Recursive computation of derivatives of elliptic functions and of incomplete elliptic integrals

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Abstract
Presented are the recurrence formulas to compute the derivatives of a general elliptic function, Weierstrass’s $\wp$ function, the Jacobian elliptic functions, and the incomplete elliptic integrals in the forms of Jacobi and Legendre with respect to the argument or the amplitude. The double precision computation by the formulas is correct with 15 digits or so for the first 10 orders of differentiation at least.

Keywords: general elliptic function; incomplete elliptic integrals; Jacobian elliptic functions; recurrence formula; Weierstrass’s $\wp$ function

1. Introduction
Taylor series expansion is a powerful tool to compute the numerical values of an analytic function, especially in multi-precision arithmetics [1, 2]. Of course, this becomes efficient only if its high-order derivatives are easy to be evaluated, say if having simple explicit expressions, or being computed recursively. Good examples are some elementary functions as the exponential, the logarithm, and the trigonometric functions [3, 4].

In this short article, a recursive scheme to compute the derivatives of the elliptic functions and of the incomplete elliptic integrals with respect to their argument or amplitude is presented. It greatly assists the numerical inversion of general incomplete elliptic integrals [5].

The classic handbook of the elliptic integrals and of the elliptic functions is Byrd and Friedman [6]. Meanwhile, the modern reference is Olver et al. [7,
Chapters 19 through 23. Visit also its WEB site: http://dlmf.nist.gov/.

2. Recurrence formulas

Throughout this article, the order of differentiation of a single variable function, \( p(z) \), is denoted by its suffix for simplicity:

\[
\begin{align*}
    p_0 & \equiv p(z), \\
    p_\ell & \equiv \frac{d^\ell p(z)}{dz^\ell}, \quad (\ell > 0)
\end{align*}
\]

2.1. General elliptic function

Historically, an elliptic function was introduced as the inverse of a standard elliptic integral [6, p.18]:

\[
I(x) = \int_a^x \frac{dt}{\sqrt{f(t)}}, \quad (3)
\]

where \( f(t) \) is a cubic or quartic polynomial of \( t \) with distinct roots. If the argument of the integral \( x \) is regarded as a function of the integral value denoted by

\[
    z \equiv I(x), \quad (4)
\]

then \( x \) is called an elliptic function of argument \( z \) and is expressed as

\[
x = p(z). \quad (5)
\]

In general, \( f(t) \) is explicitly written as

\[
f(t) = a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4. \quad (6)
\]

under the condition

\[
a_0 \neq 0, \quad \text{and/or} \quad a_1 \neq 0. \quad (7)
\]

Note that the suffix of \( a_j \) does not mean the order of differentiation.

Thus, by definition, the first-order derivative of \( p \) is expressed as

\[
p_1 = \sqrt{f(p)} = \sqrt{a_0 p^4 + 4a_1 p^3 + 6a_2 p^2 + 4a_3 p + a_4}. \quad (8)
\]
Square this, differentiate both sides of it with respect to $z$, and divide them by the common factor, $2p_1$, which is assumed to be non-zero. Then, the expression of the second-order derivative is obtained as

$$p_2 = 2a_0p^3 + 6a_1p^2 + 6a_2p + 2a_3.$$  \hspace{1cm} (9)

Thus, the second-order derivative of $p$ is expressed as an explicit function of $p$ only, more specifically, a cubic or quadratic polynomial of $p$ since $a_0 \neq 0$ and/or $a_1 \neq 0$ as assumed. Therefore, the third- and higher order derivatives can be computed by using Faa Di Bruno’s formula of synthesis derivatives [7, Formula 1.4.13].

However, when $a_0 \neq 0$ in general, the computation of trinomial coefficients and the partition of an integer into three distinct integers are significantly complicated. Thus, a different approach based on much simpler Leibniz’s derivative formula of a simple product using the binomial coefficients [7, Formula 1.4.12] is appropriate.

The above expression of $p_2$ leads to that of $p_3$ as

$$p_3 = p_1q,$$  \hspace{1cm} (10)

where $q$ is an auxiliary quadratic or linear polynomial defined as

$$q \equiv 6a_0p^2 + 12a_1p + 6a_2.$$  \hspace{1cm} (11)

Thus, the application of Leibniz’s derivative formula

$$(PQ)_\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} P_j Q_{\ell-j},$$  \hspace{1cm} (12)

for the case $P = p_1$ and $Q = q$ leads to an expression of the third- and higher order derivatives of $p$ as

$$p_{\ell+3} = \sum_{j=0}^{\ell} \binom{\ell}{j} p_{j+1} q_{\ell-j}, \quad (\ell \geq 0)$$  \hspace{1cm} (13)

This includes the equation (10) as its special case, $\ell = 0$. This expression shows that the computation of $p_\ell$ requires lower order derivatives of $p$ up to $p_{\ell-2}$ and those of $q$ up to $q_{\ell-3}$ if $\ell \geq 3$. 
On the other hand, the expression of $q_1$ becomes
\[ q_1 = b_0 p_1 + b_1 p_1, \] (14)
where
\[ b_0 \equiv 12a_0, \] (15)
\[ b_1 \equiv 12a_1. \] (16)
Note that the suffix of $b_j$ does not mean the order of differentiation. At any rate, the second- and higher order derivatives of $q$ are similarly obtained as
\[ q_{\ell+1} = b_1 p_{\ell+1} + b_0 \sum_{j=0}^{\ell} \binom{\ell - 1}{j} p_j p_{\ell-j}, \quad (\ell \geq 0) \] (17)
where the above derivative formula for the case $P = p_1$ and $Q = p_0$ is used. Namely, the computation of $q_\ell$ requires lower order derivatives of $p$ up to $p_{\ell}$ if $\ell \geq 1$.

Practically, the number of terms in the above summation can be roughly halved by using the symmetric property of the summand as
\[ q_{2\ell-1} = b_1 p_{2\ell-1} + b_0 \sum_{j=0}^{\ell-1} \binom{2\ell - 1}{j} p_j p_{2\ell-1-j}, \quad (\ell \geq 1) \] (18)
\[ q_{2\ell} = b_1 p_{2\ell} + b_0 \left[ \frac{1}{2} \binom{2\ell}{\ell} p_{\ell}^2 + \sum_{j=0}^{\ell-1} \binom{2\ell}{j} p_j p_{2\ell-j} \right], \quad (\ell \geq 1) \] (19)
where the derivative formula for the case $P = Q = p$ is used. However, the form of the equation (17) is preferable for keeping the similarity of expression with that of $p_{\ell+3}$.

At any rate, the formulas are now completed. The equations (8), (9), (11), (13), and (17) or the pair of (18) and (19) are a set of recurrence formulas to obtain the derivatives of a general elliptic function, $p(z)$, of arbitrary order by using the binomial coefficients, which can be also generated recursively. Although the serial sequence of computation is somewhat complicated as
\[ p_0, p_1, p_2, q_0, p_3, q_1, \ldots, p_\ell, q_{\ell-2}, \ldots, \]
the summation part in the computation of $p_\ell$ and $q_{\ell-2}$ when $\ell \geq 3$ can be executed simultaneously, namely in the same do-loop. This is the main reason why the summation form is kept the same.
2.2. Weierstrass elliptic function

Weierstrass’s $\wp$ function [7, Chapter 23] is a special case of $p(z)$ such that

$$z = \int_{\wp(z)}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

(20)

where $g_2$ and $g_3$ are called the invariants. Thus, its derivatives are recursively computed as

$$\wp_1 = \sqrt{4\wp^3 - g_2 \wp - g_3},$$

(21)

$$\wp_2 = 6\wp^2 - g_2/2,$$

(22)

$$\wp_{2\ell+1} = 12 \sum_{j=0}^{\ell-1} \binom{2\ell - 1}{j} \wp_j \wp_{2\ell-1-j}, \quad (\ell \geq 1)$$

(23)

$$\wp_{2\ell+2} = 6 \left( \frac{2\ell}{\ell} \right) \wp_\ell^2 + 12 \sum_{j=0}^{\ell-1} \binom{2\ell}{j} \wp_j \wp_{2\ell-j}. \quad (\ell \geq 1)$$

(24)

The auxiliary polynomial is no longer needed since

$$q = 12\wp,$$

(25)

in this case. The expressions of the first- through third-order derivatives are well known [7, Section 23.3(ii)].

2.3. Jacobian elliptic functions

The Jacobian elliptic functions [7, Section 22.1] are compactly written as

$$\varphi(u) \equiv \text{am}(u|m),$$

(26)

$$s(u) \equiv \text{sn}(u|m),$$

(27)

$$c(u) \equiv \text{cn}(u|m),$$

(28)

$$d(u) \equiv \text{dn}(u|m).$$

(29)

Their numerical values are efficiently computed by the Landen transformation [8] or by the duplication [9, 10].

Consider computing their derivatives with respect to the argument $u$ while the parameter

$$m \equiv k^2,$$

(30)
is fixed. Note that the derivatives of $\varphi(u)$ are obtained from those of $d(u)$ as

$$\varphi_{\ell+1} = d_\ell, \quad (\ell \geq 0)$$

(31)

since

$$\left( \frac{\partial \text{am}(u|m)}{\partial u} \right)_m = \text{dn}(u|m).$$

(32)

Refer to Olver et al. [7, Formula 22.16.3]. Thus, the three principal functions, $s(u)$, $c(u)$, and $d(u)$, will be focused hereafter.

Of course, the general formula described in Section 2.1 can be directly applied to them. Indeed, these elliptic functions are special cases of $p(z)$ when $f(t)$ is a quadratic polynomial of $t^2$ [7, Section 22.15(ii)] as

$$u = \int_0^{s(u)} \frac{dt}{\sqrt{(1-t^2)(1-mt^2))}},$$

(33)

$$u = \int_{c(u)}^{1} \frac{dt}{\sqrt{(1-t^2)(m_c + mt^2))}},$$

(34)

$$u = \int_{d(u)}^{1} \frac{dt}{\sqrt{(1-t^2)(t^2 - m_c))}},$$

(35)

where

$$m_c \equiv 1 - m,$$

(36)

is called the complementary parameter. As a result, their second-order derivatives are expressed as odd cubic polynomials of themselves [7, Section 22.13 (iii)]. However, the resulting formulation becomes somewhat complicated. Thus, a simpler approach is selected.

The expressions of their first-order derivatives are well known [7, Table 22.13.1]:

$$s_1 = cd,$$

(37)

$$c_1 = -sd,$$

(38)

$$d_1 = -msc.$$

(39)

Indeed, $s$, $c$, and $d$ can be defined as the solution of these ordinary differential equations. The application of Leibniz’s formula to these expressions results the expression of the second- and higher order derivatives as

$$s_{\ell+1} = \sum_{j=0}^{\ell} \binom{\ell}{j} c_j d_{\ell-j},$$

(40)
\[ c_{\ell+1} = -\sum_{j=0}^{\ell} \binom{\ell}{j} s_j d_{\ell-j}, \]  

(41)

\[ d_{\ell+1} = -m \sum_{j=0}^{\ell} \binom{\ell}{j} s_j c_{\ell-j}. \]  

(42)

This is useful for the simultaneous computation of the derivatives of three functions. Otherwise, an alternative method discussed later will be more efficient.

2.4. Jacobi’s form of incomplete elliptic integrals

Jacobi regarded incomplete elliptic integrals as functions of the normal incomplete elliptic integral of the first kind,

\[ u \equiv F(\varphi|m). \]  

(43)

In his notation, some incomplete elliptic integrals are expressed as the integrals of rational functions of Jacobian elliptic functions [6, Formulas 110.02 through 110.04]:

\[ E^*(u) \equiv \int_0^u \text{dn}^2(v|m) dv, \]  

(44)

\[ \Pi^*(u) \equiv \int_0^u \frac{dv}{1 - n \text{sn}^2(v|m)}, \]  

(45)

\[ B^*(u) \equiv \int_0^u \text{cn}^2(v|m) dv, \]  

(46)

\[ D^*(u) \equiv \int_0^u \text{sn}^2(v|m) dv, \]  

(47)

\[ J^*(u) \equiv \int_0^u \frac{\text{sn}^2(v|m)}{1 - n \text{sn}^2(v|m)} dv, \]  

(48)

where \( m \) and \( n \) are called the parameter and the characteristic. Although the standard notation is without asterisks, here the asterisks are attached in order to discriminate them from Legendre’s form of incomplete elliptic integrals since the argument is different. Among them, \( E^*(u) \) is known as Jacobi’s Epsilon function [7, Section 22.16(ii)] and \( J^*(u) \) is tightly related with Jacobi’s standard elliptic integral of the third kind [6, p.233]. The numerical value of the Jacobi’s form of incomplete elliptic integrals are efficiently computed by the procedure in our work [9].
Now, consider the numerical evaluation of their derivatives with respect to \( u \). By definition, the first-order derivatives are simply expressed as

\[
E_1^* = 1 - my, \quad (49)
\]

\[
\Pi_1^* = \frac{1}{1 - ny}, \quad (50)
\]

\[
B_1^* = 1 - y, \quad (51)
\]

\[
D_1^* = y, \quad (52)
\]

\[
J_1^* = \frac{y}{1 - ny}. \quad (53)
\]

where

\[
y = y(u) \equiv [s(u)]^2, \quad (54)
\]

is an auxiliary variable. In deriving the above relations, the following identity relations among the Jacobian elliptic functions [7, Formula 22.6.1] are used:

\[
[s(u)]^2 + [c(u)]^2 = 1, \quad (55)
\]

\[
m[s(u)]^2 + [d(u)]^2 = 1. \quad (56)
\]

Since \( y(u) \) plays the key role, its derivatives will be first considered.

Of course, they can be computed from those of \( s \equiv s(u) \) by using the relation \( y = s^2 \). However, this approach requires the simultaneous computation of three Jacobian elliptic functions. Thus, another direction is sought.

The inverse integral representation of \( y(u) \) is obtained from that of \( s(u) \) by changing the integration variable as \( \tau = t^2 \):

\[
u = \frac{1}{2} \int_0^{y(u)} \frac{d\tau}{\sqrt{\tau(1 - \tau)(1 - m\tau)}}. \quad (57)
\]

Therefore, this corresponds to a general elliptic function, \( p(z) \), of the case

\[
a_0 = a_4 = 0, \quad a_1 = \frac{m}{4}, \quad a_2 = -\left(\frac{1 + m}{6}\right), \quad a_3 = \frac{1}{4}, \quad z = 2u. \quad (58)
\]

Thus, the derivative expression of \( y \) with respect to \( not \ z \) but \( u = z/2 \) is obtained from the general formula as

\[
y_1 = 2 \sqrt{y(1 - y)(1 - my)}, \quad (59)
\]
\[ y_2 = 2 - 4(1 + m)y + 6my^2, \quad (60) \]
\[ y_{\ell+3} = -4(1 + m)y_{\ell+1} + 12m \sum_{j=0}^{\ell} \binom{\ell}{j} y_j y_{\ell+1-j}. \quad (\ell \geq 0) \quad (61) \]

Again, the number of terms in the summation can be reduced by using the symmetric nature of the summand. However, as in the case of \( q \), the present form is selected for later use.

Return to the derivative computation of incomplete elliptic integrals. Once the derivatives of \( y(u) \) are known, the second- and higher order derivatives of \( E^*(u) \), \( B^*(u) \), and \( D^*(u) \) are trivial:

\[ E^*_\ell = -my_{\ell-1}. \quad (62) \]
\[ B^*_\ell = -y_{\ell-1}. \quad (63) \]
\[ D^*_\ell = y_{\ell-1}. \quad (\ell \geq 2) \quad (64) \]

Also, the derivatives of \( II^*(u) \) are obtained from those of \( J^*(u) \) as

\[ II^*_\ell = nJ^*_\ell. \quad (\ell \geq 2) \quad (65) \]

since

\[ II^*_1 = 1 + nJ^*_1. \quad (66) \]

Thus, only the derivatives of \( J^*(u) \) are to be considered.

Rewrite the expression of \( J^*_1 \) in an implicit form as

\[ J^*_1 = y + ny^*_1. \quad (67) \]

Thus, an implicit relation of its derivatives is obtained as

\[ J^*_{\ell+1} = y_{\ell} + n \sum_{j=0}^{\ell} \binom{\ell}{j} y_j J^*_\ell+1-j = y_{\ell} + ny J^*_\ell+1 + n \sum_{j=1}^{\ell} \binom{\ell}{j} y_j J^*_\ell+1-j. \quad (68) \]

The solution of this equation with respect to the highest order derivative, \( J^*_{\ell+1} \), results its explicit expression as

\[ J^*_{\ell+1} = II^*_1 \left[ y_{\ell} + n \sum_{j=1}^{\ell} \binom{\ell}{j} y_j J^*_\ell+1-j \right], \quad (\ell \geq 1) \quad (69) \]

where the formula (50) is used. Summarizing these, the derivatives of Jacobi’s form of incomplete elliptic integrals are recursively computed by the formulas (49) through (53), (59) through (65), and (69).
2.5. Jacobian elliptic functions: alternative approach

When the derivatives of \(y\) are known as in the previous subsection, there is an alternative way to compute the derivatives of the principal Jacobian elliptic functions, \(s(u)\), \(c(u)\), and \(d(u)\). For this purpose, the expressions of their second-order derivatives \([7, \text{Section 22.13(iii)}]\) are rewritten in a slightly different manner:

\[
s_2 = -(1 + m)s + 2mys, \quad (70)
\]
\[
c_2 = -c + 2myc, \quad (71)
\]
\[
d_2 = -md + 2myd. \quad (72)
\]

Since these expressions are of similar forms as that of \(y_2\), the formula (60), the second- and higher order derivatives are similarly obtained as

\[
s_{\ell+2} = -(1 + m)s_\ell + 2m \sum_{j=0}^{\ell} \binom{\ell}{j} y_j s_{\ell-j}, \quad (\ell \geq 0) \quad (73)
\]
\[
c_{\ell+2} = -c_\ell + 2m \sum_{j=0}^{\ell} \binom{\ell}{j} y_j c_{\ell-j}, \quad (\ell \geq 0) \quad (74)
\]
\[
d_{\ell+2} = -md_\ell + 2m \sum_{j=0}^{\ell} \binom{\ell}{j} y_j d_{\ell-j}. \quad (\ell \geq 0) \quad (75)
\]

Note that these formulas include the second-order expressions, the formulas (70) through (72), as their special cases when \(\ell = 0\).

This formulation is useful if only one or two of three elliptic functions are needed and/or if they are required simultaneously with those of the incomplete elliptic integrals in Jacobi’s form.

2.6. Legendre’s form of incomplete elliptic integrals

In Legendre’s notation, the incomplete elliptic integrals discussed previously are expressed \([7, \text{Section 19.2}]\) as

\[
F(\varphi) \equiv \int_0^\varphi \frac{d\theta}{\Delta(\theta)}, \quad (76)
\]
\[
E(\varphi) \equiv \int_0^\varphi \Delta(\theta) \ d\theta, \quad (77)
\]
\[ II(\varphi) \equiv \int_0^{\varphi} \frac{d\theta}{(1 - n \sin^2 \theta) \Delta(\theta)}, \tag{78} \]
\[ B(\varphi) \equiv \int_0^{\varphi} \frac{\cos^2 \theta \ d\theta}{\Delta(\theta)}, \tag{79} \]
\[ D(\varphi) \equiv \int_0^{\varphi} \frac{\sin^2 \theta \ d\theta}{\Delta(\theta)}, \tag{80} \]
\[ J(\varphi) \equiv \int_0^{\varphi} \frac{\sin^2 \theta \ d\theta}{(1 - n \sin^2 \theta) \Delta(\theta)}, \tag{81} \]
where the dependence on \( m \) and \( n \) is omitted for simplicity and
\[ \Delta(\varphi) \equiv \sqrt{1 - m \sin^2 \varphi}, \tag{82} \]
is Jacobi’s Delta function \([6, \text{Formula 121.01}]\).

Among them, \( F(\varphi), D(\varphi), \) and \( J(\varphi) \) are closely related with Carlson’s symmetric elliptic integrals of the first, the second, and the third kind, \( R_F, R_D, \) and \( R_J \), respectively \([7, \text{Section 19.25}]\). Meanwhile, the practical usefulness of \( B(\varphi), D(\varphi), \) and \( J(\varphi) \) are emphasized in \([5, 8, 11, 12, 13, 14]\).

The numerical values of these incomplete integrals are effectively computed by the procedures developed by us: (1) \( F(\varphi) \) in \([15]\), (2) \( B(\varphi) \) and \( D(\varphi) \) in \([12]\), and (3) \( J(\varphi) \) in \([13]\). These procedures are enough since (1) \( E(\varphi) \) is computed from \( B(\varphi) \) and \( D(\varphi) \), and (2) \( II(\varphi) \) is obtained from \( F(\varphi) \) and \( J(\varphi) \).

Consider the numerical computation of the derivatives of these incomplete elliptic integrals with respect to \( \varphi \). First, the derivatives of \( F(\varphi), B(\varphi), \) and \( II(\varphi) \) are obtained from those of \( E(\varphi), D(\varphi), \) and \( J(\varphi) \) as
\[ F_\ell = E_\ell + mD_\ell, \tag{83} \]
\[ B_\ell = E_\ell - m_c D_\ell, \tag{84} \]
\[ II_\ell = F_\ell + nJ_\ell, \tag{85} \]
where \( m_c = 1 - m \). Thus, those of \( E(\varphi), D(\varphi), \) and \( J(\varphi) \) will be focused hereafter.

By definition, the first-order derivatives of \( E(\varphi), D(\varphi), \) and \( J(\varphi) \) are their integrands themselves:
\[ E_1 = \Delta, \tag{86} \]
\[ D_1 = \frac{\eta}{\Delta}, \quad (87) \]
\[ J_1 = \frac{D_1}{1 - n\eta}, \quad (88) \]

where
\[ \eta(\varphi) \equiv \sin^2 \varphi. \quad (89) \]

The computation of the second- and higher order derivatives of \( E(\varphi), D(\varphi), \) and \( J(\varphi) \) requires the recurrence formulas of the derivatives of \( \eta(\varphi) \) and \( \Delta(\varphi). \)

First, since \( \eta(\varphi) \) is a trigonometric function, its derivatives are trivial:
\[ \eta_1 = 2\sqrt{\eta(1 - \eta)}, \quad (90) \]
\[ \eta_2 = 2 - 4\eta, \quad (91) \]
\[ \eta_\ell = -4\eta_{\ell-2}. \quad (\ell \geq 3) \quad (92) \]

Next, from the relation
\[ \Delta^2 = 1 - m\eta, \quad (93) \]

an implicit relation of the derivatives of \( \Delta(\varphi) \) is derived as
\[ \left( \Delta^2 \right)_\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \Delta_j \Delta_{\ell-j} = 2\Delta_\ell \Delta + \sum_{j=1}^{\ell-1} \binom{\ell}{j} \Delta_j \Delta_{\ell-j} = -m\eta. \quad (94) \]

The solution of this equation with respect to the highest order derivative, \( \Delta_\ell \), provides its recurrence formula as
\[ \Delta_\ell = -\left( \frac{F_1}{2} \right) \left[ m\eta + \sum_{j=1}^{\ell-1} \binom{\ell}{j} \Delta_j \Delta_{\ell-j} \right], \quad (\ell \geq 1) \quad (95) \]

where the relation
\[ F_1 = \frac{1}{\Delta}, \quad (96) \]
is used. This formula includes the case \( \ell = 1 \) if the summation is ignored in that case. Of course, the number of terms in the summation can be reduced if the symmetry of summand is used. However, the present form is kept again by the same reason in the previous subsections.
At any rate, the second- and higher order derivatives of $E(\varphi)$ are obtained as

$$E_\ell = \Delta_{\ell-1}. \quad (\ell \geq 2) \quad (97)$$

As for $D(\varphi)$, from the relation

$$D_1 \Delta = \eta, \quad (98)$$

an implicit relation of its derivatives is obtained as

$$(D_1 \Delta)_\ell = \sum_{j=0}^\ell \binom{\ell}{j} \Delta_j D_{\ell+1-j} = D_{\ell+1} \Delta + \sum_{j=1}^\ell \binom{\ell}{j} \Delta_j D_{\ell+1-j} = \eta. \quad (99)$$

The solution of this equation with respect to the highest order derivative, $D_{\ell+1}$, leads to its recurrence formula as

$$D_{\ell+1} = F_1 \left[ \eta - \sum_{j=1}^\ell \binom{\ell}{j} \Delta_j D_{\ell+1-j} \right], \quad (\ell \geq 1) \quad (100)$$

As for $J(\varphi)$, the expression of $J_1$ is rewritten in an implicit form as

$$J_1 = D_1 + n\eta J_1. \quad (101)$$

Thus, an implicit relation of its derivatives is derived as

$$J_{\ell+1} = D_{\ell+1} + n (\eta J_1)_\ell = D_{\ell+1} + n\eta J_{\ell+1} + n \sum_{j=1}^\ell \binom{\ell}{j} \eta_j J_{\ell+1-j}. \quad (102)$$

The solution of this relation with respect to the highest order derivative, $J_{\ell+1}$, provides its explicit expression as

$$J_{\ell+1} = \zeta \left[ D_{\ell+1} + n \sum_{j=1}^\ell \binom{\ell}{j} \eta_j J_{\ell+1-j} \right], \quad (\ell \geq 1) \quad (103)$$

where

$$\zeta \equiv \frac{1}{1 - n\eta}. \quad (104)$$

Now the formulation is closed. The derivatives of incomplete elliptic integrals in Legendre’s form with respect to the amplitude $\varphi$ are recursively obtained from the formulas (76) through (78), (86) through (88), (90) through (92), (97), (100), and (103).
2.7. Simple cases of series expansion

As a corollary, the Maclaurin series coefficients of the Jacobian elliptic functions and the incomplete elliptic integrals are simply obtained by using their values of zero argument, which are all zero except

\[ c(0) = d(0) = \Delta(0) = 1. \]  \hspace{1cm} (105)

Similarly, their Taylor series coefficients around the other important value when \( u = K \) and \( \varphi = \pi/2 \) are easily evaluated from their values at that point as

\[ s(K) = y(K) = \eta(\pi/2) = 1, \]  \hspace{1cm} (106)
\[ c(K) = 0, \]  \hspace{1cm} (107)
\[ d(K) = \Delta(\pi/2) = k_c, \]  \hspace{1cm} (108)
\[ F(\pi/2) = K, \]  \hspace{1cm} (109)
\[ E^*(K) = E(\pi/2) = E, \]  \hspace{1cm} (110)
\[ \Pi^*(K) = \Pi(\pi/2) = \Pi, \]  \hspace{1cm} (111)
\[ B^*(K) = B(\pi/2) = B, \]  \hspace{1cm} (112)
\[ D^*(K) = D(\pi/2) = D, \]  \hspace{1cm} (113)
\[ J^*(K) = D(\pi/2) = J, \]  \hspace{1cm} (114)

where

\[ k_c \equiv \sqrt{m_c} = \sqrt{1-m}, \]  \hspace{1cm} (115)

is the complementary modulus, \( K \), \( E \), and \( \Pi \) are the complete elliptic integrals of the first kind, the second kind, and the third kind, respectively, while \( B \), \( D \), and \( J \) are the the associate complete elliptic integral of the second and the third kinds, respectively. These will be useful in developing power series approximation of these functions and integrals.

The precise and fast computation of the complete integrals are provided by a series of works of ours as (1) \( K \) and \( E \) [9], (2) \( B \) and \( D \) [11], and (3) \( J \) and \( \Pi \) [14].
3. Numerical comparison

Three tables are prepared to examine the correctness of the recurrence formulas derived in the previous section. The tables compare the results of the double precision computation by the formulas with the accurate values provided by Mathematica [16]. Notice that the values of the elliptic functions and integrals themselves used in the recursive computation of the derivatives are obtained by the fast procedures of their computation [9, 10, 11, 12, 13, 14, 15].

First, Table 4 displays the case of \(s(u), c(u), \) and \(d(u)\) computed simultaneously as described in §2.3. Second, Table 4 presents the case of \(y(u)\) and \(J^*(u)\), the main components of the incomplete elliptic integrals in Jacobi’s form given in §2.4. Finally, Table 4 shows the case of \(E(\varphi), D(\varphi), \) and \(J(\varphi)\), the principal components of the incomplete elliptic integrals in Legendre’s form explained in §2.5.

In the tables, the erroneous digits are indicated by parentheses. Their appearance supports that the recursive computation by the formulas presented in this article are sufficiently correct in the sense that the averaged number of correct digits are 15 or so in the double precision environment.

4. Conclusion

The repeated application of Leibniz’s derivative formula of a simple product provides a set of recurrence formulas to compute the derivatives of the elliptic functions and of the incomplete elliptic integrals with respect to their main argument: namely the derivatives of (1) a general elliptic function, \(p(z)\), with respect to \(z\), (2) Weierstrass’s \(\wp\) function, \(\wp(z)\), with respect to \(z\), (3) the Jacobian elliptic functions, \(am(u|m), sn(u|m), cn(u|m), \) and \(dn(u|m)\), with respect to \(u\), (4) the incomplete elliptic integrals in Jacobi’s form, \(E(am(u|m)|m), B(am(u|m)|m), D(am(u|m)|m), \) \(\Pi(am(u|m), n|m), \) and \(J(am(u|m), n|m)\), as well as \(sn^2(u|m)\), with respect to \(u\), and (5) the incomplete elliptic integrals in Legendre’s form, \(F(\varphi|m), E(\varphi|m), B(\varphi|m), D(\varphi|m), \Pi(\varphi, n|m), \) and \(J(\varphi, n|m)\), as well as Jacobi’s Delta function, \(\Delta(\varphi|m)\), with respect to \(\varphi\).

The numerical comparison with Mathematica shows that the presented recursion is sufficiently correct so as to provide around 15 correct digits for the first 10 orders of differentiation in the double precision environment. All these formulas are written as a linear sum using the multiplication and
addition operations only. The number of division and square root operation is minimized. Indeed, their usage is limited to the computation of the first order derivatives only. As a result, their numerical computation is not expected to be time-consuming.

Acknowledgments

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References


\begin{tabular}{ccc}
\hline
\( \ell \) & \( s_\ell \) & \( c_\ell \) & \( d_\ell \) \\
\hline
0 & +8.03001824895643(8)E−01 & +5.959765676721407E−01 & +8.231601635963E−01 \\
1 & +4.90584668393960(2)E−01 & −6.60997864930979E−01 & −2.39285135717885(5)E−01 \\
2 & −6.8675802080375(0)E−01 & −2.1168376368412(9)E−01 & +1.19203533993790(4)E−01 \\
3 & +2.1312753913205(5)E−01 & +7.043377137465763E−01 & +2.39285135717885(5)E−01 \\
4 & +8.6123197978570(4)E−01 & −6.60997864930979E−01 & −2.39285135717885(5)E−01 \\
5 & −4.0684501028661(5)E+00 & −2.1168376368412(9)E−01 & −2.39285135717885(5)E−01 \\
6 & +3.2554131802131(4)E+00 & +7.043377137465763E−01 & +2.39285135717885(5)E−01 \\
7 & +2.5258361291421(6)E+00 & +2.39285135717885(5)E−01 & +2.39285135717885(5)E−01 \\
8 & −1.3317708765161(2)E+02 & −2.39285135717885(5)E−01 & +2.39285135717885(5)E−01 \\
9 & +1.4498298927992(5)E+02 & −2.39285135717885(5)E−01 & +2.39285135717885(5)E−01 \\
10 & +2.1312880855117(1)E+03 & −2.39285135717885(5)E−01 & +2.39285135717885(5)E−01 \\
\hline
\end{tabular}

Table 1: Numerical values of \( s_\ell \), \( c_\ell \), and \( d_\ell \). Listed are the numerical values of \( s_\ell \), \( c_\ell \), and \( d_\ell \), the \( \ell \)-th order derivatives of \( s(u) \), \( c(u) \), and \( d(u) \) with respect to \( u \), evaluated by the recurrence formulas described in §2.3. The double precision results for a non-trivial case, \( u = 1 \) and \( m = 1/2 \), are compared with the 20-digits precision computation by Mathematica [16]. The erroneous digits are indicated by parentheses.

\begin{tabular}{ccc}
\hline
\( \ell \) & \( y_\ell \) & \( J^*_\ell \) \\
\hline
0 & +6.44811930785734(1)E−01 & +2.79121507293649(1)E−01 \\
1 & +7.87807679234(8)E−01 & +7.043377137465763E−01 \\
2 & +2.125436046352(5)E−01 & +2.39285135717885(5)E−01 \\
3 & +1.3790708765162(1)E+02 & +2.39285135717885(5)E−01 \\
4 & +5.04934873753928(0)E+00 & +2.39285135717885(5)E−01 \\
5 & −5.236022210307(2)E+00 & +2.39285135717885(5)E−01 \\
6 & −3.55684697237(6)E+01 & +2.39285135717885(5)E−01 \\
7 & +1.9316140119083(8)E+02 & +2.39285135717885(5)E−01 \\
8 & −1.8601137395204(9)E+02 & +2.39285135717885(5)E−01 \\
9 & −2.958457739134(1)E+03 & +2.39285135717885(5)E−01 \\
10 & +1.9720382891797(2)E+04 & +2.39285135717885(5)E−01 \\
\hline
\end{tabular}

Table 2: Numerical values of \( y_\ell \) and \( J^*_\ell \). Same as Table 4 but for \( y_\ell \) and \( J^*_\ell \), the \( \ell \)-th order derivatives of \( y(u) \) and \( J^*(u) \) with respect to \( u \) given in §2.4. This time, the arguments are fixed as \( u = 1 \), \( n = 1/5 \), and \( m = 1/2 \).
<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$E_\ell$</th>
<th>$D_\ell$</th>
<th>$J_\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$+9.27329883624440(0)E^{-01}$</td>
<td>$+3.11773778441457(3)E^{-01}$</td>
<td>$+5.2133479936012(8)E^{-01}$</td>
</tr>
<tr>
<td>1</td>
<td>$+8.037184151574570E^{-01}$</td>
<td>$+8.809968826393657E^{-01}$</td>
<td>$+2.032094642397759E+00$</td>
</tr>
<tr>
<td>2</td>
<td>$-2.82840796501950(4)E^{-01}$</td>
<td>$+1.44139945649937E+00$</td>
<td>$+6.73435825653646(3)E+00$</td>
</tr>
<tr>
<td>3</td>
<td>$+1.59352454407334(0)E^{-01}$</td>
<td>$-1.95727191221733(5)E^{-01}$</td>
<td>$+1.90267669323579(0)E+01$</td>
</tr>
<tr>
<td>4</td>
<td>$+1.299598880048536E+00$</td>
<td>$-7.0140028055906(5)E+00$</td>
<td>$+3.492993985348(9)E+01$</td>
</tr>
<tr>
<td>5</td>
<td>$+6.9905707224233(7)E^{-01}$</td>
<td>$-1.558742834233952E+01$</td>
<td>$-1.45156140782967(6)E+02$</td>
</tr>
<tr>
<td>6</td>
<td>$-5.872110226122627E+00$</td>
<td>$+7.9142359133204(5)E+00$</td>
<td>$-2.75157805014654(2)E+03$</td>
</tr>
<tr>
<td>7</td>
<td>$-3.135000794698644E+01$</td>
<td>$+3.734346361898833E+02$</td>
<td>$-2.53964697201035(4)E+04$</td>
</tr>
<tr>
<td>8</td>
<td>$-7.42393261602376(8)E+01$</td>
<td>$+2.355157154197709E+03$</td>
<td>$-1.53154169044662(5)E+05$</td>
</tr>
<tr>
<td>9</td>
<td>$+4.5890899336215(1)E+02$</td>
<td>$+2.33463984623143E+03$</td>
<td>$+4.727558659057(401)E+04$</td>
</tr>
<tr>
<td>10</td>
<td>$+6.81266372438607(7)E+03$</td>
<td>$-1.13572724304508(9)E+05$</td>
<td>$+2.04778637290870(6)E+07$</td>
</tr>
</tbody>
</table>

Table 3: Numerical values of $E_\ell$, $D_\ell$, and $J_\ell$. Same as Table 4 but for $E_\ell$, $D_\ell$, and $J_\ell$, the $\ell$-th order derivatives of $E(\varphi)$, $D(\varphi)$, and $J(\varphi)$ with respect to $\varphi$ given in §2.5. This time, the arguments are changed as $\varphi = 1$, $n = 4/5$, and $m = 1/2$. 