A general two-term recurrence and its solution

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Abstract
We find a general explicit formula for all sequences satisfying a two-term recurrence of a certain kind. This generalizes familiar formulas for the Stirling and Lah numbers.

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1 Introduction
In what follows, $\mathbb{N}$ and $\mathbb{P}$ will denote the non-negative and positive integers, respectively. Recall that the Stirling numbers of the second kind $S(n,k)$ count the number of partitions of an $n$-set having $k$ blocks (see, e.g., [9]) and are given by the two-term recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad \forall \ n,k \in \mathbb{P}, \quad (1)$$

with the boundary conditions $S(n,0) = \delta_{n,0}$ and $S(0,k) = \delta_{0,k}$ for all $n,k \in \mathbb{N}$. The Lah numbers $L(n,k)$ count the partitions of an $n$-set having $k$-blocks in which the elements within each block are ordered (see, e.g., [8]) and are given by the recurrence

$$L(n,k) = L(n-1,k-1) + (n+k-1)L(n-1,k), \quad \forall \ n,k \in \mathbb{P}, \quad (2)$$

with the same boundary conditions. The $S(n,k)$ and the $L(n,k)$ have the respective explicit formulas

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n, \quad \forall \ n,k \in \mathbb{N}, \quad (3)$$

and

$$L(n,k) = \frac{n!}{k!} \binom{n-1}{k-1}, \quad \forall \ n,k \in \mathbb{P}. \quad (4)$$

See, for example, (6.19) in [6] and (6.3) in [11], respectively.

Here we consider a class of sequences satisfying a two-term recurrence which generalizes both (1) and (2) and find an explicit formula for all the sequences in the class which generalizes (3) and (4). Our formula would apply to other combinatorial sequences occurring in the literature, including $q$-generalizations of the Stirling numbers. Our main result is as follows.
Theorem 1.1. Suppose \((a_i)_{i \geq 0}\) and \((b_i)_{i \geq 0}\) are sequences of complex numbers where the \(b_i\) are distinct. Let \(\{u(n,k)\}_{n,k \geq 0}\) be the array defined by the recurrence
\[
u(n,k) = u(n - 1, k - 1) + (a_{n-1} + b_k)u(n - 1, k), \quad \forall \ n, k \in \mathbb{P},
\] (5)
subject to the boundary conditions \(u(n, 0) = \prod_{i=0}^{n-1}(a_i + b_0)\) and \(u(0,k) = \delta_{0,k}\) for all \(n, k \in \mathbb{N}\). Then we have
\[
u(n,k) = \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}(b_j + a_i)}{\prod_{i \neq j}^{k}(b_j - b_i)} \right), \quad \forall \ n, k \in \mathbb{N}.
\] (6)
Using (5) and the condition \(u(0,k) = \delta_{0,k}\), one can show that \(u(n,k) = 0\) if \(k > n \geq 0\). Note that taking \(a_i = 0\) and \(b_i = i\) for all \(i\) in (5) yields the Stirling numbers, while taking \(a_i = b_i = i\) for all \(i\) yields the Lah numbers. Theorem 1.1 will follow from combining the second and third lemmas of the next section.

2 Proof of the main result

To solve recurrence (5), we first introduce two sets of polynomials as follows. Suppose \((a_i)_{i \geq 0}\) and \((b_i)_{i \geq 0}\) are given sequences of complex numbers. Define the polynomial sequences \(\{\rho_n(x)\}_{n \geq 0}\) and \(\{\varphi_n(x)\}_{n \geq 0}\) by
\[
\rho_n(x) = \prod_{i=0}^{n-1}(x + a_i), \quad n \geq 1,
\] (7)
and
\[
\varphi_n(x) = \prod_{i=0}^{n-1}(x - b_i), \quad n \geq 1,
\] (8)
with \(\rho_0(x) = \varphi_0(x) = 1\). One might call the \(\rho_n(x)\) and the \(\varphi_n(x)\), respectively, the rising factorial polynomials with respect to \((a_i)_{i \geq 0}\) and the falling factorial polynomials with respect to \((b_i)_{i \geq 0}\). In particular, when \(a_i = b_i = i\) for all \(i\), we get the usual rising and falling factorial polynomials \(\rho_n(x) = x^n\) and \(\varphi_n(x) = x^n\). Note that both \(\{\rho_n(x)\}_{n \geq 0}\) and \(\{\varphi_n(x)\}_{n \geq 0}\) are bases for the vector space \(\mathbb{C}(x)\) since both sets contain a single polynomial of each degree. Define the quantity \((b_j)_k\) by
\[
(b_j)_k := \prod_{i=0}^{k}(b_j - b_i),
\]
if \(0 \leq j \leq k\) and \(k \geq 1\), with \((b_0)_0 := 1\).

Lemma 2.1. Suppose \(P(x) \in \mathbb{C}(x)\) has degree \(n\). Then for all \(k > n\), we have
\[
\sum_{j=0}^{k} \frac{P(b_j)}{(b_j)_k} = 0.
\] (9)

Proof. To show (9), it suffices to show
\[
\sum_{j=0}^{k} \frac{b_j^r}{(b_j)_k} = 0, \quad 0 \leq r \leq n,
\] (10)
since \(P(x)\) is a polynomial. Let us define the polynomial
\[
g_r(x) = f_r(x) - x^r, \quad 0 \leq r \leq n,
\] (11)
where

\[ f_r(x) = \sum_{j=0}^{k} b_j^r \lambda_j(x) \]

and

\[ \lambda_j(x) = \frac{(x - b_0) \cdots (x - b_j)}{(x - b_j)(b_j)_n}, \quad 0 \leq j \leq k. \]

For each \( r \), note that \( g_r(b_j) = 0 \) for \( 0 \leq j \leq k \), with \( \deg(g_r) \leq k \) since \( k > n \). Since the \( b_j \)'s are distinct, we have \( g_r(x) = 0 \) and thus

\[ f_r(x) = x^r, \quad 0 \leq r \leq n, \quad (12) \]

when \( k > n \). On the other hand, the coefficient of \( x^k \) in \( f_r(x) \) is given by \( \sum_{j=0}^{k} \frac{b_j^r}{(b_j)_n} \), which must be zero by (12) since \( k > n \geq r \). \( \square \)

The following lemma shows how to express polynomials in \( \mathbb{C}(x) \) as linear combinations of the \( \varphi_k(x) \).

**Lemma 2.2.** Suppose that the terms of the sequence \((b_i)_{i \geq 0}\) are distinct. Then for all polynomials \( P(x) \in \mathbb{C}(x) \), we have

\[ P(x) = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \frac{P(b_j)}{(b_j)_n} \right) \varphi_k(x). \quad (13) \]

**Proof.** Let \( n = \deg(P) \). By Lemma 2.1 above, equality (13) is equivalent to

\[ P(x) = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \frac{P(b_j)}{(b_j)_n} \right) \varphi_k(x). \quad (14) \]

To show (14), we induct on \( n \). If \( n = 0 \), then \( P(x) = c \neq 0 \) and \( \frac{P(b_0)}{(b_0)_n} \varphi_0(x) = c \cdot 1 = c \). So suppose \( \deg(P) = n \geq 1 \) and that (14) holds for all polynomials of degree less than \( n \). By Lagrange interpolation, we may write

\[ P(x) = \sum_{j=0}^{n} P(b_j) \Lambda_j(x), \quad (15) \]

where

\[ \Lambda_j(x) = \frac{(x - b_0) \cdots (x - b_n)}{(x - b_j)(b_j)_n}. \]

Define the polynomials \( \psi_j(x) \) by

\[ \psi_j(x) = \Lambda_j(x) - \frac{\varphi_n(x)}{(b_j)_n}, \quad 0 \leq j \leq n - 1. \quad (16) \]

Note that each \( \psi_j(x) \) has degree \( n - 1 \) since it equals

\[ \frac{(x - b_0) \cdots (x - b_n)}{(x - b_j)(b_j)_n} - \frac{(x - b_0) \cdots (x - b_{n-1})}{(b_j)_n}, \]

with the \( b_i \) distinct. We may rewrite (15) as

\[ P(x) = \sum_{j=0}^{n-1} P(b_j) \left( \frac{\varphi_n(x)}{(b_j)_n} + \psi_j(x) \right) + P(b_n) \frac{\varphi_n(x)}{(b_n)_n} \quad (17) \]

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since
\[ \Lambda_n(x) = \frac{(x - b_0) \cdots (x - b_n)}{(x - b_n)(b_n)_n} = \frac{\varphi_n(x)}{(b_n)_n}. \]

We now express the \( \psi_j(x) \) occurring in (17) in a more convenient form. Applying the induction hypothesis to each \( \psi_j(x) \), we may write
\[
\psi_j(x) = \sum_{k=j}^{n-1} \left( \sum_{i=0}^{k} \psi_i(x) \right) \varphi_k(x), \quad 0 \leq j \leq n - 1. \tag{18}
\]

However, since \( \varphi_n(b_i) = 0 \) if \( i < n \), we have
\[
\psi_j(b_i) = \Lambda_j(b_i) - \frac{\varphi_n(b_i)}{(b_j)_n} = \Lambda_j(b_i) = \delta_j,i,
\]
so that (18) may be rewritten as
\[
\psi_j(x) = \sum_{k=j}^{n-1} \frac{\varphi_k(x)}{(b_j)_k}, \quad 0 \leq j \leq n - 1. \tag{19}
\]

Substituting (19) into (17), we get
\[
P(x) = \sum_{j=0}^{n-1} P(b_j) \left( \frac{\varphi_n(x)}{(b_j)_n} + \sum_{k=j}^{n-1} \frac{\varphi_k(x)}{(b_j)_k} \right) + P(b_n) \frac{\varphi_n(x)}{(b_n)_n}
\]
\[
= \sum_{j=0}^{n-1} P(b_j) \sum_{k=j}^{n} \frac{\varphi_k(x)}{(b_j)_k} + P(b_n) \frac{\varphi_n(x)}{(b_n)_n}
\]
\[
= \sum_{j=0}^{n} P(b_j) \sum_{k=j}^{n} \frac{\varphi_k(x)}{(b_j)_k} = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} P(b_j) \frac{(b_j)_k}{(b_j)_k} \right) \varphi_k(x),
\]
which completes the induction and proves (14), as required.

The next lemma provides an equivalent way of describing the array \( u(n, k) \).

**Lemma 2.3.** Given sequences \( (a_i)_{i \geq 0} \) and \( (b_i)_{i \geq 0} \) in \( \mathbb{C} \), the following conditions are equivalent ways to characterize the array \( \{u(n, k)\}_{n,k \geq 0} \):

(i) \( u(n, k) = u(n-1, k-1) + (a_{n-1} + b_k)u(n-1, k) \), \quad \forall n, k \in \mathbb{N},

subject to \( u(0, k) = \delta_{0,k} \) and \( u(n, 0) = \prod_{i=0}^{n-1} (a_i + b_0) \) for all \( n, k \in \mathbb{N} \);

(ii) \( \rho_n(x) = \sum_{k \geq 0} u(n, k) \varphi_k(x), \quad \forall n, k \in \mathbb{N} \).

**Proof.** To show (i) implies (ii), we induct on \( n \), the \( n = 0 \) case clear since
\[ \rho_0(x) = 1 = \sum_{k \geq 0} u(0, k) \varphi_k(x). \]
If \( n \geq 1 \), then we may write
\[
\rho_n(x) = \rho_{n-1}(x)(x + a_{n-1}) = \sum_{k \geq 0} u(n-1,k)\varphi_k(x)(x - b_k + a_{n-1} + b_k)
\]
\[
= \sum_{k \geq 0} u(n-1,k)\varphi_k(x)(x - b_k) + \sum_{k \geq 0} (a_{n-1} + b_k)u(n-1,k)\varphi_k(x)
\]
\[
= \sum_{k \geq 1} u(n-1,k-1)\varphi_k(x) + \sum_{k \geq 1} (a_{n-1} + b_k)u(n-1,k)\varphi_k(x) + u(n-1,0)(a_{n-1} + b_0)
\]
\[
= \sum_{k \geq 1} u(n,k)\varphi_k(x) + u(n,0) = \sum_{k \geq 0} u(n,k)\varphi_k(x),
\]
which completes the induction. For the converse, first note that taking \( u(0,k) = \delta_{0,k} \) for all \( k \in \mathbb{N} \). If \( n \geq 1 \), then, from the preceding, we may write
\[
\sum_{k \geq 1} u(n,k)\varphi_k(x) + u(n,0) = \rho_n(x)
\]
\[
= \sum_{k \geq 1} [u(n-1,k-1) + (a_{n-1} + b_k)u(n-1,k)]\varphi_k(x)
\]
\[
+ u(n-1,0)(a_{n-1} + b_0),
\]
which implies the recurrence and the other boundary condition, upon equating coefficients of \( \varphi_k(x) \) and noting \( u(0,0) = 1 \).

Our main result now follows quickly from the preceding two lemmas.

**Proof of Theorem 1.1:**

Suppose \( u(n,k) \) is defined by (5) above, with the given boundary conditions. By Lemmas 2.2 and 2.3 (where we take \( P(x) = \rho_n(x) \) in the former), we have
\[
\sum_{k \geq 0} u(n,k)\varphi_k(x) = \rho_n(x) = \sum_{k \geq 0} \left( \sum_{j=0}^{k} \frac{\rho_n(b_j)}{(b_j)_k} \right) \varphi_k(x),
\]
for all \( n \geq 0 \), from which (6) follows from comparing coefficients of \( \varphi_k(x) \).

### 3 Further remarks

We consider some particular cases of Theorem 1.1. We invite the reader to find other examples occurring in the literature as well as cases involving new sequences satisfying a recurrence having the form in (5).

We first note that the case \( a_i = 0 \) for all \( i \), with the \( b_i \) arbitrary but distinct, has been considered previously by Bach [1] and Comtet [3], where a different expression for the solution to (5) is given in this case. Here, one has \( \rho_n(x) = x^n \) and thus the \( u(n,k) \) are connection constants in \( x^n = \sum_{k \geq 0} u(n,k)\varphi_k(x) \), by (ii) in Lemma 2.3. This can be found already as (2) in Comtet [3] (see also (1.19) in [11]).

We now look at several specific cases. First, taking \( a_i = b_i = i \) for all \( i \) in (5), and observing the boundary conditions, we obtain the Lah numbers \( L(n,k) \) mentioned above. By (6), they may then be given explicitly as
\[
L(n,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}j^j(j+1)\cdots(j+n-1)}{(k-j)!j!} = \frac{n!}{k!} \sum_{j=0}^{n} (-1)^{k-j} \binom{j+n-1}{j} \binom{k}{j}.
\]
The last expression reduces, via identity (5.24) in [6], to the one in (4) above.

As another example, letting $a_i = 0$ and $b_i = i_q$ for all $i$ in (5) yields the $q$-Stirling numbers first considered by Carlitz [2] and later studied [7, 10]. Theorem 1.1 then yields a new proof of the explicit formula

$$S_q(n, k) = \frac{1}{q^k k_q!} \sum_{j=0}^{k} (-1)^i q^{i(j)} \binom{k}{j} [(k-j)_q]_n$$

given in [2]. Note that this expression reduces when $q = 1$ to (3) above.

Next, we consider the polynomials $c_q(n, k)$ defined by the recurrence

$$c_q(n, k) = c_q(n-1, k-1) + (n-q^k)c_q(n-1, k), \quad \forall \ n, k \in \mathbb{P},$$

with $c_q(n, 0) = \delta_{n,0}$ and $c_q(0, k) = \delta_{0,k}$, which were introduced by Cigler [5] who derived several properties. Note that the $c_q(n, k)$ reduce to the usual cycle numbers $c(n, k)$ when $q = 1$. Using (6) with $a_i = i + 1$ and $b_i = -q^i$ for all $i$, one obtains the following additional result:

$$c_q(n, k) = \frac{1}{q^k (1-q^k) k_q!} \sum_{j=0}^{k} (-1)^i q^{i(j)} \binom{k}{j} \prod_{q=i}^{n-1} (i + 1 - q^{k-j}), \quad \forall \ n, k \in \mathbb{N}.$$ 

Furthermore, taking $a_i = q^i$ and $b_i = -1$ for all $i$ yields a similar formula for the $q$-analogue of $S(n, k)$ introduced by Cigler in the same paper.

On the other hand, if the $b_i$ are not all distinct in the recurrence (5) above, then it is unclear if a general formula can be found. In the case when $b_i = 0$ for all $i$, then $u(n, k)$ is given by the $(n-k)$-th symmetric function on the set $\{a_0, a_1, \ldots, a_{n-1}\}$. This may be seen directly from (5), the first term accounting for those products not including $a_{n-1}$, the second term for those that do. Furthermore, choosing $a_i = -i$ for all $i$ gives the well known representation of the Stirling number $s(n, k)$ of the first kind as a symmetric function on $\{1, 2, \ldots, n-1\}$ (see, e.g., p. 213 of [4]).

Finally, the recurrence for $u(n, k)$ may be iterated to yield the equivalent form

$$u(n, k) = \sum_{j=k}^{n} u(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k), \quad \forall \ n, k \in \mathbb{P},$$

subject to the same boundary conditions. Taking $a_i = 0$ and $b_i = i$ for all $i$ in (20) yields, for example, recurrence (6.20) in [6] for $S(n, k)$.

References


