Note

New permutation statistics: Variation and a variant

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\begin{abstract}
For each permutation \( \pi \) we introduce the variation statistic of \( \pi \), as the total number of elements on the right between each two adjacent elements of \( \pi \). We modify this new statistic to get a slightly different variant, which behaves more closely like Mahonian statistics such as \( \text{maj} \). In this paper we find an explicit formula for the generating function for the number of permutations of length \( n \) according to the variation statistic, and for that according to the modified version.
\end{abstract}

\section{1. Introduction}

Almost a century ago, MacMahon \cite{9} founded the theory of permutation statistics by studying the number of descents in a permutation. This statistic still plays an important role in the theory. The descent set (respectively, rise set) of a permutation \( \pi = \pi_1\pi_2 \cdots \pi_n \) is the set of indices \( i \) for which \( \pi_i > \pi_{i+1} \) (respectively, \( \pi_i < \pi_{i+1} \)), and the number of descents (respectively, rises) in a permutation \( \pi \) is the cardinality of the descent set (respectively, rise set). The distribution of the number of descents (rises) in a set of permutations of length \( n \) is given by the Eulerian numbers \( A(n, k) \). More precisely, the number of permutations of length \( n \) with exactly \( k \) descents (rises) is given by the Eulerian number \( A(n, k) \); see \cite{6}. Moreover, the sum of the elements in the descent set, i.e. \( \sum_{i<j<\pi_{i+1}} i \), denoted by \( \text{maj}(\pi) \) is defined to be the major statistic. The major statistic has a nice distribution over the symmetric group \( S_n \):

\[ \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = (1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1}). \]

We call permutation statistics with the same distribution Mahonian.

In this paper we consider a refinement of the notions of descents and rises by considering a new statistic on the set of permutations. This statistic can be defined as follows. For any sequence \( a = a_1a_2 \cdots a_n \), the reduced form of \( a \) is defined to be a word \( b = b_1b_2 \cdots b_n \) such that

- \( a_i < a_j \) if and only if \( b_i < b_j \),
- there exists an \( \ell \) such that \( \{b_1, b_2, \ldots, b_{\ell}\} = \{1, 2, \ldots, \ell\} \).

Let \( \pi = \pi_1\pi_2 \cdots \pi_n \) be any permutation of length \( n \); the \( i \)-th suffix of \( \pi \) is defined by the reduced form of \( \pi_i\pi_{i+1} \cdots \pi_n \) and denoted by \( \text{suffix}(\pi; i) \). Clearly, \( \text{suffix}(\pi; i) \) is a permutation of length \( n - i + 1 \) for any \( \pi \) and \( i \).

\begin{itemize}
\item \( a_i < a_j \) if and only if \( b_i < b_j \),
\item there exists an \( \ell \) such that \( \{b_1, b_2, \ldots, b_{\ell}\} = \{1, 2, \ldots, \ell\} \).
\end{itemize}
We define the variation statistic on the set of permutations as

\[ v(\pi) = \sum_{i=1}^{n-1} (|\text{suffix}(\pi; i)_2 - \text{suffix}(\pi; i)_1| - 1). \]

See Table 1. In other words, the variation of \( \pi \) is the sum of the number of elements on the right between each two adjacent elements of \( \pi \). For example, the permutations 123, 132, 213, 231, 312 and 321 have the variations 0, 1, 0, 1, 0 and 0, respectively. One of our aims is to find an explicit formula for the generating function for the number of permutations of length \( n \) according to the variation statistic. This is achieved by Theorem 2.5.

Our statistic can be formulated in terms of counting occurrences of generalized patterns of type \( ab - c \). We say that a permutation \( \pi_1 \pi_2 \cdots \pi_n \in S_n \) has an occurrence of the pattern \( ab - c \), if there exist \( i, j \) such that \( i + 1 < j \) and the reduced form of \( \pi_i \pi_{i+1} \pi_j \) is \( abc \) (similarly we may define an occurrence of pattern \( a - bc \)). For example, the pattern 41523 \( \in S_5 \) has three occurrences of the pattern 31 − 2, namely 412, 413 and 523. Several papers deal with the enumeration of the number of permutations with fixed number of occurrences of a generalized pattern \( ab - c \) (for instance, see [2–4]). By definition, the variation statistic can be characterized in terms of counting occurrences of generalized patterns 31 − 2 and 13 − 2 as follows:

\[ v(\pi) = \text{occ}_{31-2}(\pi) + \text{occ}_{13-2}(\pi), \]

where \( \text{occ}_\tau(\pi) \) is the number of occurrences of the pattern \( \tau \) in \( \pi \). Finding the generating function \( \sum_{n=0}^{\infty} \sum_{\pi \in S_n} x^n q^{\text{occ}_{ab-c}(\pi)} \) with \( abc \in S_3 \) is a well-known hard problem in the area of permutation patterns. In fact so far only two cases are known. By a result of Clarke, Steingrimsson and Zeng [5, Corollary 11], we have that

\[ \sum_{n=0}^{\infty} \sum_{\pi \in S_n} x^n q^{\text{occ}_{31-2}(\pi) + \text{occ}_{31-2}(\pi)} = \frac{1}{1 - x - \frac{2q}{1 - \frac{2q}{1 - \frac{3q}{1 - \frac{4q}{1 - \frac{5q}{1 - \frac{6q}{1 - \frac{7q}{1 - \frac{8q}{\ldots}}}}}}}}}. \]

and

\[ \sum_{n=0}^{\infty} \sum_{\pi \in S_n} x^n q^{\text{occ}_{13-2}(\pi)} = \frac{1}{1 - x - \frac{2q + 1 + \text{occ}_{31-2}(\pi)}{1 - \frac{2q}{1 - \frac{3q}{1 - \frac{4q}{1 - \frac{5q}{1 - \frac{6q}{1 - \frac{7q}{1 - \frac{8q}{\ldots}}}}}}}}}. \]

In this paper we are interested in finding a formula for the generating function

\[ \sum_{n=0}^{\infty} \sum_{\pi \in S_n} x^n q^{v(\pi)} = \sum_{n=0}^{\infty} \sum_{\pi \in S_n} x^n q^{\text{occ}_{31-2}(\pi) + \text{occ}_{31-2}(\pi)}, \]

which we think is combinatorially meaningful and methodologically interesting.

Nonetheless, the variation has a defect: its maximum value is \((n-1)(n-2)/2 \) instead of \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \). Accordingly, it will be convenient to define the modified variation statistic \( v' \) by letting \( v'(\pi) = v(\pi) + \pi_1 - 1 \), where \( \pi_1 \) is the initial element of \( \pi \). See Table 2. In Corollary 2.4 we shall present an explicit formula for the generating function for the number of permutations of length \( n \) according to the modified variation statistic.

2. Main theorems and proofs

In this section we present our major theorems together with proofs. First we need the following notation. Let \( a_{n, m} \) be the number of permutations of length \( n \) having exactly variation \( m \). More generally, let \( a_{n, i_1, i_2, \ldots, i_k} \) be the number of permutations \( \pi = \pi_1 \pi_2 \cdots \pi_n \) of fixed length \( n \) such that \( v(\pi) = m \) and \( \pi_1 \pi_2 \cdots \pi_k = i_1 i_2 \cdots i_k \). The corresponding polynomials are defined as
Table 2
Number of permutations of length \( n \) according to the statistic \( \nu' \).

<table>
<thead>
<tr>
<th>( \nu' )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>20</td>
<td>50</td>
<td>105</td>
<td>196</td>
<td>336</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5</td>
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<td>70</td>
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<td>546</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>26</td>
<td>105</td>
<td>350</td>
<td>1050</td>
</tr>
</tbody>
</table>

\[
F_n(q) = \sum_{m \geq 0} a_{n,m} x^n q^m \quad \text{and} \quad F_n(q|i_1, i_2, \ldots, i_k) = \sum_{m \geq 0} a_{n.m}(i_1, i_2, \ldots, i_k)x^n q^m.
\]

First, let us find a recurrence relation for the polynomials \( F_n(q|i) \).

**Lemma 2.1.** For all \( 1 \leq i \leq n \),

\[
F_n(q|i) = q^{-2} F_{n-1}(q|1) + \cdots + q^0 F_{n-1}(q|i-1) + q^0 F_{n-1}(q|i) + \cdots + q^{n-1-i} F_{n-1}(q|n-1).
\]

**Proof.** Since the variation statistic is defined via the reduced form of a permutation, we obtain that \( F_n(q|ij) = q^{i-j-1} F_{n-1}(q|ji) \), where \( j' = j \) if \( i > j \) and \( j' = j - 1 \) otherwise. Also, by definition we have that

\[
F_n(q|i) = \sum_{j=1}^{i-1} F_n(q|ij) + \sum_{j=i+1}^{n} F_n(q|ij).
\]

Hence,

\[
F_n(q|i) = \sum_{j=1}^{i-1} q^{i-j} F_{n-1}(q|j) + \sum_{j=i+1}^{n} q^{n-1-j} F_{n-1}(q|j - 1),
\]

as required. \( \square \)

Define \( F_n(q; t) = \sum_{i=1}^{n} F_n(q|i) t^{i-1} \). Then Lemma 2.1 implies that

\[
F_n(q; t) = \sum_{i=1}^{n} t^{i-1} \left( \sum_{j=1}^{i-1} q^{i-j} F_{n-1}(q|j) + \sum_{j=i+1}^{n} q^{n-1} F_{n-1}(q|j) \right)
\]

\[
= \sum_{j=1}^{n-1} \frac{t^j - t^j q^{n-j}}{1-tq} F_{n-1}(q|j) + \sum_{j=1}^{n-1} \frac{t^j - q^j}{t-q} F_{n-1}(q|j)
\]

\[
= \frac{1}{1-tq} (F_{n-1}(q; t) - (tq)^{n-1} F_{n-1}(q; 1/t)) + \frac{1}{t-q} (tF_{n-1}(q; t) - qF_{n-1}(q; q)).
\]

for all \( n \geq 1 \). From the definition we have \( F_0(q; t) = 1 \).

Let \( F(x, q; t) \) be the generating function for the number of permutations of length \( n \) according to the variation statistic and the initial element, i.e.,

\[
F(x, q; t) = \sum_{n \geq 0} F_n(q; t)x^n.
\]

Therefore, rewriting the above recurrence relation in terms of generating functions we arrive at

\[
F(x, q; t) = 1 + \frac{xt}{1-tq} (F(x, q; t) - F(xtq, q; 1/t)) + \frac{xt}{t-q} (F(x, q; t) - \frac{xq}{t-q} F(x, q; q)). \quad (2.1)
\]

In order to solve this functional equation we use a symmetric tool on our problem as follows.

**Lemma 2.2.** We have

\[
F(x, q; t) - 1 = \frac{1}{t} (F(xt, q; 1/t) - 1).
\]

**Proof.** Note that the complement symmetry operation,

\[
c : \pi = \pi_1 \cdots \pi_n \rightarrow (n+1-\pi_1) \cdots (n+1-\pi_n),
\]
on the set of permutations $S_n$ satisfies $c(\pi) = n + 1 - \pi_1$ and $v(\pi) = v(c(\pi))$. Thus

$$\frac{1}{t}(F(x, q; 1/t) - 1) = \frac{1}{t} \sum_{n \geq 1} \sum_{\pi \in S_n} q^{v(\pi)} (1/t)^{\pi_1 - 1} (xt)^n$$

$$= \sum_{n \geq 1} \sum_{\pi \in S_n} q^{v(c(\pi))} t^{n-\pi_1} x^n$$

$$= \sum_{n \geq 1} \sum_{\pi \in S_n} q^{v(\pi')} t^{\pi_1} x^n$$

$$= F(x, q; t) - 1,$$

as claimed. □

**Lemma 2.2** implies that $F(xtq, q; 1/q) = 1 - q + qF(x, q; q)$. Hence, Eq. (2.1) can be written as

$$\left(1 - \frac{xt}{1 - tq} - \frac{xt}{t - q}\right) F(x, q; t) = 1 - \frac{xt(1 - q)}{1 - tq} - \frac{xtq}{1 - tq} F(x, q; q) - \frac{xq}{t - q} F(x, q; q).$$

(2.2)

This type of functional equation, namely (2.2), can be solved systematically using the kernel method (see [1,8]). In this case, if we assume that there is a generating function $t_0 = t_0(x, q)$ with $1 - \frac{xq_0}{1 - t_0q} - \frac{xq_0}{t_0 - q} = 0$ then we obtain

$$F(x, q; q) = \frac{1 - t_0q - xt_0 + xt_0q(t_0 - q)}{xq(1 - t_0q)} - \frac{t_0(t_0 - q)}{1 - t_0q} F(xt_0, q; q)$$

(2.3)

with

$$t_0 = \frac{q}{1 + q^2 - x(1 - q)} C \left( \frac{q(q + x(1 - q))}{(1 + q^2 - x(1 - q))^2} \right),$$

where $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ is the generating function for the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$. Define

$$P(x, q) = (1/x + 1/q - 1)t_0(x, q) - q/x \quad \text{and} \quad Q(x, q) = \frac{1}{x^2q^2} (q + (x - 1)t_0(x, q)).$$

**Theorem 2.3.** We have

$$F(x, q; q) = \sum_{j \geq 0} q^{2j} P(u_j(x, q), q) \prod_{i=0}^{j-1} (u_i(x, q)Q(u_i(x, q), q)),$$

where $u_0(x, q) = x$ and $u_j(x, q) = u_{j-1}(xt_0(x, q), q)$ for $j \geq 1$.

**Proof.** Note that $P(x, q)$ and $Q(x, q)$ are two power series functions of $x$ such that $\lim_{x \to 0} P(x, q) = 1$ and $\lim_{x \to 0} Q(x, q) = \frac{x^2}{1 - x}$, and they are two power series functions of $q$ such that $\lim_{q \to 0} P(x, q) = \frac{1}{1-x}$ and $\lim_{q \to 0} Q(x, q) = \frac{1}{1-x^2}$. Hence, rewriting (2.3) we obtain that

$$F(x, q; q) = P(x, q) + xq^2Q(x, q)F(xt_0(x, q), q),$$

where $P(x, q)$ and $Q(x, q)$ are two power series. Using the above equation an infinite number of times we get the desired result. □

Note that by definition, $F(x, q; q)$ is just the generating function

$$G(x, q) \triangleq \sum_{n \geq 0} x^n \sum_{\pi \in S_n} q^{v'(\pi)}$$

for the number of permutations $\pi$ of length $n$ regarding the modified variation statistic $v'!$. The modified statistic has the advantage that its maximum value is $\binom{n}{2}$, uniquely evidenced by $\pi = (n1n - 12\ldots)$, as compared to the maximum value of $\binom{n-1}{2}$ for the ordinary variation statistic $v$. Hence, there is potentially more to investigate in $v'$ (although it had different distribution from inv and maj). To summarize, Theorem 2.3 has the following corollary:

**Corollary 2.4.** The generating function $G(x, q) = \sum_{n \geq 0} x^n \sum_{\pi \in S_n} q^{v'(\pi)}$ for the number of permutations $\pi$ of length $n$ with respect to the modified variation statistic $v'(\pi)$ is given by

$$\sum_{j \geq 0} q^{2j} P(u_j(x, q), q) \prod_{i=0}^{j-1} (u_i(x, q)Q(u_i(x, q), q)),$$

where $u_0(x, q) = x$ and $u_j(x, q) = u_{j-1}(xt_0(x, q))$ for $j \geq 1$. 

Now, (2.2) with the substitution $t = 1$ gives
\[
\left(1 - \frac{2x}{1 - q}\right) F(x, q; 1) = 1 - x - \frac{2xq}{1 - q} F(x, q; q),
\]
which is equivalent to
\[
F(x, q; 1) = \frac{(1 - q)(1 - x) - 2xq}{1 - q - 2x} F(x, q; q).
\]

Hence, the next theorem follows from the above equation together with Theorem 2.3.

**Theorem 2.5.** The generating function $F(x, q) = \sum_{n \geq 0} x^n \sum_{\pi \in S_n} q^{v(\pi)}$ for the number of permutations $\pi$ of length $n$ with respect to the variation statistic $v(\pi)$ is given by
\[
\frac{(1 - q)(1 - x) - 2xq}{1 - q - 2x} \sum_{j \geq 0} q^j p(u_j(x, q), q) \prod_{i=0}^{j-1} (u_i(x, q) Q(u_i(x, q), q)),
\]
where $u_0(x, q) = x$ and $u_j(x, q) = \underbrace{u_{j-1}(x, q) \cdots u_0(x, q)}$ for $j \geq 1$.

**Remark 2.6.** It is possible to derive Corollary 2.4 directly. Let $b_{n,m}$ be the number of permutations of length $n$ such that $v'(\pi) = m$; let $a_{n,m}(i_1, i_2, \ldots, i_k)$ be the number of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ of length $n$ such that $v'(\pi) = m$ and $\pi_1 \pi_2 \cdots \pi_k = i_1 i_2 \cdots i_k$. The corresponding polynomials are defined as
\[
G_n(q) = \sum_{m \geq 0} a_{n,m} x^n q^m \quad \text{and} \quad G_n(q|i_1, i_2, \ldots, i_k) = \sum_{m \geq 0} a_{n,m}(i_1, i_2, \ldots, i_k) x^n q^m.
\]

Then it is easy to find a recurrence relation for the polynomials $G_n(q|i)$ and proceed similarly from there:
\[
G_n(q|i) = \sum_{j=1}^{i-1} q^{2(i-j)-1} G_{n-1}(q|j) + \sum_{j=i}^{n-1} G_{n-1}(q|j).
\]

To summarize, we have obtained two major results. The first, Theorem 2.3 (or Corollary 2.4), presents the generating function for the number of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ of length $n$ according to the modified variation statistic $v'(\pi) = v(\pi) + \pi_1 - 1$; and the second, Theorem 2.5, gives the generating function for the number of permutations of length $n$ according to the standard variation statistic $v(\pi)$.

Programming our theorems in either Maple or Mathematica, the following results are derived.

**Corollary 2.7.** Let $F_m(x)$ and $F'_m(x)$ be the generating functions for the number of permutations of length $n$ having $v(\pi) = m$ and $v'(\pi) = m$, respectively. Then:
1. $F_0(x) = \frac{1-x}{1-2x}, F'_0(x) = \frac{1}{1-x}$;
2. $F_1(x) = \frac{2x^2}{(1-x)(1-2x^2)}, F'_1(x) = \frac{x^2}{1-x^2}$;
3. $F_2(x) = \frac{2x^2(2x-3)}{(1-x)^2(1-2x^2)}, F'_2(x) = \frac{x^2(1+x)}{1-x}$;
4. $F_3(x) = \frac{2x^6(1-2x+2x^2-7x^3+8x^4)}{(1-x)^3(1-2x^2)}, F'_3(x) = \frac{x^6(6x^2-2x+1)}{(1-x)^3}$.

**Remark 2.8.** In [7], a coding version of the maj statistic is introduced. There, each position of the code is obtained from an operation comparable to our variation statistic. The essential difference is that our $v$ statistic does not care about the order in each pair: we only need the number of integers between $a$ and $b$ to the right of the adjacent pair, either $(ab)$ or $(ba)$, and $(ab)$ and $(ba)$ generate the same result; while in [7] the order does matter.

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**References**