Minimal permutations with $d$ descents

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Abstract. Recently, Bouvel and Pergola initiated the study of a special class of permutations, minimal permutations with a given number of descents, which arise from the whole genome duplication-random loss model of genome rearrangement. In this paper, we show that the number of minimal permutations of length $2d - 1$ with $d$ descents is given by $2^{d-3}(d-1)c_d$, where $c_d$ is the $d$-th Catalan number. We also derive a recurrence relation on the generating functions for the number of minimal permutations $\pi$ of length $n$ with respect to the number of descents, and the values of the first and second elements of $\pi$. Furthermore, we show that given $d \geq 1$ there exists a constant $a_d$ such that the number of minimal permutations of length $n$ with $n - d$ descents is asymptotically equivalent to $a_dd^n$, as $n \to \infty$.

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1 Introduction

Bouvel and Pergola [3] initiated the study of a special class of permutations, minimal permutations with $d$ descents, arising from the combinatorial analysis of the whole genome duplication-random loss model of genome rearrangement. In the last decades, the genome rearrangement has been extensively studied in computational biology, see [2, 6, 12].

1.1 The tandem duplication - random loss model for genome rearrangement

In the tandem duplication - random loss model, or simply the duplication-loss model, a genome is considered as a permutation of the integers from 1 to $n$, see [5] for the original biological motivations. One step of tandem duplication - random loss, or simply duplication-loss consists in (i) a fragment of consecutive elements of the permutation is duplicated, and the duplicated fragment is inserted immediately after the original copy, and (ii) one copy of every duplicated element is lost. The width of a duplication-loss step is defined to be the number of elements duplicated. For instance, a duplication-loss step of width 3 is shown as follows:

$$12\overline{345}6 \leadsto 12\overline{345}\overline{345}6 \leadsto 12\overline{345}\overline{345}\overline{345}6 \leadsto 124356.$$
Various duplication-loss models can be defined depending on the cost function \( c \in \mathbb{R}_+^N \) that is chosen. Usually, the cost \( c(k) \) of a duplication-loss step is assumed to be dependent only on the width \( k \) of this step. For instance, Chaudhuri, Chen, Mihaescu and Rao [5] defined the cost of a duplication-loss step of width \( k \) as \( c(k) = \alpha^k \) with \( \alpha \geq 1 \). In [4], Bouvel and Rossin considered the cost function \( c(k) = 1 \) if \( k \leq K \), and \( c(k) = \infty \) otherwise, for a parameter \( K \in \mathbb{N} \cup \{ \infty \} \setminus \{0, 1\} \). In [3], Bouvel and Pergola studies the model with a very simple cost function, namely \( c(k) = 1 \), for all \( k \in \mathbb{N} \). This particular model is called the whole genome duplication - random loss model. The permutations obtainable form the identity permutation \( 12 \ldots n \) after a certain number \( p \) of duplication-loss steps in this model were characterized as follows:

**Theorem 1.1** (see [4, 5]) The permutations that can be obtained in at most \( p \) steps in the whole genome duplication - random loss model are exactly those whose number of descents is at most \( 2^p - 1 \).

### 1.2 Basic concepts and previous results

Let \( S_n \) denote the set of permutations of \( [n] = \{1, 2, \ldots, n\} \), also called permutations of length \( n \), written in one-line notation. We say that a subsequence of \( \pi \) has type \( \sigma \) whenever it has all of the same pairwise comparisons as \( \sigma \). For example, the subsequence 2968 of the permutation 214539768 \( \in S_9 \) has type 1423. We say that the permutation \( \pi \in S_n \) avoids \( \tau \in S_k \) if there is no subsequence of \( \pi \) that has type \( \tau \), otherwise we say that \( \pi \) contains \( \tau \). In this context, \( \tau \) is called a pattern. For example, the permutation 214539768 \( \in S_9 \) avoids 4321 and contains 2143. We write \( \tau \prec \pi \) to denote that \( \pi \) contains \( \tau \). We denote the class of all permutations (of all lengths including the empty permutation) avoiding the patterns \( \tau^1, \tau^2, \ldots, \tau^k \) by \( S(\tau^1, \tau^2, \ldots, \tau^k) \). We say that \( S(\tau^1, \tau^2, \ldots, \tau^k) \) is a class of pattern-avoiding permutations of basis \( \{\tau^1, \tau^2, \ldots, \tau^k\} \).

In a permutation \( \pi = \pi_1\pi_2\cdots\pi_n \in S_n \), a descent is a position \( i \) such that \( \pi_i > \pi_{i+1} \), and an ascent is a position \( i \) where \( \pi_i < \pi_{i+1} \). For example, the permutation 12537648 \( \in S_8 \) has three descents (see the positions 3, 5 and 6) and has four ascents (see the positions 1, 2, 4 and 7). A permutation \( \pi \) is a minimal permutation with \( d \) descents if \( \pi \) has exactly \( d \) descents and it is minimal in the sense of \( \prec \), i.e., there exists no permutation \( \sigma \) with exactly \( d \) descents such that \( \sigma \prec \pi \). Denote by \( B_d \) the set of minimal permutations with \( d \) descents. The length of a minimal permutation with \( d \) descents is at least \( d + 1 \) and at most \( 2d \) [3]. Thus, the set \( B_d \) is finite. When \( d = 2^p \), \( B_d \) is the basis of the class of permutations obtainable in at most \( p \) steps in the whole genome duplication - random loss model.

**Theorem 1.2** (for implicit proof see [4] and for explicit proof see [3]) The class of permutations obtainable in at most \( p \) steps in the whole genome duplication - random loss model is a class of pattern-avoiding permutations whose basis is finite and composed of the minimal permutations with \( 2^p \) descents.

In [3], Bouvel and Pergola gave a local characterization of the permutations of \( B_d \) as follows:

**Theorem 1.3** (see Theorem 4 in [3]) Let \( \pi = \pi_1\pi_2\cdots\pi_n \) be any permutation of length \( n \). Then \( \pi \) is a minimal permutation with \( d \) descents if and only if \( \pi \) is a permutation with \( d \) descents such that

- it starts and ends with a descent;
• if $\pi_i \pi_{i+1}$ is an ascent, then $i \in \{2, 3, \ldots, n - 2\}$ and $\pi_i \pi_{i+1} \pi_{i+2}$ has type either 2143 or 3142.

Let $f_d(n)$ be the number of minimal permutations of length $n$ with $d$ descents. Clearly, $f_d(n) = 0$ for all $d \leq 0$ or $n \leq d$, and $f_d(d + 1) = 1$ for all $d \geq 1$. Bouvel and Pergola [3] found that the number of minimal permutations with $d$ descents in $S_{2d}$ is given by the $d$-th Catalan number $c_d$, that is,

$$f_d(2d) = \frac{1}{d+1} \binom{2d}{d}, \quad d = 1, 2, \ldots.$$ 

Also, they showed that the number of minimal permutations with $d$ descents in $S_{d+2}$ is given by

$$f_d(d + 2) = 2^{d+2} - (d + 1)(d + 2) - 2, \quad d = 1, 2, \ldots.$$ 

1.3 Outline of our results

In this paper, we extend the above results. In Section 2, we aim at finding combinatorially a formula for the number of of minimal permutations of length $2d - 1$ with $d$ descents.

**Theorem 1.4** For $d \geq 1$, the number of minimal permutations of length $2d - 1$ with $d$ descents is equal to $2^{d-3}(d - 1)c_d$.

In Section 3, our goal is to find a recurrence relation on the generating functions for the number of minimal permutations $\pi$ of length $n$ with respect to the number of descents, and the values of the first and second elements of $\pi$. Denote by $f_d(n; i, j)$ the number of minimal permutations $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ with $d$ descents such that $\pi_1 = i$ and $\pi_2 = j$. Let

$$F(n; v, w, q) = \sum_{d \geq 1} \sum_{i=2}^{n} \sum_{j=1}^{i-1} f_d(n; i, j)v^{i-1}w^{j-1}q^d$$

be the generating function for the number $f_d(n; i, j)$. Since a minimal permutation of length $n$ with $d$ descents must have $\lceil \frac{n}{2} \rceil \leq d \leq n - 1$, the number $f_d(n; i, j) = 0$ if $d > n - 1$ or $d < \lceil \frac{n}{2} \rceil$. From this we can deduce that the generating function $F(n; v, w, q)$ is a finite polynomial on $v, w, q$.

**Theorem 1.5** The polynomial $F(n; v, w, q)$ on $v, w, q$ satisfies the following recurrence relation

$$F(n; v, w, q) = v^{n-1}qF(n - 1; w, 1, q) + \frac{vq}{1-vw}(F(n - 1; vw, 1, q) - v^{n-2}F(n - 1; w, 1, q))$$

$$+ vq(n - 2; 1, 1, q) - \frac{v^2wq}{(1-w)(1-vw)}(F(n - 2; 1, 1, q) - F(n - 2; 1, vw, q))$$

$$+ \frac{v^2wq}{(1-v)(1-w)}(F(n - 2; 1, 1, q) - wF(n - 2; 1, vw, q) - F(n - 2; v, 1, q) + wF(n - 2; v, w, q))$$

(1.1)

with the initial conditions $F(2; v, w, q) = vq$ and $F(3; v, w, q) = v^2wq^2$.

This recurrence relation derives the known formula for $f_d(2d)$.

In Section 4, we obtain a recurrence relation for the generating function $G_d(t, v, w)$ for minimal permutations $\pi$ of length $n$ with $n - d$ descents with respect to the length, and the values of the first and second elements of $\pi$. Using the recurrence relation, we get an explicit formula for the generating function $G_d(t, 1, 1)$ for $d = 2, 3, 4, 5$. Finally, based on the recurrence relation for $G_d(t, v, w)$, we find the asymptotic behavior for the number $f_{n-d}(n)$ as $n \to \infty$. 

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Theorem 1.6  Given \( d \geq 1 \), there exists a constant \( a_d \) such that the number of minimal permutations of length \( n \) with \( n - d \) descents is asymptotically equivalent to \( a_d n^d \), as \( n \to \infty \).

2  Minimal permutations of length \( 2d - 1 \) with \( d \) descents

The maximum length of a minimal permutation with \( d \) descents is \( 2d \). Bouvel and Pergola [3] proved that the number of minimal permutations of length \( 2d \) with \( d \) descents is equal to the Catalan number \( c_d \). In this section, we aim at getting an explicit formula for the number of minimal permutations of length \( 2d - 1 \) with \( d \) descents. The case for \( d = 1 \) is very trivial since there is no minimal permutation of length one with one descent. For our convenience, let \( d = n + 1 \) for \( d \geq 2 \) and we come to study minimal permutations of length \( 2n + 1 \) with \( n + 1 \) descents for \( n \geq 1 \).

Recall that a matching on a set \([2n] = \{1, 2, \ldots, 2n\}\) is a partition of \([2n]\) in which every block contains exactly two elements. In this paper, we identify a matching as a graph on the \( 2n \) points on a horizontal line in the increasing order in which every edge \((i, j), i < j\), is drawn as an arc between the nodes \( i \) and \( j \) above the horizontal line. Let \( e = (i, j) \) and \( e' = (i', j') \) be two edges of a matching \( M \), we say that \( e \) nests \( e' \) if \( i < i' < j' < j \). In this case, the pair of edges \( e \) and \( e' \) is called a nesting of the matching. Otherwise, \( e \) and \( e' \) are said to be nonnesting. For example, in a matching on the set \([6] = \{1, 2, 3, 4, 5, 6\}\) consisting of \((1, 4)(2, 6)(3, 5)\), the pair of edges \((2, 6)\) and \((3, 5)\) is a nesting, while the pair of edges \((1, 4)\) and \((2, 6)\) is nonnesting. A matching is said to be nonnesting if and only if there are no two edges that are nesting [10].

Let \( \pi = \pi_1 \pi_2 \ldots \pi_{2n+1} \) be a minimal permutation of length \( 2n + 1 \) with \( n + 1 \) descents. Then there are only two consecutive descents in \( \pi \) and the other descents are separated by ascents. Suppose that \( \pi_{p-1} \pi_p \) and \( \pi_p \pi_{p+1} \) are the consecutive descents. It is convenient to identify the minimal permutation \( \pi \) with a graph \( G(\pi) \). By convention, the \( 2n + 1 \) vertices are numbered from 1 to \( 2n + 1 \) and represented in this order on a horizontal line. In the graph, an edge connecting the nodes \( i \) and \( j \) is drawn as an arc above the line if \( ji \) is a descent left to \( \pi_p \) in \( \pi \) and as an arc below the line, if \( ji \) is a descent right to \( \pi_p \). We denote such an edge \( e \) by a pair \((i, j)\) with \( i < j \), and say that the node \( i \) is the left endpoint of \( e \) and the node \( j \) is the right endpoint of \( e \). An illustration is given in Figure 1, where the horizontal line is drawn by a dotted line and edges are drawn by solid lines. Denote by \( G^+ (\pi) \) (resp. \( G^- (\pi) \)) the subgraph of \( G(\pi) \) consisting of all the edges above (resp. below) the line. We claim that \( G(\pi) \) is a graph with \( n + 1 \) edges, where

(a). the last left endpoint of the subgraph \( G^+ (\pi) \) and the first right endpoint of the subgraph \( G^- (\pi) \) coincide;

(b). each node other than \( \pi_p \) has degree one, and the node \( \pi_p \) has degree two;

(c). both the subgraph \( G^+ (\pi) \) and the subgraph \( G^- (\pi) \) are nonnesting matchings.

Figure 1: The graph \( G(\pi) \) of \( \pi = 3174106285119 \)
Since there is only two consecutive descents in \( \pi \). Hence, if \( \pi_{i-1} \pi_i \) is a descent of \( \pi \) for \( i < 2n + 1 \)
and \( i \neq p \), then \( \pi_i \pi_{i+1} \) is an ascent. Recall that (see Theorem 1.3) if \( \pi \pi_{i+1} \) is an ascent in \( \pi \), then
\( \pi_{i-1} \pi_i \pi_{i+1} \pi_{i+2} \) has type either 3142 or 2143, which implies that \( \pi_{i+1} \pi_{i+2} \) is also a descent of \( \pi \), and
\( \pi_1 < \pi_{i+2} \) and \( \pi_{i-1} < \pi_{i+1} \). Therefore, the edges of \( G^+ (\pi) \) are \( (\pi_2, \pi_1), (\pi_4, \pi_3), \ldots, (\pi_p, \pi_{p-1}) \) such
that \( \pi_1 < \pi_3 < \ldots < \pi_{p-1} \) and \( \pi_2 < \pi_4 < \ldots < \pi_p \), which implies that \( G^+ (\pi) \) is a nonnesting
matching. Also, the edges of \( G^- (\pi) \) are \( (\pi_{p+1}, \pi_p), (\pi_{p+3}, \pi_{p+2}), \ldots, (\pi_{2n+1}, \pi_{2n}) \) such that \( \pi_p < \pi_{p+2} < \ldots < \pi_{2n} \), and \( \pi_{p+1} < \pi_{p+3} < \ldots < \pi_{2n+1} \), which implies that \( G^- (\pi) \) is also a nonnesting
matching. It is trivial to check that the node \( \pi_p \) is both the last left endpoint of the subgraph \( G^+ (\pi) \)
and the first right endpoint of the subgraph \( G^- (\pi) \). Furthermore, each node other than \( \pi_p \) has degree
one, and the node \( \pi_p \) has degree two. It is easy to check that we can retrieve a minimal permutation
\( \pi \) from its corresponding graph \( G(\pi) \).

There is a well known bijection between the set of nonnesting matchings on \([2n]\) and the set of Dyck
paths of length \( 2n \) [15]. Recall that a \( \text{Dyck path} \) of length \( 2n \) is a lattice path on the plane from
\((0, 0)\) to \((2n, 0)\) that does not go below the \( x \)-axis and consists of up steps \( U = (1, 1) \) and down steps
\( D = (1, -1) \). Here we give a brief description of this bijection. Given a nonnesting matching \( M \), read
\( M \) from left to right, for each left endpoint, we adjoin an up step and for each right endpoint, we
adjoin a down step. A matching is said to be with a \textit{heading} of length \( m \) if there are exactly \( m \) left
endpoints left to the first right endpoint. Similarly, A matching is said to be with a \textit{tail} of length \( m \) if
there are exactly \( m \) right endpoints right to the last left endpoint. It is clear that the corresponding
Dyck path of a nonnesting matching with a heading (resp. tail) of length \( m \) is starting (resp. ending)
with \( m \) consecutive up (resp. down) steps. A nonnesting matching with \( 5 \) edges with a heading of
length \( 4 \) and a tail of length \( 3 \) and its corresponding Dyck path of length \( 10 \) starting with \( 4 \) consecutive
up steps and ending with \( 3 \) consecutive down steps are shown in Figure 2. Dyck paths of length \( 2n \)
starting (resp. ending) with \( m \) consecutive up (resp. down) steps are counted by \( \frac{m}{2n-m} \binom{2n-m}{n-m} \) [7].

![Figure 2: A matching and its corresponding Dyck path.](image)

Given a graph \( G(\pi) \) with \( n+1 \) edges of a minimal permutation \( \pi \) of length \( 2n+1 \) with \( n+1 \) descents,
suppose that \( G(\pi) \) satisfies the following conditions:

1. a node \( p \) is both the last left endpoint of the subgraph \( G^+ (\pi) \) and the first right endpoint of the
subgraph \( G^- (\pi) \);
2. each node other than the node \( p \) has degree one, and the node \( p \) has degree two;
3. the subgraph \( G^+ (\pi) \) is a nonnesting matching with \( i \) edges and with a tail of length \( k \) and the
subgraph \( G^- (\pi) \) is a nonnesting matching with \( n+1-i \) edges and with a heading of length \( j \).

It is clear that \( p = 2i - k + j \). Denote by \( S(n, i, j, k) \) the set of all such graphs. For example, Figure
1 is an illustration of a graph with 6 edges in which \( p = 6, i = 3, k = 2 \) and \( j = 2 \). In order to get
a graph $G \in S(n, i, j, k)$, we should first choose $2i - k - 1$ nodes from the nodes $1, 2, \ldots, p - 1$ and choose $k$ nodes from the nodes $p + 1, p + 2, \ldots, 2n + 1$ to be the endpoints of $G^+$. The number of ways to choose these $2i - 1$ nodes is equal to

$$\binom{2i - k + j - 1}{2i - k - 1} \binom{2n - 2i + k - j + 1}{k} = \binom{2i - k + j - 1}{j} \binom{2n - 2i + k - j + 1}{k}.$$ \hspace{1cm} (1)

The number of nonnesting matchings on the $2i - 1$ chosen nodes together with the node $p$ ending with a tail of length $k$ is equal to $\frac{k}{2i - k} \binom{2n + 1 - i}{j}$. Also, the number of nonnesting matchings on the remaining $2n - 2i + 1$ nodes together with the node $p$ ending with a tail of length $j$ is equal to $\frac{j}{2(n + 1 - i) - j} \binom{2(n + 1 - i) - j - 1}{k - 1}$. Hence, the cardinality of $S(n, i, j, k)$ is given by

$$\binom{2i - k}{i - k} \binom{2(n + 1 - i) - j}{n + 1 - i - j} \binom{2i - k + j - 1}{j - 1} \binom{2(n + 1 - i) - j + 1}{k - 1}.$$ \hspace{1cm} (2)

When $i$ ranges from $1$ to $n$, $k$ from $1$ to $i$ and $j$ from $1$ to $n + 1 - i$, we get the graphs of all minimal permutations of length $2n + 1$ with $n + 1$ descents. Hence, minimal permutations of length $2n + 1$ with $n + 1$ descents are counted by

$$f_{n+1}(2n+1) = \sum_{i=1}^{n} \sum_{k=1}^{i} \binom{n+1-i}{i-k} \binom{2(n+1-i)-j}{n+1-i-j} \binom{2(n+1-i)-j+1}{j-1} \binom{2(n+1-i)-j-1+k}{k-1} \binom{2i-k}{i-k} \binom{2(n+1-i)-j}{n+1-i-j} \binom{2(n+1-i)-j+1}{j-1} \binom{2(n+1-i)-j-1+k}{k-1}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{i} \binom{n+1-i}{i-k} \binom{2(n+1-i)-j}{n+1-i-j} \binom{2(n+1-i)-j+1}{j-1} \binom{2(n+1-i)-j-1+k}{k-1} \binom{2i-k}{i-k} \binom{2(n+1-i)-j}{n+1-i-j} \binom{2(n+1-i)-j+1}{j-1} \binom{2(n+1-i)-j-1+k}{k-1}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{i} \binom{2i-k}{i-k} \binom{n+1-i}{n-i+k} \binom{2(n+1-i)-j-1+k}{n+i+k} \binom{2(n+1-i)-j-1+k}{k-1} \binom{2(n+1-i)-j-1+k}{n+i+k} \binom{2(n+1-i)-j-1+k}{n+i+k}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{i} \binom{2i-k}{i-k} \binom{n+i+k}{n-i+k} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i}$$

$$= \sum_{i=1}^{n} \binom{n+i+1}{i+1} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i}$$

$$= \sum_{i=1}^{n} \binom{n+1}{i+1} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i} \binom{2n+1}{n-i}$$

$$= 2^{n+1} \binom{2n+1}{n-i} = 2^{n+1} \binom{2n+1}{n-i}.$$
where the Vandermonde convolution formula [13] is given by

\[
\binom{m+n+k-1}{k} = \sum_{i+j=k} \binom{m+i-1}{i} \binom{n+j-1}{j}.
\]

Let \( n = d - 1 \), then we get Theorem 1.4. Note that the binomial \( \binom{2n+1}{n-1} \) counts the number of ways to draw \( n - 1 \) noncrossing diagonals in a convex \( (n+3) \)-gon [11] and also counts the number of standard Young tableaux of shape \((n, n, 1)\) [14]. It would be interesting to find a bijective proof of the formula for \( f_{n+1}(2n+1) \).

### 3 Minimal permutations of length \( n \) with \( d \) descents

In this section, we aim at getting a recurrence relation for the generating function \( F(n; v, w, q) \). Based on the recurrence relation, we recover the enumeration of minimal permutations of length \( 2d \) with \( d \) descents.

#### 3.1 Proof of Theorem 1.5

In order to prove our theorem we need the following notations. Denote by \( f_d(n; i_1, i_2, \ldots, i_s) \) the number of minimal permutations \( \pi = \pi_1 \pi_2 \cdots \pi_n \in S_n \) with \( d \) descents such that \( \pi_j = i_j \) for all \( j = 1, 2, \ldots, s \). We denote the corresponding generating function by \( f(n; i_1, i_2, \ldots, i_s; q) \), that is,

\[
f(n; i_1, i_2, \ldots, i_s; q) = \sum_{d \geq 1} f_d(n; i_1, i_2, \ldots, i_s) q^d.
\]

**Lemma 3.1** Let \( n \geq 2 \).

(i) For all \( 1 \leq i < j \leq n \),

\[
f(n; i, j; q) = 0.
\]

(ii) For all \( 1 \leq j < k < i \leq n - 1 \),

\[
f(n; i, j, k; q) = 0.
\]

(iii) For all \( 1 \leq k < j < i \leq n - 1 \),

\[
f(n; i, j, k; q) = q f(n - 1; j, k; q).
\]

(iv) For all \( 1 \leq j < i \leq n \),

\[
\sum_{h=i+1}^{n} f(n; i, h; q) = q \sum_{k=i-1}^{n-2} \sum_{\ell=j}^{k-1} f(n-2; k, \ell; q).
\]
Proof. Immediately, Theorem 1.3 gives (i). Since there is no minimal permutation \( \pi = ijk\pi' \) of length \( n \) that satisfies \( 1 \leq j < k < i \leq n - 1 \), (ii) holds.

Let \( \pi = \pi_1\pi_2 \cdots \pi_n \) be a minimal permutation in \( S_n \) such that \( \pi_1 > \pi_2 > \pi_3 \). Denote by \( \pi' \) the permutation in \( S_{n-1} \) obtained from \( \pi_2\pi_3 \cdots \pi_n \) by replacing each \( \pi_j \) by \( \pi_j - 1 \) if \( \pi_j > \pi_i \) for \( j = 2, 3, \ldots, n \). Clearly, the first two elements of \( \pi' \) are \( \pi_2 \) and \( \pi_3 \). Then \( \pi \) is a minimal permutation with \( d \) descents if and only if the permutation \( \pi' \) is a minimal permutation of length \( n - 1 \) with \( d - 1 \) descents. Thus, \( f_d(n; i, j, k) = f_{d-1}(n - 1; j; k) \) for all \( d \). Multiplying this recurrence by \( q^d \) and summing over \( d \geq 1 \), we obtain (iii).

Let us write an equation for \( R = \sum_{h=1}^{n} f(n; i, j, h; q) \). Let \( \pi = \pi_1\pi_2 \cdots \pi_n \) be a minimal permutation in \( S_n \) such that \( \pi_3 > \pi_1 = i > \pi_2 = j \). From Theorem 1.3 we have that \( n \geq 4 \) and either (1) \( \pi_3 > \pi_1 > \pi_4 > \pi_2 \) or (2) \( \pi_3 > \pi_4 > \pi_1 > \pi_2 \) holds. Denote by \( \pi'' \) the permutation in \( S_{n-2} \) obtained from \( \pi_3\pi_4 \cdots \pi_n \) by replacing each \( \pi_j \) by \( \pi_j - 2 \) if \( \pi_j > \pi_1 \) and by \( \pi_j - 1 \) if \( \pi_2 < \pi_j < \pi_1 \) for \( j = 3, 4, \ldots, n \). In both cases, \( \pi \) is a minimal permutation with \( d \) descents if and only if the permutation \( \pi'' \in S_{n-2} \) is a minimal permutation with \( d - 1 \) descents. Clearly, the first two values of \( \pi'' \) are \( \pi_3 - 2 \) and \( \pi_4 - 1 \) in the former case and \( \pi_3 - 2 \) and \( \pi_4 - 2 \) in the latter case. Thus, the former case contributes

\[
R_1 = q \sum_{h=i+1}^{n} \sum_{\ell=j+1}^{i-1} f(n - 2; h - 2, \ell - 1; q)
\]

to \( f(n; i, j, q) \), while the latter case contributes

\[
R_2 = q \sum_{h=i+1}^{n} \sum_{\ell=i+1}^{k-1} f(n - 2; h - 2, \ell - 2; q)
\]

to \( f(n; i, j, q) \). Combining the above two cases, we get that \( R = R_1 + R_2 \) as claimed in (iv).

Now let us write a recurrence relation for the polynomials \( F(n; v, w, q) \). Let \( 1 \leq j < i \leq n - 1 \) and \( n \geq 4 \). From the definitions we have that

\[
f(n; i, j; q) = \sum_{k=1}^{j-1} f(n; i, j, k; q) + \sum_{k=j+1}^{i-1} f(n; i, j, k; q) + \sum_{k=i+1}^{n} f(n; i, j, k; q),
\]

which by Lemma 3.1(ii)-(iv) implies that

\[
f(n; i, j; q) = q \sum_{k=1}^{j-1} f(n - 1; j, k; q) + q \sum_{k=i+1}^{n-2} \sum_{\ell=j}^{k-1} f(n - 2; k, \ell; q).
\]

(3.1)

Note that, when \( i = n > j \geq 1 \), the generating function for the number of all minimal permutations \( \pi_1\pi_2 \cdots \pi_n \in S_n \) with \( d \) descents such that \( \pi_1 = n \) is given by

\[
\sum_{j=1}^{n-1} f(n; n, j; q)v^{n-j}w^{j-1} = v^{n-1}q \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} f(n - 1; j, k; q)w^{j-1}1^{k-1} = v^{n-1}qF(n - 1; w, 1, q).
\]
Thus, multiplying (3.1) by $v^{i-1}w^{j-1}$ and summing over all $i, j, n \geq i > j \geq 1$, by using Lemma 3.1(i)-(ii), we obtain that

$$F(n; v, w, q) = v^{n-1}qF(n - 1; w, 1, q)$$

$$+ q \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \left( \sum_{k=1}^{j-1} f(n - 1; j, k; q) \right) v^{i-1}w^{j-1} + q \sum_{i=2}^{n-1} \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{n-k} f(n - 2; k, \ell; q) \left( \sum_{i=1}^{n-k-1} v^{i-1}w^{\ell-1} \right) v^{i-1}w^{j-1}. \quad (3.2)$$

Now let us express Expressions (a) and (b) in the above equation in terms of the polynomials $F(n; v, w, q)$. Expression (a) can be written as

$$n-1 \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \left( \sum_{k=1}^{j-1} f(n - 1; j, k; q) \right) v^{i-1}w^{j-1} \quad (3.3)$$

$$= \sum_{j=2}^{n-1} \sum_{k=1}^{j-1} f(n - 1; j, k; q) \left( w^{j-1} \sum_{i=j}^{n-2} v^{i} \right) = \sum_{j=2}^{n-1} \sum_{k=1}^{j-1} f(n - 1; j, k; q)w^{j-1}v - \frac{v^{n-1}}{1-v}$$

$$= \frac{v}{1-v} (F(n - 1; vw, 1, q) - (vw)^{n-2}) - \frac{v^{n-1}}{1-v} (F(n - 1; w, 1, q) - w^{n-2})$$

$$= \frac{v}{1-v} (F(n - 1; vw, 1, q) - v^{n-2}F(n - 1; w, 1, q)),$$

and Expression (b) as

$$n-2 \sum_{i=1}^{n-1} \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{n-k} f(n - 2; k, \ell; q) v^{i-1}w^{j-1}$$

$$= vF(n - 2; 1, 1, q) + \sum_{i=3}^{n-1} \sum_{k=i-1}^{n-2} \sum_{j=1}^{i-1} f(n - 2; k, \ell; q) v^{i-1}w^{j-1}$$

$$= vF(n - 2; 1, 1, q) + \sum_{i=3}^{n-1} \sum_{k=i-1}^{n-2} \min(i-1, \ell) f(n - 2; k, \ell; q) v^{i-1}w^{j-1}$$

$$= vF(n - 2; 1, 1, q) + \sum_{k=2}^{n-1} \sum_{\ell=1}^{k-1} \min(i-1, \ell) f(n - 2; k, \ell; q) v^{i-1}w^{j-1}$$

$$= vF(n - 2; 1, 1, q) + \sum_{k=2}^{n-1} \sum_{\ell=1}^{k-1} f(n - 2; k, \ell; q) \left( \sum_{i=3}^{k+1} \min(i-1, \ell) v^{i-1}w^{j-1} \right)$$

$$= vF(n - 2; 1, 1, q) + \sum_{k=2}^{n-1} \sum_{\ell=1}^{k-1} f(n - 2; k, \ell; q) \left( \sum_{i=3}^{k+1} v^{i-1}w^{j-1} \right) \left( \frac{1 - w^{\ell}}{1-w} \right).$$

Rewriting the internal sum as

$$\frac{v^2 - v^{\ell+1}}{(1-v)(1-w)} - \frac{(vw)^{2} - (vw)^{\ell+1}}{(1-w)(1-vw)} + \frac{(v^{\ell+1} - v^{k+1})(1 - w^{\ell})}{(1-v)(1-w)},$$

9
we get that Expression (b) is given by

\[
\begin{align*}
\sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \left( \sum_{k=1}^{n-2} \sum_{\ell=1}^{k-1} f(n-2; k, \ell; q) \right) v^{i-1} w^{j-1} \\
= vF(n-2; 1, 1, q) + \frac{v^2}{(1-v)(1-w)} (F(n-2, 1, 1, q) - F(n-2, 1, v, q)) \\
- \frac{v^2 w^2}{(1-w)(1-vw)} (F(n-2; 1, 1, q) - F(n-2; 1, vw, q)) \\
+ \frac{v^2}{(1-v)(1-w)} (F(n-2; 1, v, q) - wF(n-2; 1, vw, q) - F(n-2; v, 1, q) + wF(n-2; v, w, q)).
\end{align*}
\]  

Combining the above results, (3.3) and (3.4), with substituting in (3.2), we get Recurrence (1.1). □

Theorem 1.5 performs an algorithm for finding \( M_n(q) = F(n; 1, 1, q) \), where the algorithm implements in Maple as follows:

```maple
restart: B:=v*q: print(M(2,q)=q); A:=v^2*w*q^2: print(M(3,q)=q^2); for n from 4 to m do C:=simplify(q*v^(n-1)*subs(v=w,subs(w=1,A)) + q*v/(1-v)*(subs(v=v*w,subs(w=1,A))-v^(n-2)*subs(v=w,subs(w=1,A))) + q*v*subs(v=1,w=1,B) - q*v^2*w^2/(1-w)/(1-vw)*(subs(v=1,w=1,B)-subs(w=v*w,subs(v=1,B))) + q*v^2/(1-v)/(1-w)*(subs(v=1,w=1,B)-w*subs(w=v*w,subs(v=1,B))-subs(w=1,B)+w*B)):
print(M(n,q)=subs(v=1,w=1,simplify(factor(C)))); B:=simplify(A):
A:=simplify(C):
od:
```

By applying the above algorithm for \( m = 17 \) and extracting the coefficients of \( q^k \), we obtain Table 1.

### 3.2 Minimal permutations of length 2d with d descents

In this section we recover the enumeration of the number of minimal permutations of length 2d with d descents. Using the fact that each minimal permutation \( \pi = \pi_1 \pi_2 \cdots \pi_{2d} \) of length 2d with d descents satisfies that \( \pi_2 = 1 \), yields that

\[
F(2d; v, 1, q) = F(2d; v, 0, q).
\]  

(3.5)

Since a minimal permutation never has two consecutive ascents, we have that the minimal degree of \( q \) in the polynomial \( F(n; v, w, q) \) is \( \lceil n/2 \rceil \). Hence, if we denote the coefficient of \( q^{\lceil n/2 \rceil} \) in the polynomial
Table 1: The number $f_{n-d}(n)$ of minimal permutations of length $n$ with $n-d$ descents, where $d \geq 1$ and $n = 1, 2, \ldots, 17$.

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$F(n; v, w, q)$ by $Q(n; v, w)$, then (1.1) gives that

$$Q(2d; v, w) = vQ(2d - 2; 1, 1) - \frac{v^2w^2}{(1-v)(1-vw)}(Q(2d - 2; 1, 1) - Q(2d - 2; 1, vw))$$

$$+ \frac{v^2}{(1-v)(1-w)}(Q(2d - 2; 1, 1) + wQ(2d - 2; v, w) - Q(2d - 2; v, 1) - wQ(2d - 2; 1, vw)).$$

Note that since the minimal degree of $q$ in the polynomial $F(2d; v, w, q)$ is $d$, we obtain that the minimal degree of $q$ in the generating function

$$R = v^{2d-1}qF(2d - 1; w, 1, q) + \frac{vq}{1-v}(F(2d - 1; vw, 1, q) - v^{2d-2}F(2d - 1; w, 1, q))$$

is $d+1$, thus the term $R$ in (1.1) does not contribute to the generating function $Q(2d; v, w)$.

Substituting $w = 0$ and using (3.5) we obtain that

$$Q(2d; v, 1) = vQ(2d - 2; 1, 1) + \frac{v^2}{1-v}Q(2d - 2; 1, 1) - \frac{v^2}{1-v}Q(2d - 2; v, 1).$$

Let $Q(t, v) = \sum_{n \geq 1} Q(2n; v, 1)t^n$. Multiplying the above equation by $t^d$, summing over $d \geq 2$ and using the initial condition $Q(2; v, 1) = v$, we get that

$$\left(1 + \frac{v^2t}{1-v}\right)Q(t, v) = tv + \frac{vt}{1-v}Q(t, 1).$$

This type of functional equation can be solved systematically using the kernel method (see [1, 9]). In this case, if we assume that $v = C(t) = \frac{1-\sqrt{1-4t}}{2}$, where $v = C(t)$ satisfies the equation $1 + \frac{v^2t}{1-v} = 0$, then we obtain that $Q(t, 1) = C(t) - 1$ which implies the following result.
Corollary 3.2 (see [3]) For $d \geq 1$, the number of minimal permutations of length $2d$ with $d$ descents is given by the $d$-th Catalan number $c_d$.

4 Minimal permutations of length $n$ with $n - d$ descents

Let us define the polynomials

$$F_d(n; v, w) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} f_d(n; i, j)v^{i-1}w^{j-1}$$
and

$$G_d(n; v, w) = F_{n-d}(n; v, w),$$

namely $G_d(n; v, w)$ is the generating function for the number of minimal permutations $\pi$ of length $n$ with $n - d$ descents according to the values of the first and second elements of $\pi$. Clearly, $G_d(n; v, w) = 0$ for all $n \leq d$. We denote by $G_d(t, v, w)$ the corresponding generating function, with respect to the length, that is,

$$G_d(t, v, w) = \sum_{n \geq d+1} G_d(n; v, w)t^n = \sum_{n \geq d+1} F_{n-d}(n; v, w)t^n.$$

The main goal of this section is to find a recurrence relation for $G_d(t, v, w)$.

4.1 Recurrence relation for $G_d(t, v, w)$

Finding the coefficient of $q^n$ in (1.1), we obtain that

$$F_m(n; v, w) = v^{n-1}F_{m-1}(n-1; w, 1) + \frac{v}{1-v}(F_{m-1}(n-1; v; w, 1) - v^{n-2}F_{m-1}(n-1; w, 1))$$
$$+ vF_{m-1}(n-2; 1, 1) - \frac{v^2w}{(1-w)(1-vw)}(F_{m-1}(n-2; 1, 1) - F_{m-1}(n-2; 1, vw))$$
$$+ \frac{v^2}{(1-v)(1-w)}(F_{m-1}(n-2; 1, 1) - wF_{m-1}(n-2; 1, vw) - F_{m-1}(n-2; 1, v) + wF_{m-1}(n-2; v, w)),$$

for all $m \geq 2$. Substituting $m = n - d$ and using the definition of $G_d(n; v, w)$ we get that

$$G_d(n; v, w) = v^{n-1}G_d(n-1; v; w, 1) + \frac{v}{1-v}(G_d(n-1; v; w, 1) - v^{n-2}G_d(n-1; w, 1))$$
$$+ vG_{d-1}(n-2; 1, 1) - \frac{v^2w}{(1-w)(1-vw)}(G_{d-1}(n-2; 1, 1) - G_{d-1}(n-2; 1, vw))$$
$$+ \frac{v^2}{(1-v)(1-w)}(G_{d-1}(n-2; 1, 1) - wG_{d-1}(n-2; 1, vw) - G_{d-1}(n-2; v, 1) + wG_{d-1}(n-2; v, w)),$$

for all $d \geq 1$ and $n \geq d + 1$. Note that there is only one minimal permutation of length $n$ with $n - 1$ descents, namely $n(n-1)\cdots 1$. Hence, we get that $G_1(n; v, w) = v^{n-1}w^{n-2}$ for all $n \geq 2$. Thus, multiplying the above equation by $t^n$ and summing over all $n \geq d + 1$ we have the following result.

Theorem 4.1 The generating function $G_d(t, v, w)$ satisfies

$$G_d(t, v, w) = -\frac{tv}{1-v}G_d(vt, 1, 1) + \frac{tv}{1-v}G_d(t, vw, 1) + H_{d-1}(t, v, w),$$

where

$$H_{d-1}(t, v, w) = vt^2G_{d-1}(t, 1, 1) - \frac{v^2w^2}{(1-w)(1-vw)}(G_{d-1}(t, 1, 1) - G_{d-1}(t, 1, vw))$$
$$+ \frac{v^2}{(1-v)(1-w)}(G_{d-1}(t, 1, 1) - wG_{d-1}(t, 1, vw) - G_{d-1}(t, v, 1) + wG_{d-1}(t, v, w)),$$

with the initial condition $G_1(t, v, w) = \frac{vt}{1-vw}.$
4.2 Formulas for the number of minimal permutations of length $n$ with $n - d$ descents, where $d = 2, 3, 4, 5$

Since the recurrence relation or the expression of $G_d(t, v, w)$ are long to present in this paper for $d = 3, 4, 5$, we present only the final answer by applying the following steps (the case $d = 2$ is discussed with more details):

- Assume that $G_{d-1}(t, v, w)$ is given.
- Hence, the following expression is defined:

$$H_{d-1}(t, v, w) = vt^2G_{d-1}(t, 1, 1) - \frac{v^2w^2t^2}{(1-v)(1-vw)}(G_{d-1}(t, 1, 1) - G_{d-1}(t, 1, vw)) + \frac{v^2w^2}{(1-v)(1-w)}(G_{d-1}(1, 1, 1) - wG_{d-1}(t, 1, vw) - G_{d-1}(t, v, 1) + wG_{d-1}(t, v, w)).$$

(4.1)

- Theorem 4.1 yields the following functional equation for $G_d(t, v, 1)$:

$$G_d(t, v, w) = \frac{tv}{1-v}G_d(vt, w, 1) + \frac{tv}{1-v}G_d(t, vw, 1) + H_{d-1}(t, v, w).$$

(4.2)

- Substituting $w = 1$ and $v = \frac{1}{t+1}$ in (4.2) we obtain that

$$G_d(t/(t+1), 1, 1) = H_{d-1}(t/(t+1), 1, 1)$$

which is equivalent to

$$G_d(t, 1, 1) = H_{d-1}(t/(1-t), 1-t, 1).$$

(4.3)

- At the end, by (4.2) for $w = 1$ and using the generating function $G_d(t, 1, 1)$ we get a formula for the generating function $G_d(t, v, 1)$.

- Using the expressions of the generating functions $G_d(t, v, 1)$ and $G_d(t, 1, 1)$, we get from (4.2) an explicit formula for $G_d(t, v, w)$.

Below we present several explicit calculations.

4.2.1 The case $d = 2$

Theorem 4.1 for $d = 2$ gives that

$$G_2(t, v, w) = -\frac{tv}{1-v}G_2(vt, w, 1) + \frac{tv}{1-v}G_2(t, vw, 1) + \frac{vt^4(1 + v - vt)}{(1-vwt)(1-t)(1-vt)}.$$

(4.4)

If we substitute $w = 1$, then

$$G_2(t, v, 1) = -\frac{tv}{1-v}G_2(vt, 1, 1) + \frac{tv}{1-v}G_2(t, v, 1) + \frac{vt^4(1 + v - vt)}{(1-t)(1-vt)^2}.$$

(4.5)

Again, this type of functional equation can be solved systematically using the kernel method (see [1, 9]). In this case, if we assume that $v = v_0 = 1/(1+t)$, then $G_2(t/(1+t), 1, 1) = 2t^4/(1+t)$, which
implies that the generating function for the number of minimal permutations of length \( n \) with \( n - 2 \) descents is given by

\[
G_2(t, 1, 1) = \frac{2t^4}{(1 - 2t)(1 - t)^3}, \tag{4.6}
\]

as found in [3]. Now let us find \( G_2(t, v, w) \). By (4.5) we get that

\[
G_2(t, v, 1) = \frac{1}{1 - \frac{v}{1-v}} \left( -\frac{vt}{1-v} G_2(vt, 1, 1) + \frac{vt^4(1 + v - vt)}{(1-t)(1-vt)^2} \right),
\]

which is equivalent to

\[
G_2(t, v, 1) = \frac{v(1 + v + v(2v - 3)(v + 1)t + 2v^2(1 - v)t^2)t^4}{(1 - 2vt)(1 - vt)^3(1 - t)}.
\]

Therefore, by (4.4),

\[
G_2(t, v, w) = \frac{g_2vt^4}{(1-t)(1-vt)(1-2vvt)(1-vvt)^3}, \tag{4.7}
\]

where

\[
g_2 = 1 + v - v(1 + 3vw - v^2w^2 + 3w - vw^2 - v^2w)t \\
+ v^2w(2v^2w^3 - v - vw^2 + 2w - 3v^2w + 3 - v^2w^2 + vw)t^2 \\
- v^3w^2(2v^2w^2 + 2 + 2vw^2 - 2v^2w - 3v - vw)t^3 - 2v^5w^3(1 - w)t^4.
\]

### 4.2.2 The cases \( d = 3, 4, 5 \)

Note that in order to apply our algorithm for \( d \), we have to know explicit formula for the generating function \( G_{d-1}(t, v, w) \). Thus, applying the above algorithm for \( d = 3, d = 4 \), and then for \( d = 5 \) (and using (4.7) when \( d = 3 \)) we get the following result.

**Theorem 4.2** The generating function \( G_d(t, 1, 1) \) for the number of minimal permutations of length \( n \) with \( n - d \) descents, \( d = 3, 4, 5 \), is given by

- \( G_3(t, 1, 1) = \frac{(5 + 14t - 65t^2 + 2t^3 + 12t^4 - 24t^5)t^6}{(1 - 3t)(1 - 2t)^3(1 - t)^5} \),

- \( G_4(t, 1, 1) = \frac{2g_4t^8}{(1 - 4t)(1 - 3t)(1 - 2t)^4(1 - t)^7} \), where

  \[
g_4 = 7 + 126t - 899t^2 - 1630t^3 + 23865t^4 - 61182t^5 + 33779t^6 + 94158t^7 - 159664t^8 \\
  + 62656t^9 + 26640t^{10} - 19872t^{11} + 3456t^{12}.
\]

- \( G_5(t, 1, 1) = \frac{g_5t^{10}}{(1 - 5t)(1 - 4t)(1 - 3t)(1 - 2t)^5(1 - t)^7} \), where

  \[
g_5 = 42 + 2970t + 18237t^2 + 494153t^3 - 6722930t^4 + 28774836t^5 + 20486822t^6 \\
  - 719335338t^7 + 3254406822t^8 - 6843365454t^9 + 3450078253t^{10} + 19103420745t^{11} \\
  - 56125793234t^{12} + 74480865084t^{13} - 48521224680t^{14} + 2190310864t^{15} \\
  + 20428550944t^{16} - 13901832384t^{17} + 2953994112t^{18} + 505474560t^{19} \\
  - 354170880t^{20} + 49766400t^{21}.
\]
From (4.3) we get that the generating function $G_d(t, 1, 1)$ for $n = 1, 2, \ldots, 17$ and $d = 3, 4, 5$, we obtain the $d$-th column in Table 1.

### 4.3 Proof of Theorem 4.4

The aim of this section is to prove Theorem 4.4. In order to do that we need the following lemma.

**Lemma 4.3** Fix $m \geq 1$. Then there exist two polynomials $A_m(t, v, w)$ and $B_m(t, v, w)$ on $t, v, w$ such that $G_m(t, v, w) = \frac{A_m(t, v, w)}{B_m(t, v, w)}$, $m \geq 1$, where $G_m(t, v, w)$ defined when either $vw = 1$, or $v = 1$, or $w = 1$ and $v = 1/(1 + t)$.

**Proof.** We proceed the proof by induction on $m$. From $m = 1$ we have that $G_1(t, v, w) = \frac{v^2}{1 - vw}$ (see Theorem 4.1) which implies that the claim holds for $m = 1$. Assume that the claim holds for $m - 1$ and let us prove it for $m$. From Theorem 4.1 we have that

$$H_{m-1}(t, v, w) = vtG_{m-1}(t, 1, 1) - \frac{\alpha^2 \alpha^2}{(1 - w)(1 - vw)} G_{m-1}(t, 1, 1) - G_{m-1}(t, v, w))$$

$$+ \frac{\alpha^2 \alpha^2}{(1 - v)(1 - vw)} (G_{m-1}(t, 1, 1) - wG_{m-1}(t, 1, vw) - G_{m-1}(t, v, 1) + wG_{m-1}(t, v, w)),$$

In order to show that the generating function $H_{m-1}(t, v, w)$ is a rational function we need to show it is defined when either $vw = 1$, or $v = 1$, or $w = 1$, or $v = 1/(1 + t)$. Now, let us prove that the generating function $H_{m-1}(t, v, w)$ is defined when $vw = 1$, which is equivalent to show the limit

$$\lim_{vw \to 1} \frac{G_{m-1}(t, 1, 1) - G_{m-1}(t, 1, vw)}{1 - vw}$$

exists. Rewriting this expression in terms of $A_{m-1}$ and $B_{m-1}$ we obtain that

$$\lim_{vw \to 1} \frac{G_{m-1}(t, 1, 1) - G_{m-1}(t, 1, vw)}{1 - vw} = \lim_{z \to 1} \frac{A_{m-1}(t, 1, 1) B_{m-1}(t, 1, z) - A_{m-1}(t, 1, z) B_{m-1}(t, 1, z)}{(1 - z) B_{m-1}(t, 1, 1) B_{m-1}(t, 1, z)}.$$  

Using Lopital’s rule we obtain that the generating function $H_{m-1}(t, v, w)$ is defined when $vw = 1$. Similar arguments (only complicated) show that the generating function $H_{m-1}(t, v, w)$ is defined when either $v = 1$, or $w = 1$ or $w = 1$ and $v = 1/(1 + t)$, which implies that the function $H_{m-1}(t, v, w)$ is a rational function and $H_{m-1}(t, 1/(1 + t), 1)$ is a rational function on $t$.

From (4.3) we get that the generating function $G_m(t, 1, 1)$ is a rational function on $t$. Using

$$G_m(t, v, 1) = \frac{tv}{1 - v} G_m(tv, 1, 1) + \frac{tv}{1 - v} G_m(t, v, 1) + H_m(t, v, 1),$$

(see Theorem 4.1) we get that the generating function $G_m(t, v, 1)$ is a rational function on $t$ and $v$. Also the generating function $G_m(t, v, 1)$ is defined when $v = 1$. Hence, combining this result together with the fact that (see Theorem 4.1)

$$G_m(t, v, w) = \frac{tv}{1 - v} (G_m(t, vw, 1) - G_m(tv, w, 1)) + H_m(t, v, w),$$

and using Lopital’s rule, we obtain that the generating function $G_m(t, v, w)$ is a rational function and it is defined when either $v = 1$, or $w = 1$, or $vw = 1$, or $w = 1$ and $v = 1/(1 + t)$, which completes the induction. 

Now we state Theorem 4.4 and prove it.
Theorem 4.4 Given $d \geq 1$, there exists a constant $a_d$ such that the number of minimal permutations of length $n$ with $n - d$ descents is asymptotically equivalent to $a_d d^n$, as $n \to \infty$.

Proof. In order to prove this theorem we show that the generating function $G_d(t, v, w)$ is given by $G_d(t, v, w) = \frac{K_d(t, v, w)}{1 - dvtw}$ such that the function $K_d(t, v, w)$ is an analytic function on the disk $|t| \leq 1/d$ (here $|v|, |w| \leq 1$). We proceed the proof by induction on $d$. Since $G_1(t, v, w) = \frac{t^2}{1 - tw}$, the claim holds for $d = 1$. Assume that the claim holds for $d - 1$ and let us prove it for $d$. By Theorem 4.1, Lemma 4.3 and the induction hypothesis we obtain that the function $H_{d-1}(t, v, w)$ can be written as

$$H_{d-1}(t, v, w) = \frac{L_{d-1}(t, v, w)}{(1 - (d - 1)t)(1 - (d - 1)vt)(1 - (d - 1)vvt)},$$

where the function $L_{d-1}(t, v, w)$ is an analytic rational function on the disk $|t| \leq 1/(d - 1)$ (and it is defined when either $v = 1$, or $w = 1$, or $vw = 1$, or $w = 1$ and $w = 1/(1 + t)$). From (4.3) we have that $G_d(t/(1 + t), 1, 1) = H_{d-1}(t, 1/(1 + t), 1)$ which is equivalent to $G_d(t, 1, 1) = H_{d-1}(t/(1 - t), 1 - t, 1)$ (note that $|t/(1 - t)| < \frac{d - 1}{d}$ implies that $|t| < 1/d$). Thus the function $G_d(t, 1, 1)$ can be expressed as $K_d(t, 1, 1)$, where $K_d(t, 1, 1)$ is an analytic rational function on the disk $|t| \leq 1/d$. Now, if we substitute $w = 1$ in (4.2), then we get that

$$\left(1 - \frac{tv}{1 - v}\right)G_d(t, v, 1) = -\frac{vt}{1 - v}G_d(vt, 1, 1) + H_{d-1}(t, v, 1),$$

which gives that the function $G_d(t, v, 1)$ can be written as $\frac{K_d(t, v, 1)}{1 - dvt}$ where $K_d(t, v, 1)$ is an analytic function on the disk $|t| \leq 1/d$. Hence, by (4.2) with $|w| \leq 1$, we obtain that the generating function $G_d(t, v, w)$ is given by $\frac{K_d(t, v, w)}{1 - dvtw}$ where $K_d(t, v, w)$ is an analytic rational function on the disk $|t| \leq 1/d$, which completes the induction.

Hence, the generating function $G_d(t, v, w)$ is an analytic rational function on the disk $|t| < 1/d$ and it has a simple pole at $t = 1/d$, which implies (see [8, Theorem IV.7]) that there exists a constant $a_d$ such that the number of minimal permutations of length $n$ with $n - d$ descents is asymptotically equivalent to $a_d d^n$, as $n \to \infty$. 

5 Conclusion and open problems

For $d = 2^p$, $B_d$ is the basis of the class of permutations obtainable in at most $p$ steps in the whole genome duplication - random loss model (for local characterization of the permutations of $B_d$ (finite set), see Theorem 1.3). The aim of this paper to study the number $f_d(n)$ of minimal permutations of length $n$ with $d$ descents. We extend the known results (see [3]) by showing, combinatorially, that $f_d(2d - 1) = 2d - 3(d - 1)c_d$, see Theorem 1.4. The general problem for finding explicit formula for the sequence $f_d(n)$ remains an open problem. However, we present a recurrence relation (on $n$) for the generating functions for the number of minimal permutations $\pi$ of length $n$ according to the number of descents, and the values of the first and second elements of $\pi$, see Theorem 1.5. Moreover, we studied the generating function for the number of minimal permutations $\pi$ of length $n - d$ with $d$ descents according to $n$ and the first two values of $\pi$, see Theorem 4.1. In particular, our formulas (Theorem 1.5 and Theorem 4.1) can be used (together with any Mathematical programming) to find explicit formulas for the polynomial $M_n(q) = F(n; 1, 1, q)$, for any given $n$, and the generating function
$G_d(t, 1, 1)$, for any given $d$, see Table 1 and Section 4.2, respectively. In particular, Theorem 4.1 gives an explicit formula for the generating function $G_d(t, v, w)$ for $d \leq 5$. Note that $G_1(n; v, w) = v^{n-1}w^{n-2}$ and the expression of $G_2(t, v, w)$ is presented in (4.7). For $d \geq 3$ the expressions involved become extremely cumbersome. So we just derived the generating functions $G_d(t, 1, 1)$ for $d = 3, 4, 5$, as shown in the Section 4.2. Based on the expressions of $G_d(t, v, w)$ for $d = 2, 3, 4, 5$, we derived the following conjecture.

**Conjecture 5.1** Let $P_d(t) = \prod_{j=1}^{d} (1 - (d - j)t)^{2j-1}$. Then there exists a polynomial $A_d(t, v, w)$ with integer coefficients of degree $3d^2 - 4d$ (as a polynomial of $t$) such that

$$G_d(t, v, w) = \frac{vt^{2d}A_d(t, v, w)}{P_d(t)P_d(vt)P_{d+1}(vwt)}.$$  

Moreover, there exists a polynomial $B_d(t)$ with integer coefficients of degree $d^2 - 4$ such that

$$G_d(t, 1, 1) = \frac{t^{2d}B_d(t)}{P_{d+1}(t)}.$$  

Note that the above conjecture shows that the smallest simple pole of the generating function $G_d(t, 1, 1)$ is $t = 1/d$, which implies Theorem 4.4.

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**References**


