THE $q$-CALKIN–WILF TREE

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Abstract. We define a $q$-analogue of the Calkin–Wilf tree and the Calkin–Wilf sequence. We show that the $n$th term $f(n; q)$ of the $q$-analogue of the Calkin–Wilf sequence is the generating function for the number of hyperbinary expansions of $n$ according to the number of powers that are used exactly twice. We also present formulae for branches within the $q$-analogue of the Calkin–Wilf tree and predecessors and successors of terms in the $q$-analogue of the Calkin–Wilf sequence.

1. Introduction

A plane tree $T$ can be defined recursively as a finite set of vertices, such that one vertex $r$ is called the root of $T$, and the remaining vertices form an ordered partition $(T_1, T_2, \ldots, T_m)$ of $m$ disjoint non-empty sets, each of which is a plane tree. We will draw plane trees with the root on the top level, the first level. The edges connecting the root of the tree to the roots of $T_1, T_2, \ldots, T_m$, will be drawn from left to right on the second level. For each vertex $v$, the vertices in the next lower level adjacent to $v$ are called the children of $v$, and $v$ is called their parent. Clearly, each vertex other than $r$ has exactly one parent. A vertex of $T$ is called a leaf if it has no children. The empty tree, formed by a single vertex, has no children, otherwise it is said to be an internal vertex. The outdegree of a vertex $v$ is the number of its children, and is denoted by $\deg(v)$. A binary tree is a plane tree in which each vertex has outdegree two.

The Calkin–Wilf tree is a binary tree in which the vertices correspond one-to-one to the positive rational numbers. This tree can be defined recursively as follows: The root of the tree is 1, and each vertex $\frac{a}{b}$ has two children: $\frac{a}{a+b}$ (the left one), and $\frac{a+b}{b}$ (the right one). See Figure 1.

```
1/1
/  \
1/2  2/1
|    |
1/3  3/2  2/3  3/1
|    |    |    |
1/4  4/3  3/5  5/2  2/5  5/3  3/4  4/1
```

Figure 1. The first four levels of the Calkin–Wilf tree.

Calkin and Wilf [4] have shown that this tree contains every positive rational number once and only once, with each rational number being represented as a reduced fraction. Reading the tree line by line...
line, the Calkin–Wilf sequence of the enumeration of \( \mathbb{Q}^+ \) starts with
\[
\begin{array}{cccccccccccc}
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 & \ldots
\end{array}
\]
As pointed out by Reznick \[8\], this sequence was already investigated by Stern \[9\] in 1858. This sequence satisfies the iteration
\[
x_1 = 1, \\
x_{n+1} = \frac{1}{2 \lfloor x_n \rfloor + 1 - x_n}.
\]
This observation is due to Newman (see Knuth \[7\]).

The Calkin–Wilf sequence has many interesting properties. For example, it encodes the hyperbinary representations of all positive integers (see \[4\]); it can be used as a model for the game Euclid, first formulated by Cole and Davie \[5\] (see also Hofmann et al. \[6\]); and several statistical properties of the Calkin–Wilf tree have been considered in Alkauskas and Steuding \[2\].

In this paper we define the \( q \)-analogue of the Calkin–Wilf tree and the \( q \)-analogue of the hyperbinary expansion. We then show at Theorem 2 that this tree defines the \( q \)-Calkin–Wilf sequence of polynomials \( f(n; q) \), where \( f(n; q) \) is the generating function for the number of hyperbinary expansions according to the number of powers that are used exactly twice. We also derive results for branches in the \( q \)-analogue of the Calkin–Wilf tree and for predecessors and successors in the \( q \)-analogue of the Calkin–Wilf sequence. In what follows we designate \( \mathbb{N} \) as the set of all positive integers \( \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 \) as the set of all non-negative integers \( \{0, 1, 2, \ldots\} \).

2. The \( q \)-Calkin–Wilf Tree and the \( q \)-Calkin–Wilf Sequence

The following definition of the \( q \)-Calkin–Wilf tree generalises the definition of the Calkin–Wilf tree found in Bates et al. \[3\].

**Definition 1.** *(q-Calkin–Wilf Tree).* The \( q \)-Calkin–Wilf tree is a binary tree with root \( \frac{1}{1} \). Each vertex \( \frac{a}{b} \) from level 1 onwards is a parent to two children: a left child \( \frac{aq+b}{b} \) and a right child \( \frac{aq}{b} \), where \( q \in \mathbb{N} \). Each of these children are located one level below their parent in the tree.

Figure 2 shows the first four levels of the \( q \)-Calkin–Wilf tree.

![Figure 2. The first four levels of the \( q \)-Calkin–Wilf tree.](image-url)
Clearly, by induction on \( j \), the left most and right most vertices in level \( j \) of the \( q \)-Calkin–Wilf tree are respectively,

\[
\frac{1}{1 + (j - 1)q} \quad \text{and} \quad \frac{1 + q + \cdots + q^j}{1}.
\]

The \( q \)-Calkin–Wilf tree is related to the concept of a \( q \)-hyperbinary expansion and the \( q \)-hyperbinary sequence as shown in the following two definitions.

**Definition 2.** (\( q \)-Hyperbinary Expansion). The hyperbinary expansion of a number \( n \) is an expansion of \( n \) as a sum of powers of 2, each power being used at most twice. We denote the set of all hyperbinary expansions of \( n \) by \( \mathbb{H}_n \), and the total number of powers that are used exactly twice in the hyperbinary expansion \( x \in \mathbb{H}_n \) by \( h_n(x) \). The \( q \)-hyperbinary expansion of \( x \) is defined as \( q^{h_n(x)} \).

**Definition 3.** (\( q \)-Hyperbinary Sequence and \( q \)-Calkin–Wilf Sequence). Let \( f(n; q) \) be the polynomial of the sum of \( q \)-hyperbinary expansions of \( n \) with \( f(0; q) = 1 \) and \( f(-1; q) = 0 \). Then the sequence \( \{ f(n; q) \}_{n \in \mathbb{N}_0} \) is called the \( q \)-hyperbinary sequence and the sequence \( \{ f(n + 1; q) \} \) is called the \( q \)-Calkin–Wilf sequence.

**Example 1.** The hyperbinary expansions of 5 are \( 4 + 1 \) and \( 2 + 2 + 1 \). Thus \( q \)-hyperbinary expansions of 5 are \( q^0 \) and \( q^1 \). Accordingly, \( f(5; q) = 1 + q \). Similarly, the hyperbinary expansions of 10 are \( 8 + 2 \), \( 8 + 1 + 1 \), \( 4 + 4 + 2 \), \( 4 + 4 + 1 + 1 \) and \( 4 + 2 + 2 + 1 + 1 \), which implies that the \( q \)-hyperbinary expansions of 10 are \( q^1 \), \( q^1 \), \( q^1 \), \( q^2 \) and \( q^2 \), respectively. Accordingly, \( f(10; q) = 1 + 2q + 2q^2 \).

Following [3], we define the \( q \)-analogue of the branches and diagonals of the \( q \)-Calkin–Wilf tree.

**Definition 4.** (Branches and diagonals). Let \( v \) be any vertex of the \( q \)-Calkin–Wilf tree and \( j \in \mathbb{N} \). The set of all vertices that is generated when an infinite number of right (left) movements proceed from the left (right) child of \( v \) is denoted by \( L_v \) (\( R_v \)) and its called the left (right) branch of \( v \). The set \( L_v \) (\( R_v \)) includes the left (right) child of \( v \).

The \( j \)th left (right) diagonal \( L_j \) (\( R_j \)) of the \( q \)-Calkin–Wilf tree is the set of all vertices found at the \( j \)th leftmost (rightmost) position in each level of the tree beginning at level \( \lceil \log_2 j \rceil + 1 \).

**Example 2.** We have for \( m \in \mathbb{N}_0 \),

\[
L_1 = \left\{ \frac{1}{1 + mq} \right\},
\]

\[
R_1 = \left\{ \frac{1 + q}{1 + mq + mq^2} \right\},
\]

\[
R_1 = \left\{ \frac{\sum_{i=0}^{q} q^i}{1} \right\},
\]

\[
L_2 = \left\{ \frac{1 + (m + 1)q}{1 + mq} \right\}.
\]
and
\[ R_2 = \left\{ \frac{\sum_{i=0}^{m} q^i}{\sum_{i=0}^{m+1} q^i} \right\}. \]

The proof of the following theorem when \( q = 1 \) can be found in [4] and an inductive proof is found in Aigner and Ziegler [1]. Recently, Bates et al. [3] offered a proof based on branching in the tree which can be readily generalised to establish the following theorem relating to the \( q \)-Calkin–Wilf tree.

**Theorem 1.** Let the concatenation of successive levels of the \( q \)-Calkin–Wilf tree form a sequence. The denominator of each vertex in this sequence is the numerator of the next vertex in the sequence.

**Remark 1.** The sequence described in Theorem 1 is of the form
\[ \left\{ \frac{g(n; q)}{g(n + 1; q)} \right\}_{n \in \mathbb{N}_0} \]
for some function \( g \) with \( g(0; q) = g(1; q) = 1 \). Now the left and right children of
\[ \left\{ \frac{g(n; q)}{g(n + 1; q)} \right\} \]
are respectively,
\[ \left\{ \frac{g(2n + 1; q)}{g(2n + 2; q)} \right\} \quad \text{and} \quad \left\{ \frac{g(2n + 2; q)}{g(2n + 3; q)} \right\}. \]
Thus by Definition 1, we obtain for all \( n \in \mathbb{N}_0 \),
\[ g(2n + 1; q) = g(n; q) \]
and
\[ g(2n + 2; q) = gg(n; q) + g(n + 1; q). \]

**Theorem 2.** The \( q \)-Calkin–Wilf sequence is the concatenation of successive levels of the \( q \)-Calkin–Wilf tree. That is, for all \( n \in \mathbb{N}_0 \),
\[ g(n; q) = f(n; q). \]

**Proof.** We proceed by induction on \( n \).

The theorem is true for \( n = 0 \). Assume that the theorem holds for all integers 1, 2, . . . , \( 2n \). We now prove it for the cases \( 2n + 1 \) and \( 2n + 2 \).

**Case 1.** \( 2n + 1 \). Using the proof of the case \( f(2n + 1; q) = f(n; q) \) under the restriction \( q = 1 \) in [3, Theorem 2], there exists a bijection \( \alpha : \mathbb{H}_{2n+1} \rightarrow \mathbb{H}_n \) such that \( h_{2n+1}(x) = h_n(\alpha(x)) \). Accordingly,
\[ f(2n + 1; q) = \sum_{x \in \mathbb{H}_{2n+1}} q^{h_{2n+1}(x)} \]
\[ = \sum_{y \in \mathbb{H}_n} q^{h_n(y)} \]
\[ = f(n; q). \]

By our induction hypothesis, \( f(2n + 1; q) = g(n; q) = g(2n + 1; q) \).
Case 2. : $2n + 2$. From the proof of [3, Theorem 2], it follows that each hyperbinary expansion $x \in \mathbb{H}_{2n+2}$ can be mapped to either the hyperbinary expansion $x'$ of $n$ or the hyperbinary expansion $x''$ of $n + 1$ such that $h_n(x) = h_{n+1}(x') + 1$ and $h_n(x) = h_{n+1}(x'')$. Accordingly,

$$f(2n + 2; q) = \sum_{x \in \mathbb{H}_{2n+2}} q^{h_{2n+2}(x)}$$

$$= \sum_{x' \in \mathbb{H}_n} q^{h_n(x')} + \sum_{x'' \in \mathbb{H}_{n+1}} q^{h_{n+1}(x'')}$$

$$= qf(n; q) + f(n + 1; q).$$

By our induction hypothesis, $f(2n + 2; q) = qg(n; q) + g(n + 1; q) = g(2n + 2; q)$.

The result follows.

**Corollary 1.** For all $n \in \mathbb{N}_0$,

$$f(2n + 1; q) = f(n; q)$$

and

$$f(2n + 2; q) = qf(n; q) + f(n + 1; q).$$

**Proof.** This follows from Remark 1 and Theorem 2.

We now present alternative recurrence relations to those given in Corollary 1.

**Theorem 3.** For $n \in \mathbb{N}$,

$$f(n; q) = \begin{cases} f(n - 1; q) - qf(n - 2; q) & n \text{ odd} \\ qf(n - 1; q) + f(\frac{n}{2}; q) & n \text{ even.} \end{cases}$$

**Proof.** We have the following cases:

(1) $n$ odd: We prove this by induction on $n$.

We have $f(1; q) = f(0; q) - qf(-1; q)$.

Let for some $k$, $f(2k + 1; q) = f(2k; q) - qf(2k - 1; q)$. By Corollary 1,

$$f(2k + 2; q) = qf(k; q) + f(k + 1; q)$$

$$= qf(k; q) + f(2k + 3; q).$$

That is, $f(2k + 3; q) = f(2k + 2; q) - qf(2k + 1; q)$.

(2) $n$ even: By Corollary 1,

$$f(2k + 2; q) = qf(k; q) + f(k + 1; q)$$

$$= qf(2k + 1; q) + f(k + 1; q).$$

Letting $n = 2k + 2$ gives the result.
Definition 5. \textit{(Generating function for the q-hyperbinary sequence).} The generating function for the $q$-hyperbinary sequence \( \{f(n; q)\}_{n \in \mathbb{N}_0} \) is given by

\[
F(x, q) = \sum_{n \in \mathbb{N}_0} f(n; q)x^n.
\]

Theorem 4. The generating function \( F(x, q) \) is given by

\[
F(x, q) = \prod_{j \in \mathbb{N}_0} (1 + x^{2^j} + qx^{2^{j+1}}).
\]

Proof. Let \( F(x, q) = F_{\text{odd}}(x, q) + F_{\text{even}}(x, q) \) where

\[
F_{\text{odd}}(x, q) = \sum_{n \in \mathbb{N}_0} f(2n + 1; q)x^{2n+1} \quad \text{and} \quad F_{\text{even}}(x, q) = \sum_{n \in \mathbb{N}_0} f(2n; q)x^{2n}.
\]

By Corollary 1, and since \( f(-1; q) = 0 \),

\[
F_{\text{odd}}(x, q) = xF(x^2, q) \quad \text{and} \quad F_{\text{even}}(x, q) = (1 + qx^2)F(x^2, q).
\]

And so,

\[
F(x, q) = (1 + x + qx^2)F(x^2, q) = (1 + x + qx^2)(1 + x^2 + qx^4)F(x^4, q) = \cdots = \prod_{j \in \mathbb{N}_0} (1 + x^{2^j} + qx^{2^{j+1}}),
\]

as claimed. \( \square \)

Theorem 5. We have

\[
f(2^k - 2; q) = \begin{cases} 
\frac{q^{k-1}}{q-1} & \text{for } q > 1 \\
k & \text{for } q = 1.
\end{cases}
\]

Proof. If \( f(n; 1) = 1 \) then \( n \) has only one hyperbinary expansion \( h \) which is the binary expansion of \( n \), which implies that \( h_n(h) = 0 \) and \( f(n; q) = 1 \). It follows from [3, Theorem 4] that \( f(2^k - 1; q) = 1 \).

By Corollary 1,

\[
f(2^k - 2; q) = qf(2^{k-1} - 2; q) + f(2^{k-1} - 1; q) = qf(2^{k-1} - 2; q) + 1.
\]

(2.1)

By repeated use of (2.1),

\[
f(2^k - 2; q) = 1 + q(1 + qf(2^{k-2} - 2; q)) = 1 + q + q^2 + \cdots + q^{k-1}f(0; q) = 1 + q + q^2 + \cdots + q^{k-1},
\]

as claimed. \( \square \)

The following theorem is a generalisation of a result found in [3, Theorem 3]. It shows that \( f(n; q) \) is even only when \( n \equiv 2 \mod 3 \) and \( q \) is odd.
Theorem 6. We have

i) \( f(3n; q) \) and \( f(3n + 1; q) \) are odd

ii) \( f(3n + 2; q) \) is even (odd) for \( q \) odd (even).

Proof. The result follows by induction on \( n \). \( \Box \)

3. Branches

We now derive results for left and right branches of the \( q \)-Calkin–Wilf tree.

Theorem 7. For \( k \in \mathbb{N}_0 \),

(i) \( R_{\frac{a}{b}} = \left\{ \frac{qa + b}{kq(qa + b) + b} \right\} \),

(ii) \( L_{\frac{a}{b}} = \begin{cases} \left\{ \frac{q^k a + (qa + b) \sum_{i=0}^{k-2} q^i}{qa + b} \right\} & \text{for } q > 1 \\ \left\{ k + \frac{a}{q + b} \right\} & \text{for } q = 1 \end{cases} \)

Proof. We consider \( R_{\frac{a}{b}} \) and \( L_{\frac{a}{b}} \) respectively.

(i) \( R_{\frac{a}{b}} \): The right child of \( \frac{a}{b} \) is \( \frac{c}{b} \) where \( c = qa + b \). Consecutive left descendants of \( \frac{c}{b} \) are

\[
\frac{c}{qc + b}, \frac{c}{2qc + b}, \frac{c}{3qc + b}, \ldots
\]

That is, the \( k \)th term, \( k > 0 \), in \( R_{\frac{a}{b}} \) is

\[
\frac{qa + b}{(k - 1)q(qa + b) + b}.
\]

(ii) \( L_{\frac{a}{b}} \): The left child of \( \frac{a}{b} \) is \( \frac{d}{b} \) where \( d = qa + b \). Consecutive right descendants of \( \frac{d}{a} \) are

\[
\frac{qa + d}{d}, \frac{q^2 a + (q + 1) d}{d}, \frac{q^3 a + (q^2 + q + 1) d}{d}, \ldots
\]

That is, the \( k \)th term, \( k > 1 \), in \( L_{\frac{a}{b}} \) is

\[
\frac{q^{k-1} a + (qa + b) \sum_{i=0}^{k-2} q^i}{qa + b} = \begin{cases} \frac{q^{k-1} a + (qa + b) \sum_{i=0}^{k-2} q^i}{qa + b} & \text{for } q > 1 \\ k - 1 + \frac{a}{q + b} & \text{for } q = 1 \end{cases}
\]

\( \Box \)

Corollary 2. We have

\[
L_{\frac{a}{b}} = \left\{ \frac{q^k a + (qa + b) \sum_{i=0}^{k-2} q^i}{qa + b} \right\}_{k \in \mathbb{N}_0}
\]

Proof. The result follows from Theorems 7 ii) and 5. \( \Box \)
4. Predecessors and Successors

We now generalise some of the results for succession and precession found in [3] by determining successors and predecessors in the $q$-Calkin–Wilf sequence.

**Theorem 8. (Successors).** Let $x_n$ be the $n$th term in the $q$-Calkin–Wilf sequence where $n \in \mathbb{N}$. Then

\[
x_{2n+1} = \frac{1}{1 - qx_n}.
\]

**Proof.** By Definition 3 and Corollary 1,

\[
x_{2n} = \frac{f(2n+1;q)}{f(2n+2;q)} = \frac{f(n;q)}{qf(n;q)+f(n+1;q)} = \frac{m}{p},
\]

say, and

\[
x_{2n+1} = \frac{f(2n+2;q)}{f(2n+3;q)} = \frac{qf(n;q)+f(n+1;q)}{f(n+1;q)}.
\]

That is,

\[
x_{2n+1} = \frac{p}{p - qm},
\]

and

\[
\frac{1}{x_{2n+1}} = 1 - q \frac{m}{p} = 1 - qx_n.
\]

The result follows. \(\square\)

**Theorem 9. (Predecessors).** Let $x_n$ be the $n$th term in the $q$-Calkin–Wilf sequence where $n = 2, 3, 4, \ldots$. Then,

i) For $q > 1$,

\[
x_{n-1} = \begin{cases} \frac{q-1}{q} & \text{for } \frac{1}{x_n} \equiv 1 \mod q, \\ \frac{1}{x_n} - 1 \left(1 - \left\{ \frac{1}{x_n} \right\} \right) + \frac{q-1}{q} & \text{otherwise}. \end{cases}
\]

ii) For $q = 1$,

\[
x_{n-1} = \begin{cases} \frac{q-1}{q} & \text{for } \frac{1}{x_n} \in \mathbb{N} \\ \left\{ \frac{1}{x_n} \right\} + 1 - \left\{ \frac{1}{x_n} \right\} & \text{otherwise.} \end{cases}
\]

where $\left\lfloor \frac{1}{x_n} \right\rfloor$ denotes the integer part, and $\left\{ \frac{1}{x_n} \right\}$ the fractional part, of $\frac{1}{x_n}$.

**Proof.** There are three cases to consider.
(1) For $q > 1$, if $x_n = \frac{1}{1 + kq}$, $k \in \mathbb{N}$, then $x_n \in L_1$, for which

$$x_{n-1} = \frac{1 + q + q^2 + q^3 + \cdots + q^{k-1}}{1}$$

found on $R_1$. Thus

$$k = \frac{\frac{1}{x_n} - 1}{q}$$

and the result follows since $1 + q + q^2 + q^3 + \cdots + q^{k-1}$ is a geometric progression with common ratio $q$, and $\frac{1}{x_n} = 1 + kq$.

(2) For $q > 1$, let $\frac{a}{b}$ be the root of $x_{n-1}$ and $x_n$. Then $x_{n-1}$ and $x_n$ are the $k$th terms respectively in the left and right branches of $\frac{a}{b}$. From Theorem 7, for $k \in \mathbb{N}_0$,

$$x_{n-1} = \frac{q^k a + (qa + b) \frac{a^k - 1}{q-1}}{qa + b}$$

and

$$x_n = \frac{qa + b}{kq (qa + b) + b}.$$

Let

$$\frac{m}{p} = \frac{1}{x_n} = \frac{kq (qa + b) + b}{qa + b}.$$

Then $\left\lfloor \frac{1}{x_n} \right\rfloor = \left\lfloor \frac{m}{p} \right\rfloor = kq$. That is, $k = \left\lfloor \frac{m}{q} \right\rfloor$.

Also $p = qa + b, b = p \left\{ \frac{m}{p} \right\}$ and $qa = p - p \left\{ \frac{m}{p} \right\}$. Thus

$$x_{n-1} = \frac{\left\lfloor \frac{m}{p} \right\rfloor - 1}{p} \left( p - p \left\{ \frac{m}{p} \right\} \right) + \frac{\left\lfloor \frac{m}{q} \right\rfloor - 1}{q} \left( q - 1 \right)$$

$$= \frac{\left\lfloor \frac{m}{p} \right\rfloor - 1}{q} \left( 1 - \left\{ \frac{m}{p} \right\} \right) + \frac{\left\lfloor \frac{m}{q} \right\rfloor - 1}{q} \left( q - 1 \right)$$

$$= q \frac{\left\lfloor \frac{m}{q} \right\rfloor - 1}{q} \left( 1 - \left\{ \frac{1}{x_n} \right\} \right) + q \frac{\left\lfloor \frac{m}{q} \right\rfloor - 1}{q} \frac{\left\lfloor \frac{m}{q} \right\rfloor}{q - 1}.$$

(3) The $q = 1$ case is proven in [3, Theorem 18].

\[ \square \]

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