to find the recurrence relation for the total weight $V$. Proposition 2.11 yields

$$V_k(q_1, \ldots, q_k) = 1 + \sum_{\emptyset \neq \pi' \in S(1-3-2)} \prod_{j=1}^{k} b_j^\pi \cdot q_j$$

$$= 1 + \sum_{\emptyset \neq \pi' \in S(1-3-2)} \prod_{j=1}^{k} b_j^\pi \cdot q_1 \cdot \prod_{j=2}^{k-1} \prod_{j=1}^{k} (q_j b_j) \cdot q_k$$

$$+ \sum_{\pi' \in S(1-3-2)} \prod_{j=1}^{k} b_j^\pi \cdot \prod_{j=2}^{k-1} (q_j b_j) \cdot \prod_{j=2}^{k} (q_j b_j).$$

Hence

$$V_k(q_1, \ldots, q_k) = 1 + q_1 V_k(q_1, q_2, q_3, \ldots, q_{k-1} q_k, q_k)$$

$$+ q_1 q_2 V_k(q_1, q_2, q_3, \ldots, q_{k-1} q_k, q_k)(V_k(q_1, q_2, \ldots, q_k) - 1). \quad (6)$$

Observe now that $V_k(q_1, \ldots, q_k) = V_k(q_1, q^{0,2}, \ldots, q^{0,k})$ and by (6) and Proposition 2.4 we get

$$V_k(q_1, q^{d,2}, \ldots, q^{d,k}) = 1 + q_1 V_k(q_1, q^{d+1,2}, \ldots, q^{d+1,k})$$

$$+ q_1 q^{d,2} V_k(q_1, q^{d+1,2}, \ldots, q^{d+1,k})(V_k(q_1, q^{d,2}, \ldots, q^{d,k}) - 1).$$

To obtain the continued fraction representation for $F_{21-3-\ldots-k}(x, y)$ it is sufficient to use Proposition 2.10 and to observe that

$$q_1 q^{d,2} \bigg|_{q_1 = x, q_2 = \ldots = q_{k-1} = 1, q_k = y} = xy^{(d-1) \choose k-2}.$$

COROLLARY 2.12.

$$F_{21}(x, y) = \frac{1 - x + xy - \sqrt{(1 - x)^2 + 4x(1 + x)y + x^2 y^2}}{2xy},$$

in other words, for any $r \geq 1$

$$f_{21,r}(n) = \frac{r + 1}{n(n - r)} \left( \frac{n}{r + 1} \right)^2.$$
Now, we are ready to find an explicit expression for \( F_{21,3,...,k,r}(x) \) where \( 0 \leq r \leq k - 2 \).

Consider a recurrence relation

\[
T'_j = 1 - \frac{x}{x - \frac{1}{T'_{j-1}(x)}}, \quad j \geq 1.
\]  

(7)

The solution of (7) with the initial condition \( T'_0 = 0 \) is given by \( R_j(x) \) (Lemma 2.13), and the solution of (7) with the initial condition \( T'_0 = 0 \) is denoted by \( S'_j(x, y; k) \), or just \( S'_j \) when the value of \( k \) is clear from the context. Our interest in (7) is stipulated by the following relation, which is an easy consequence of Theorem 2.9:

\[
F_{21,3,...,k}(x, y) = S'_j(x, y; k).
\]  

(8)

First of all, we find an explicit formula for the functions \( T'_j(x) \) in (7).

**Lemma 2.13.** For any \( j \geq 1 \),

\[
T'_j(x) = R_j(x).
\]  

**Proof.** Indeed, it follows immediately from (7) that \( T'_0(x) = 0 \) and \( T'_1(x) = 1 \). Let us induce, we assume \( T'_{j-1}(x) = R_{j-1}(x) \), and prove that \( T'_j(x) = R_j(x) \). By use of (7)

\[
T'_j(x) = 1 - \frac{x}{x - \frac{1}{R'_{j-1}(x)}}.
\]

On the other hand, following [12], \( R_j(x) = \frac{1}{1 - xR_{j-1}(x)} \) which means that \( R_j(x) = 1 + xR_{j-1}(x)R_j(x) \), hence \( T'_j(x) = R_j(x) \). \( \square \)

Next, we find an explicit expression for \( S'_j \) in terms of \( G \) and \( R_j \).

**Lemma 2.14.** For any \( j \geq 2 \) and any \( k \geq 2 \)

\[
S'_j(x, y; k) = R_j(x) \frac{1 - xR_{j-1}(x)G_{21,3,...,k}(x, y; k)}{1 - xR_j(x)G_{21,3,...,k}(x, y)}.
\]  

(10)

As a corollary from Lemma 2.14 and (6) we get the following expression for the generating function \( F_{21,3,...,k}(x, y) \).

**Corollary 2.15.** For any \( k \geq 3 \)

\[
F_{21,3,...,k}(x, y) = R_k(x) + \left( R_{k-2}(x) - R_{k-3}(x) \right) \sum_{m \geq 1} \left( xR_{k-2}(x)G_{21,3,...,k}(x, y) \right)^m.
\]

Now we are ready to express the generating function \( F_{21,3,...,k,r}(x) \) where \( 0 \leq r \leq k - 2 \), via Chebyshev polynomials.
THEOREM 2.16. For any \( k \geq 3 \), \( F_{21, \ldots, k, r}(x) \) is a rational function given by
\[
F_{21, \ldots, k, r}(x) = x^{\frac{r-1}{2}} U_{k-2}^{r-2}\left(\frac{1}{2\sqrt{r}}\right) U_k^{r-2}\left(\frac{1}{2\sqrt{r}}\right), \quad 1 \leq r \leq k - 2,
\]
\[
F_{21, \ldots, k, 0}(x) = U_{k-1}\left(\frac{1}{2\sqrt{r}}\right) \sqrt{r} U_k\left(\frac{1}{2\sqrt{r}}\right),
\]
where \( U_j \) is the \( j \)th Chebyshev polynomial of the second kind.

PROOF. Observe that \( G_{21, \ldots, k}(x, y) = 1 + \frac{x}{1-x-xy} + y^{k-1} P(x, y) \), so by Corollary 2.15 we get
\[
F_{21, \ldots, k}(x, y) = R_k(x) + (R_{k-2}(x) - R_{k-3}(x)) \sum_{m=1}^{k-1} \left( x R_{k-2}(x) \left( 1 + \frac{x}{1-x-xy} \right) \right)^m,
\]
where \( P(x, y) \) and \( P'(x, y) \) are formal power series. To complete the proof, it suffices to use (9) together with the identity \( U_{n-1}(z) - U_n(z) U_{n-2}(z) = 1 \).

REMARK 2.17. Theorem 2.16 and [12] yield the number of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the pattern 21-3-\ldots-k \( \) is the same number of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the pattern 1-2-3\ldots-k, for all \( r = 0, 1, 2, \ldots, k - 2 \). However, the question is if there exists a natural bijection between the set of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the generalized pattern 21-3\ldots-k, and the set of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the classically pattern 1-2-3\ldots-k.

2.3. Patterns: \( \tau = 12 \ldots k \) and \( \tau = k \ldots 21 \). Let \( \pi \in S_n \); we say \( \pi \) has \( d \)-increasing canonical decomposition if \( \pi \) has the following form
\[
\pi = (\pi^1, \pi^2, \ldots, \pi^d, a_d, \ldots, a_2, a_1, n, \pi^{d+1}),
\]
where all the entries of \( \pi^i \) are greater than all the entries of \( \pi^{i+1} \), and \( a_d < a_{d-1} < \cdots < a_1 < n \). We say \( \pi \) has \( d \)-decreasing canonical decomposition if \( \pi \) has the following form
\[
\pi = (\pi^1, n, a_1, \ldots, a_d, \pi^{d+1}, \pi^d, \ldots, \pi^1),
\]
where all the entries of \( \pi^i \) are greater than all the entries of \( \pi^{i+1} \), and \( a_d < a_{d-1} < \cdots < a_1 < n \). The following proposition is the basis of all other results in this section.

PROPOSITION 2.18. Let \( \pi \in S_n(1-3-2) \). Then there exists unique \( d \geq 0 \) and \( e \geq 0 \) such that \( \pi \) has a \( d \)-increasing canonical decomposition, and has \( e \)-decreasing canonical decomposition.

PROOF. Let \( \pi \in S_n(1-3-2) \), and let \( a_d, a_{d-1}, \ldots, a_1, n \) a maximal increasing subsequence of \( \pi \) such that \( \pi = (\pi', a_d, \ldots, a_1, n, \pi'') \). Since \( \pi \) avoids 1-3-2 there exists \( d \) subsequences \( \pi^j \) such that \( \pi = (\pi^1, \pi^d, a_d, \ldots, a_1, n, \pi^{d+1}) \), and all the entries of \( \pi^i \) are greater than all the entries of \( \pi^{i+1} \), and all the entries of \( \pi^d \) are greater than all entries of \( \pi' \). Hence, \( \pi \) has \( d \)-increasing canonical decomposition. Similarly, there exist \( e \) unique such that \( \pi \) is \( e \)-decreasing canonical decomposition.

\[\square\]
Let us define $I_r(x, y; d)$ (respectively, $J_r(x, y; e)$) as the generating function for all $d$-increasing (respectively, $e$-decreasing) canonical decomposition of permutations in $S_n(1-3-2)$ with exactly $r$ occurrences of $\tau$. The following proposition is implied immediately by the definitions.

**Proposition 2.19.**

$$F_\tau(x, y) = 1 + \sum_{d \geq 0} I_r(x, y; d) = 1 + \sum_{e \geq 0} J_r(x, y; e).$$

**Proof.** Immediately, by definitions of the generating functions and Proposition 2.18 (1 for the empty permutation).

Now, we present examples for Propositions 2.18 and 2.19.

**First example**

**Theorem 2.20.** $F_{k-21}(x, y) = F_{12\ldots k}(x, y)$, such that

$$F_{12\ldots k}(x, y) = \sum_{n=0}^{k-2} x^n F_{12\ldots k}^n(x, y) + \frac{x^{k-1} F_{12\ldots k}^{k-1}(x, y)}{1 - xy F_{12\ldots k}(x, y)}.$$

**Proof.** By Proposition 2.18 and definitions it is easy to obtain for all $d \geq 0$

$$I_{12\ldots k}(x, y; d) = x^{d+1} y^d F_{12\ldots k}^{d+1}(x, y),$$

where $s_d = d + 1 - k$ for $d \geq k - 1$, and otherwise $s_d = 0$. So by Proposition 2.19 the theorem holds.

Similarly, we obtain the same result for $F_{k-21}(x, y)$.

As a remark, by the above theorem, it is easy to obtain the same results for Corollaries 2.5 and 2.12.

**Second example**

**Theorem 2.21.** Let $1 \leq l \leq k - 1$. Then $F_{1-2\ldots-(l-1)-l(l+1)\ldots k}(x, y) = U_l(x, 1, \ldots, 1, y)$ where

$$U_l(q_1, \ldots, q_l) = 1 + \sum_{d \geq 0} \left( q_d \prod_{j=1}^{l-1} \left( \frac{d+1}{j+1} \right) \prod_{j=0}^{d} U_l(p_{1:j}, \ldots, p_{l:j}) \right),$$

and for $i = 1, 2, \ldots, l$, $p_{i;j} = \prod_{m=1}^{l-1} q_j \left( \frac{d+1}{j+1} \right)$, $p_{1:j} = q_l$ for all $0 \leq j \leq k - l$, and $p_{i;j} = \prod_{m=1}^{l} p_{1:k-l}$ for all $j \geq k - l + 1$.

**Proof.** Following [12] we define $\gamma_j(\pi)$, $j \leq l - 1$, as the number of occurrences of $1-2\ldots-j$ in $\pi$. Define $\gamma_l(\pi)$ for any $\pi$, as the number of occurrences of $1-2\ldots-(l-1)-l(l+1)\ldots k$ in $\pi$, and $\gamma_0(\pi) = 1$ for any $\pi$, which means that the empty pattern occurs exactly once in each permutation. The weight of a permutation $\pi$ is a monomial in $l$ independent variables $q_1, \ldots, q_l$ defined by

$$u_l(\pi) = \prod_{j=1}^{l} q_j^{\gamma_j(\pi)}.$$
Continued fractions and generalized patterns

The total weight is a polynomial

\[ U_l(q_1, \ldots, q_l) = \sum_{\pi \in S(1-3-2)} u_l(\pi). \]

The following proposition is implied immediately by the definitions and Proposition 2.18.

**Proposition 2.22.** \( F_{1,2,\ldots,(l-1)-(l+1)\ldots,k}(x, y) = U_k(x, 1, \ldots, 1, y) \) for \( k > l \geq 1 \), and \( U_l(q_1, \ldots, q_l) = 1 + \sum_{d \geq 0} \sum_{\pi \in A_d} u_l(\pi) \), where \( A_d \) is the set of all \( d \)-increasing canonical decomposition permutations in \( S(1-3-2) \).

Let us denote \( U_{l,d}(q_1, \ldots, q_l) = \sum_{\pi \in A_d} u_l(\pi) \).

**Proposition 2.23.** For any \( d \geq 0 \),

\[ U_{l,d}(q_1, \ldots, q_l) = q_l \left( \frac{d+1-l}{d} \right)^{l-1} \prod_{j=1}^{d} q_j \prod_{j=0}^{d} U_l(p_{1;j}, \ldots, p_{l;j}). \]

**Proof.** Let \( \pi \) be \( d \)-increasing canonical decomposition, that is,

\[ \pi = (\pi^1, \pi^2, \ldots, \pi^d, a_d, \ldots, a_2, a_1, n, \pi^{d+1}), \]

where the numbers \( a_d < a_{d-1} < \cdots < a_1 < n \) appear as consecutive numbers in \( \pi \), all entries of \( \pi^j \) are greater than all the entries of \( \pi^{j+1} \), and all entries of \( \pi^d \) are greater than \( a_d \).

So, by calculating \( u_l(\pi) \) and summing over all \( \pi \in A_d \) we have that

\[ U_{l,d}(q_1, \ldots, q_d) = q_l \left( \frac{d+1-l}{d} \right)^{l-1} \prod_{j=1}^{d} q_j \prod_{j=0}^{d} U_l(p_{1;j}, \ldots, p_{l;j}). \]

Therefore, Theorem 2.21 holds, by using Propositions 2.22 and 2.23.

Now, let \( l = k - 1 \) and by using Theorem 2.21, it is easy to obtain the following.

**Corollary 2.24.** For \( k \geq 3 \),

\[ F_{1,2,\ldots,(k-2)-(k-1)k}(x, y) = \sum_{j=0}^{k-1} (x F_{1,2,\ldots,(k-2)-(k-1)k}(x, y))^j. \]

**Remark 2.25.** Similarly, the argument of \( d \)-increasing canonical decomposition, or the argument \( d \)-decreasing canonical decomposition yields other formulae, for example, the formula for \( F_{12,3,45}(x, y) \).

3. Three Letters Pattern Without Internal Dashes

In this section, we give a complete answer for \( F_\tau(x, y) \) where \( \tau \) is a generalized pattern without internal dashes; that is, \( \tau \) is 123, 213, 231, 312, and 321, by the following four subsections.
3.1. Patterns 123 and 321.

**Theorem 3.1.**

\[ F_{123}(x, y) = F_{321}(x, y) = \frac{1 + xy - x - \sqrt{1 - 4x + 4x^2 - 4xy(2 - xy)}}{2xy(x + y - xy)}. \]

**Proof.** Theorem 2.20 yields, \( F_{123}(x, y) = F_{321}(x, y) = H \) where

\[ H = 1 + xH + \frac{x^2H^2}{1 - xyH}, \]

so the theorem holds. \( \square \)

3.2. Pattern 231.

**Theorem 3.2.**

\[ F_{231}(x, y) = \frac{1 - 2x + 2xy - \sqrt{1 - 4x + 4x^2 - 4x^2y}}{2xy}, \]

that is, for all \( r, n \geq 0 \)

\[ F_{231,r}(x) = \frac{1}{r+1} \left( \frac{2r}{r} \right) \frac{x^{2r+1}}{(1-2x)^{2r+1}}, \quad f_{231,r}(n) = \frac{2n-2r-1}{r+1} \left( \frac{n-1}{2r} \right) \left( \frac{2r}{r} \right). \]

**Proof.** Let \( l = \pi^{-1}(n) \). Since \( \pi \) avoids 1-3-2, each number in \( \pi' \) is greater than any of the numbers in \( \pi'' \). Therefore, \( \pi' \) is a 1-3-2-avoiding permutation of the numbers \( \{n - l + 1, n - l + 2, \ldots, n - 1\} \), while \( \pi'' \) is a 1-3-2-avoiding permutation of the numbers \( \{1, 2, \ldots, n - l\} \). On the other hand, if \( \pi' \) is an arbitrary 1-3-2-avoiding permutation of the numbers \( \{n - l + 1, n - l + 2, \ldots, n - 1\} \) and \( \pi'' \) is an arbitrary 1-3-2-avoiding permutation of the numbers \( \{1, 2, \ldots, n - l\} \), then \( \pi = (\pi', n, \pi'') \) is 1-3-2-avoiding.

Now let us observe all the possibilities that \( \pi' \) and \( \pi'' \) is empty or not. This yields

\[ F_{231}(x, y) = 1 + x + 2x(F_{231}(x, y) - 1) + xy(F_{231}(x, y) - 1)^2, \]

hence the theorem holds. \( \square \)

3.3. Pattern 213.

**Theorem 3.3.**

\[ F_{213}(x, y) = \frac{1 - x^2 + x^2y - \sqrt{1 + 2x^2 - 2x^2y + x^4 - 2x^4y + x^4y^2 - 4x}}{2x(1 + xy - x)}. \]

**Proof.** Let \( D(x, y) \) be the generating function of all 1-3-2-avoiding permutations \( (\alpha', n) \in S_n \) such that contain 213 exactly \( r \) times. Let \( \alpha = (\alpha', n, \alpha'') \); if we consider the two cases \( \alpha' \) empty or not we have \( F_{213}(x, y) = 1 + D(x, y)F_{213}(x, y) \). Let \( \alpha = (\alpha', n) \); if we observe the two cases \( \alpha' \) empty or not, then (similarly)

\[ D(x, y) = x + x^2 + x^2y(F_{213}(x, y) - 1) + x^2(D(x, y) - 1) + x^2(D(x, y) - 1)(F_{213}(x, y) - 1). \]

However,

\[ F_{213}(x, y) = 1 + xF_{213}(x, y) \frac{1 + x - xy + x(y - 1)F_{213}(x, y)}{1 - xF_{213}(x, y)}, \]

hence, the theorem holds. \( \square \)
3.4. Pattern 312.

THEOREM 3.4.

\[ F_{312}(x, y) = \frac{1 - x^2 + x^2y - \sqrt{1 + 2x^2 - 2x^2y + x^4 - 2x^4y + x^4y^2 - 4x}}{2x(1 + xy - x)} . \]

PROOF. Let \( \alpha \in S(1-3-2) \); if \( \alpha = \varnothing \), then there is one permutation, otherwise by Proposition 2.18 we can write \( \alpha = (\alpha^1, n, a_1, a_2, \ldots, a_d, \alpha^{d+1}, \alpha^d, \ldots, \alpha^2) \) where all the entries of \( \alpha^j \) are greater than all the entries of \( \alpha^{j+1} \), and \( n > a_1 > a_2 > \cdots > a_d \). Hence, for any \( d = 0, 1 \) the generating function of these permutations in these cases is \( x^{d+1}F_{312}(x, y) \).

Let \( d \geq 2 \); if \( \alpha^{d+1} = \varnothing \), then the generating function of these permutations in this case is \( x^{d+1}F_{312}^d(x, y) \), otherwise the generating function is \( x^{d+1}yF_{312}^d(x, y)(F_{312}(x, y) - 1) \).

Hence

\[
F_{312}(x, y) = 1 + (x + x^2)F_{312}(x, y) + \sum_{d \geq 2} x^{d+1}F_{312}^d(x, y) + \sum_{d \geq 2} F_{312}^{d+1}(x, y)F_{312}(x, y) - 1, \]

which means that

\[
F_{312}(x, y) = 1 + xF_{312}(x, y) + \frac{x^2F_{312}(x, y)}{1 - xF_{312}(x, y)} + \frac{x^2yF_{312}(x, y)(F_{312}(x, y) - 1)}{1 - xF_{312}(x, y)}, \]

so the rest is easy to see. \( \Box \)

4. THREE LETTERS PATTERN WITH ONE DASH

In this section, we present examples \( F_{\tau}(x, y) \) where \( \tau \) is a generalized pattern with one dash. Theorem 2.1 yields

THEOREM 4.1. The generating function \( F_{12-3}(x, y) \) is given by the continued fraction

\[
\frac{1}{1 - x - xy - x^2 - \frac{x}{1 - x - xy - x^2 - \frac{x}{1 - x - xy - x^2 - \cdots}}} .
\]

Theorem 2.9 yields

THEOREM 4.2. For any \( k \geq 2 \),

\[
F_{21-3}(x, y) = 1 - \frac{x}{1 - \frac{1}{x - \frac{1}{y - \frac{1}{x - \frac{1}{y - \frac{1}{x - \cdots}}}}}} .
\]

For \( k = 3 \) and \( l = 2 \) Theorem 2.24 yields
THEOREM 4.3.

\[ F_{1-23}(x, y) = 1 + x F_{1-23}(x, y) + \sum_{d \geq 1} x^{d+1} y^{(d)} F_{2-13}(x, y) + \prod_{j=0}^{d-1} F_{1-23}(x y^j, y). \]

COROLLARY 4.4.

\[ F_{1-23;0}(x) = 1 - x - \frac{2x - 3x^2}{2x^2}; \]
\[ F_{1-23;1}(x) = \frac{x - 1}{2x} + \frac{1 - 2x - x^2}{2x \sqrt{1 - 2x - 3x^2}}; \]
\[ F_{1-23;2}(x) = \frac{x^4}{(1 - 2x - 3x^2)^{3/2}}; \]
\[ F_{1-23;3}(x) = x^2 - 1 + \frac{11x^2 + 43x^6 + 41x^5 - 7x^4 - 25x^3 + x^2 + 5x - 1}{(1 - 2x - 3x^2)^{3/2}}. \]

PROOF. By Theorem 4.3 and by \( F_{1-23}(x, 0) = F_{1-23;0}(x) \) we get

\[ F_{1-23;0}(x) = 1 + x F_{1-23;0}(x) + x^2 F_{1-23;0}(x), \]

which means the first formula holds.

By Theorem 4.3 we get

\[ \frac{d}{dy} F_{1-23}(x, 0) = x \frac{d}{dy} F_{1-23}(x, 0) + 2x^2 F_{1-23}(x, 0) \frac{d}{dy} F_{1-23}(x, 0) + x^3 F_{1-23}(x, 0)^2 F_{1-23}(0, 0), \]

and by \( F_{1-23;1}(x) = \frac{d}{dy} F_{1-23}(x, y) \big|_{y=0} \) and the first formula, we get the second formula.

Similarly, by Theorem 4.3 and by \( F_{1-23;0}(x) = \frac{1}{n!} \frac{d^n}{dx^n} F_{1-23}(x, y) \big|_{y=0} \) the other formulae holds.

THEOREM 4.5.

\[ F_{2-13}(x, y) = \frac{1}{1 - x - \frac{x^2}{y} \cdot \frac{x^3}{y^3} \cdot \frac{x^4}{y^4} \cdot \frac{x^5}{y^5} \cdot \frac{x^6}{y^6} \cdot \cdot \cdot}. \]

PROOF. By Propositions 2.18 and 2.19, we obtain

\[ F_{2-13}(x, y) = 1 + x F_{2-13}(x, y) \sum_{d \geq 0} x^d F_{2-13}(x y^d, y), \]

and the rest is easy to see. \( \square \)

5. FURTHER RESULTS

First of all, let us denote by \( G_{\tau; \phi}(x, y) \) the generating function for the number of permutations in \( S_n(1-3-2, \tau) \) such that contain \( \phi \) exactly \( r \) times; that is

\[ G_{\tau; \phi}(x, z) = \sum_{n \geq 0} x^n \sum_{\pi \in S_n(1-3-2, \tau)} \gamma^{a \phi(\pi)}. \]
where \( a_\phi(\pi) \) is the number of occurrences of \( \phi \) in \( \pi \). In this section, (similar to previous sections) we find \( G_{\tau, \phi}(x, y) \) in terms of continued fractions or by explicit formulae, for some cases of \( \tau \) and \( \phi \).

**Theorem 5.1.** The generating functions \( G_{123; 213}(x, y) \) and \( G_{321; 312}(x, y) \) are given by

\[
\frac{1}{1 - x - x^2(1 - y) - \frac{x^2y}{1 - x - x^2(1 - y) - \frac{x^2y}{1 - x - x^2(1 - y) - \ldots}}}.
\]
equivalently,

\[
1 - x - x^2 + x^2y - \sqrt{(1 - x - x^2)^2 - 2xy(1 + x + x^2) + x^4y^2}
\]

\[
2x^3y.
\]

**Theorem 5.2.**

\( G_{123; 231}(x, y) = H(x, y) + x^2(1 - y)H(x, y)^2, \)

where \( H(x, y) = \frac{1}{1 - x - x^2yH(x, y)} \), which means the number of permutations in \( S_n(1-3-2, 123) \) such that contain 231 exactly \( r \geq 0 \) times is given by

\[
(C_{r+1} - C_r) \left( \frac{n - 1}{2r + 1} \right) + C_r \left( \frac{n}{2r + 1} \right).
\]

where \( C_m \) is the \( m \)th Catalan number.

**Theorem 5.3.** The generating functions \( G_{213; 123}(x, y) \) and \( G_{312; 321}(x, y) \) are given by

\[
\frac{1 - x - x^2 + xy - \sqrt{(1 - x - x^2)^2 - 2xy(1 - x + x^2) + x^4y^2}}{2xy(1 - x)}.
\]

As a concluding remark we note that there are many questions left to answer such as: if there exists a bijection between, for example, the set of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the generalized pattern 21-3-\ldots-k, and the set of 1-3-2-avoiding permutations in \( S_n \) such that contain exactly \( r \) times the classical pattern 1-2-3-\ldots-k, where \( r = 0, 1, \ldots, k - 2 \).

**References**


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