Sphere Packing Optimization and EXIT Chart Analysis for Multi–Dimensional QAM Signaling

Toshiaki Koike-Akino and Vahid Tarokh

Abstract—We investigate on multi–dimensional QAM constellations optimized by sphere packing with the known densest lattices. We propose a greedy design method assisted by the sphere detection. It is demonstrated that the optimized constellations can significantly increase the squared minimum distance in comparison to the conventional QAM constellations. In addition, we analyze the optimized QAMs through the use of an extrinsic information transfer (EXIT) chart for iterative decoding.

Index Terms—Multi–dimensional QAM, sphere packing, signal constellation design, EXIT chart analysis

I. INTRODUCTION

A number of communication systems use a high–level $M$–ary quadrature amplitude modulation (QAM) to accommodate high data rate of multi–bps/Hz [1–4]. We typically use a square–grid $M$–QAM because we can demodulate it by separating in–phase (I) and quadrature–phase (Q) independently. However, it is known that a hexagonal lattice provides denser packing than does the square–grid lattice as in [5] and is more energy efficient than some standard QAMs as discussed in [6]. In this paper, we present the advantage of multi–dimensional modulations [7, 8], which is optimized by sphere packing with the known densest lattices [5].

It is known that the hexagonal lattice is optimal for one dimensional complex field. For higher dimensions, the optimality has not been proven yet, but we know near–optimal lattices; for example, the checkerboard lattice for two dimension, the diamond lattice for three and four dimensions, the Coxeter–Todd lattice for six dimension, the Barnes–Wall lattice for eight dimension and the Leech lattice for twelve dimension as given in [5]. In this paper, we adopt such dense lattices to optimize multi–dimensional modulations. Although multi–dimensional modulations offer an excellent performance, it has a drawback of high demapping complexity in general because the size of searching space increases exponentially with the dimension. Forney has partially resolved the complexity issue by the trellis–state structure for Voronoi constellations. As an alternative method, we can use a complexity–efficient sphere detection algorithm [9] for any arbitrary subset in lattice, which can asymptotically reduce the computational complexity into a polynomial order of the dimensionality. We optimize a multi–dimensional QAM mapper, which is compatible to exploit the sphere detection for low–complexity demapping.

We introduce a greedy packing algorithm assisted by the sphere detection to design the signal constellation. The proposed scheme is applicable to some precoding techniques and to some practical raw data with non–ideal compression, in which data distribution is not equiprobable. In addition, we make an analysis of the optimized QAMs through the use of an extrinsic information transfer (EXIT) chart [10, 11] for iterative decoding applications. The chief contribution of this paper is two–fold as follows: We adopt the known densest lattice to a design of energy–efficient multi–dimensional modulations for any arbitrary $a$ priori probability, and we analyze the optimized QAMs through the EXIT chart.

II. SIGNAL FORMULATIONS

Notations: We let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ an integer group, a real–number field and a complex field, respectively. We denote $\mathbb{Z}_+$ and $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$ as a positive integer set and a non–negative integer ring whose entry is less than $q \in \mathbb{Z}_+$. We describe vectors and matrices in bold italic face. The notations $X^\dagger$, $[x]_n$, $\Re[x]$, and $\Im[x]$ refer to as the conjugate transpose of a matrix $X$, the $n$–th entry of a vector $x$, the real and imaginary parts of a complex number $x$, respectively. The imaginary unit is denoted as $j = \sqrt{-1}$.

A. QAM Signaling in Hexagonal Lattice

Let $s \in \mathbb{Z}_M$ be an information data to be transmitted in a symbol instance, where $M = 2^B$ is an alphabet size and $B \in \mathbb{Z}_+$ is the number of bits per symbol. The data $s$ is transmitted in a QAM signal via a modulation mapper $M(\cdot)$. The transmitting signal is thus written as $x = M(s)$. For high–level modulations, we generally use a square $M$–QAM although these are not optimal.

In terms of energy efficiency, the optimal constellation for higher–level modulations (in single dimension) is given by hexagonal lattice as discussed in [6]. For example, the hexagonal 16–QAM is better than the square 16–QAM by approximately 0.58 dB, and the hexagonal 256–QAM has an approximately 0.81 dB gain over the square one. In Table I, we list the comparison between the hexagonal lattice and the conventional square–grid lattice for high–level QAM signaling. For odd $B$ bits, we have several types of constellations such as rectangular, shifted square, and cross QAM, as depicted in Fig. 1 for 128–ary signaling. We also present the optimally–packed disc–shaped QAM in square–grid lattice. One can see that the hexagonal lattice is the best constellation for any $M$–ary QAM. In the hexagonal lattice, an additional one bit transmission per symbol necessitates the increased signal–to–noise ratio (SNR) by almost exactly 3 dB each; more specifically, the minimum distance has 12.47 dB loss over BPSK for $B = 5$ ($M = 32$) whereas 15.47 dB for $B = 6$ ($M = 64$). In this paper, we discuss the advantages of such a good lattice in multi–dimensional signaling.
white Gaussian noise (AWGN) channel. The vector signals vector, which is modeled as

\[ \mathbf{y} = \mathbf{x} + \mathbf{w} \]

in general because the number of the possible points is increased to a polynomial order from an exponential order with the increase of the dimension. The demapping of (2) into

\[ \hat{s} = \arg \min_{s' \in \mathbb{Z}_M} \| y' - s'L \|_2^2, \]

where \( y' = yU^\dagger \). Here, the matrices \( U \in \mathbb{C}^{N \times N} \) and \( L \in \mathbb{C}^{N \times N} \) are the unitary matrix and the lower–triangular matrix, respectively, given by the LQ decomposition of \( M \). The diagonal entries of \( L \) are all non–negative real numbers.

The sphere detection can efficiently estimate the source data successively from \( s_{N-1} \) to \( s_0 \) by using the sphere constraint. Thanks to the triangular property of \( L \), we can readily exclude unlikely hypotheses \( s' = [s'_0, s'_1, \ldots, s'_{N-1}] \) outside a sphere of a certain radius \( \rho \) as follows

\[ \| y' - s'L \|_2^2 = \sum_{n=N-1}^{0} \left| y'_n - \sum_{i=N-1}^{n+1} s'_i L_{i,n} - s'_n L_{n,n} \right|^2 \]

\[ \geq \left| y'_{N-1} - s'_{N-1} L_{N-1,N-1} \right|^2 \]

\[ + \left| (y'_{N-2} - s'_{N-1} L_{N-2,N-2} - s'_n L_{n,n} \right| \geq \rho^2, \]

where \( L_{i,j} \) is the \((i, j)\)–th entry of \( L \). This implies that we can search for the closest hypothesis point to the received signal along the tree–searching diagram. At the first branch of the tree, we can constrain the possible candidates for the integer \( s_{N-1} \in \mathbb{Z}_M \) only within the finite region \([y' - \rho)/L_{N-1,N-1}, (y' + \rho)/L_{N-1,N-1}\]\. In an analogous way, the closest hypothesis point can be efficiently found by setting a proper radius. Some literatures have confirmed that it can reduce the average amount of computational complexity to a polynomial order from an exponential order with the dimension for a reasonable SNR regime.

### C. Sphere Detection with Matrix Decomposition

We suppose that \( M^N(\cdot) \) can be expressed in a linear form:

\[ M^N(s) = sM, \]

where \( M \in \mathbb{C}^{N \times N} \) denotes the mapping matrix. It simplifies demapping of (2) into

\[ \hat{s} = \arg \min_{s' \in \mathbb{Z}_M} \| y' - s'L \|_2^2, \]

where \( y' = yU^\dagger \). Here, the matrices \( U \in \mathbb{C}^{N \times N} \) and \( L \in \mathbb{C}^{N \times N} \) are the unitary matrix and the lower–triangular matrix, respectively, given by the LQ decomposition of \( M \). The diagonal entries of \( L \) are all non–negative real numbers.

The sphere detection can efficiently estimate the source data successively from \( s_{N-1} \) to \( s_0 \) by using the sphere constraint. Thanks to the triangular property of \( L \), we can readily exclude unlikely hypotheses \( s' = [s'_0, s'_1, \ldots, s'_{N-1}] \) outside a sphere of a certain radius \( \rho \) as follows

\[ \| y' - s'L \|_2^2 = \sum_{n=N-1}^{0} \left| y'_n - \sum_{i=N-1}^{n+1} s'_i L_{i,n} - s'_n L_{n,n} \right|^2 \]

\[ \geq \left| y'_{N-1} - s'_{N-1} L_{N-1,N-1} \right|^2 \]

\[ + \left| (y'_{N-2} - s'_{N-1} L_{N-2,N-2} - s'_n L_{n,n} \right| \geq \rho^2, \]

where \( L_{i,j} \) is the \((i, j)\)–th entry of \( L \). This implies that we can search for the closest hypothesis point to the received signal along the tree–searching diagram. At the first branch of the tree, we can constrain the possible candidates for the integer \( s_{N-1} \in \mathbb{Z}_M \) only within the finite region \([y' - \rho)/L_{N-1,N-1}, (y' + \rho)/L_{N-1,N-1}\]\. In an analogous way, the closest hypothesis point can be efficiently found by setting a proper radius. Some literatures have confirmed that it can reduce the average amount of computational complexity to a polynomial order from an exponential order with the dimension for a reasonable SNR regime.

### D. Densest Lattice for Sphere Packing

As in [5], for two–dimensional real field (one–dimensional complex field), the hexagonal lattice is the densest packing. It is given by using the lattice generator matrix \( G \) as follows:

\[ \lambda = zG = z \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}, \]

where \( \lambda \in \mathbb{R}^2 \) is the possible lattice points and \( z \in \mathbb{Z}^2 \) is any arbitrary two–dimensional integer coordinate. Setting an identity matrix for \( G \) corresponds to a square lattice. For
four-, eight-, and sixteen-dimensional lattices, the generator matrices of the best known lattice are written as

\[
G_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},
\]

\[
G_8 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},
\]

\[
G_{16} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},
\]

respectively, each of which is referred to as the checkerboard lattice, the diamond lattice and the Berness–Wall lattice. (We can find the densest lattices for the other dimensions in [5].) In particular, the eight-dimensional diamond lattice and the twenty-four-dimensional Leech lattice offer an excellent density performance, which is close to the Roger’s bound.

III. MAPPPING OPTIMIZATION

It is natural to use the densest lattices for multi-dimensional QAMs because of its energy-efficient feature. Moreover, since the lattice generator matrix \( G \) is lower–triangular, it is suited for sphere detection. However, it is not straightforward to optimize a subset selection from infinite lattice points, for maximizing squared distance with a symbol energy constraint. It is particularly true when the source information has non-uniform \( a priori \) probability distribution, which is resulted from some precoding techniques and non-ideal compressions.

A. Multi–Dimensional QAM Signaling in Densest Lattice

For general applications, we formulate the way of relating multi-dimensional modulations to the densest lattices in considering the case where the source data may follow a non–equiprobable distribution. Let \( p(s) \) be the probability that a source data \( s \in \mathbb{Z}_M^N \) appears (of course, we have \( \sum_{s' \in \mathbb{Z}_M^N} p(s') = 1 \)). When the data is equiprobable, it simplifies \( p(s) = 1/2^{BN} \) for all \( s \). We suppose that the modulation has unity energy per symbol in average as follows:

\[
\frac{1}{N} \sum_{s' \in \mathbb{Z}_M^N} p(s') \|\mathcal{M}_N(s')\|^2 = 1.
\]

In this paper, we exploit the \( 2N \)-dimensional lattice generator matrix \( G_{2N} \) for the \( N \)-dimensional \( M^N \)-QAM design, whose real components correspond to the even entries and imaginary components the odd entries as follows:

\[
\Re \left[ \mathcal{M}_N(s) \right] = \gamma L^{2N}(s)G_{2N} - c_0 \align{1}{2},
\]

\[
\Im \left[ \mathcal{M}_N(s) \right] = \gamma L^{2N}(s)G_{2N} - c_{0_{2n+1}},
\]

for the \( n \)-th dimensional signaling \( (n \in \mathbb{Z}_N) \), where \( \gamma \in \mathbb{R} \), \( c_0 \in \mathbb{R}^{2N} \), and \( L^{2N}() \) are the normalization factor, the centroid vector, and the one–to–one labelling function from a data vector \( s \in \mathbb{Z}_M^N \) into a subset of the integer lattice coordinates \( z \in \mathbb{Z}^{2N} \). From the energy constraint in (6), the centroid and the normalization factor are given as

\[
c_0 = \sum_{s' \in \mathbb{Z}_M^N} p(s') L^{2N}(s')G_{2N},
\]

\[
\gamma = \sqrt{N} \left( \sum_{s' \in \mathbb{Z}_M^N} p(s') \|L^{2N}(s')G_{2N}\|^2 - \|c_0\|^2 \right)^{-\frac{1}{2}}.
\]

Note that the centroid is the best offset of the lattice to improve energy efficiency.

B. Greedy Sphere Packing for Constellation Design

Here, we propose a modulation design method using the sphere detection approach in the densest lattices. To achieve the minimum error-rate performance, we need to optimize the labelling function \( L^{2N}() \) which selects the best \( M^N \) points from the densest lattice alignment \( \gamma = zG_{2N} \), so as to minimize the total energy from its centroid. It suggests that higher probable data should be assigned at close to the centroid because they may consume more energy cost.

The main idea of the proposed design method is as follows: We begin by sorting the possible source data \( s \in \mathbb{Z}_M^N \) in the descending order of its probability \( p(s) \), for simplicity, we let \( p(s_0) \geq p(s_1) \geq \cdots \geq p(s_{M^N-1}) \), and next choose the origin \( z = 0_{2N} \) as the corresponding label of the most probable data \( s_0 \), namely, \( L^{2N}(s_0) = 0_{2N} \). We then perform the sphere detection to find the closest lattice points to the origin for the second probable data \( s_1 \) by using a proper radius, after which we update the centroid. Subsequently, we search for the third hypothesis point near to the updated centroid by using the sphere detection. In a similar manner, the procedure is successively iterated until we select all the \( M^N \) lattice points. At last, the selected lattice points are subtracted by the eventual centroid \( c_0 \) and normalized by the energy factor \( \gamma \).

The proposed design algorithm for multi–dimensional modulations is summarized as follows:

1: Sort the possible source data \( s \in \mathbb{Z}_M^N \) in descending order of their occurrence probability \( p(s) \): We let \( p(s_0) \geq p(s_1) \geq \cdots \geq p(s_{M^N-1}) \).
TABLE II

<table>
<thead>
<tr>
<th>B</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>−3.91</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>−6.35</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>−9.42</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>−12.47</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>−15.47</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>−18.48</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>−21.49</td>
</tr>
</tbody>
</table>

Achievable Gain over BPSK of Optimized N–Dimensional QAM

The gains of multi–dimensional signaling shown in Table II can be achieved in a very large SNR regime in which the minimum distance dominates the error rate performance. Meanwhile, the visible gains may be small in reasonable SNR regions. Now, we evaluate the bit error rate (BER) performance of the designed multi–dimensional QAMs in an SNR from 0 dB to 13 dB. Fig. 3 shows the BER performance as a function of the received SNR in AWGN channels. We plot the performance curves of $N$–dimensional $2^{N}$–QAM (1 bps/Hz) and $4^{N}$–QAM (2 bps/Hz) for $N \in \{1, 2, 3, 4, 8, 12\}$. One can see that the BER performance is significantly improved as the dimensionality increases. The $2^{N}$–QAMs achieve the gains over single–dimensional BPSK by about 0.9 dB, 1.8 dB and 3.3 dB for 2, 4 and 12 dimensions, respectively, at a BER of $10^{-5}$. Likewise, this figure confirms a significant advantage of $4^{N}$–QAMs. Note that the curves of $4^{N}$–QAM are shifted leftwards by 3.0 dB when we plot the BER versus $E_b/N_0$.

C. Advantage in Minimum Squared Distance

In Table II, we show the achievable gain (over single–dimensional BPSK) of the multi–dimensional QAM optimized by the proposed mapping design method. As we have seen in Table I, the single–dimensional hexagonal lattice itself has some gains over the standard QAM signaling, whereas the multi–dimensional signaling can enjoy additional benefits as shown in Table II. Note that the four–dimensional QAM has an approximately 3 dB gain, which implies that it can transmit one additional bit compared to the single–dimensional QAM; for example, the 4–dimensional $16^{4}$–QAM (achieving a maximum of 4 bps/Hz) may exhibit an error–rate performance comparable to the one–dimensional 8–QAM (achieving a maximum of 3 bps/Hz). Shown in this table, the Leech lattice for 12–dimensional $2^{12}$–QAM signaling offers an excellent performance of about 5.9 dB gain over the single–dimensional BPSK. For high–level modulations to accommodate $B = 2$ bps/Hz spectral efficiency, the optimized 8–dimensional $4^{8}$–QAM has 1.25 dB gain over BPSK in per–symbol energy efficiency, which corresponds to 4.26 dB gain in per–bit energy efficiency (or, $E_b/N_0$).

In Fig. 2, we illustrate one of the optimized multi–dimensional QAMs, 4–dimensional $4^{4}$–QAM signaling, which achieves the same 1 bps/Hz at maximum per symbol as the single–dimensional BPSK, yet whose minimum squared distance is larger than that of BPSK by 3.01 dB. This signaling comes from the best diamond lattice $G_8$ in real–valued 8–dimension (or, complex–valued 4–dimension). Since the number of constellation points in one symbol becomes 6 to convey 1 bit per symbol, it can be regarded as some sort of block–coded modulations with 4–symbol block length, 6–ary QAM and a coding rate of $1/\log_2(6) \simeq 0.39$. However, it should be noticed that it differs from the conventional block coding because it jointly optimizes the coding and the modulation. Since the optimized coding has non–linearity, we use a look–up table to describe the labelling function in this paper. An efficient labelling remains as a future work.

IV. PERFORMANCE EVALUATIONS

A. Bit Error Rate (BER) Performance

The gains of multi–dimensional signaling shown in Table II can be achieved in a very large SNR regime in which the minimum distance dominates the error rate performance. Meanwhile, the visible gains may be small in reasonable SNR regions. Now, we evaluate the bit error rate (BER) performance of the designed multi–dimensional QAMs in an SNR from 0 dB to 13 dB. Fig. 3 shows the BER performance as a function of the received SNR in AWGN channels. We plot the performance curves of $N$–dimensional $2^{N}$–QAM (1 bps/Hz) and $4^{N}$–QAM (2 bps/Hz) for $N \in \{1, 2, 3, 4, 8, 12\}$. One can see that the BER performance is significantly improved as the dimensionality increases. The $2^{N}$–QAMs achieve the gains over single–dimensional BPSK by about 0.9 dB, 1.8 dB and 3.3 dB for 2, 4 and 12 dimensions, respectively, at a BER of $10^{-5}$. Likewise, this figure confirms a significant advantage of $4^{N}$–QAMs. Note that the curves of $4^{N}$–QAM are shifted leftwards by 3.0 dB when we plot the BER versus $E_b/N_0$.

B. EXIT Chart Analysis

The optimized multi–dimensional modulations can contribute to an energy–efficient transmission as we have seen so far. However, we cannot guarantee that they also provide a significant gain when we use an error–correcting code because we require soft–output demodulations. Some wireless communication systems have recently introduced an iterative decoding principle because of its outstanding error–correction performance and its low–complexity feature. In order to examine whether the multi–dimensional QAMs work well in such an iterative decoding scenario, we make use of the EXIT chart analysis [10,11], which has been invented to predict
the convergence behavior of log–likelihood ratio (or L–value) messages to be exchanged between inner and outer decoders.

In [10], it has been revealed that the well–known Gray mapping for single–dimensional 16QAM signaling is not advantageous for iterative decoding through the EXIT chart analysis because it has almost no iterative gain even when reliable a priori L–value can be given. It implies that the EXIT curve should have a steep gradient as a function of the mutual information of the a priori L–values. We evaluate the EXIT curve in Fig. 4 for N–dimensional $2^N$–QAM signaling in AWGN channels at an SNR of $-3$ dB. All the curves become almost straight lines (except for the 12 dimension), that is alike the results in [10]. The single–dimensional BPSK has a flat property with the increased a priori mutual information, which results in no achievable iterative gain like the Gray–mapped 16QAM. It is confirmed that the increase of the dimensionality can improve the steepness and the area of the extrinsic mutual information curve because of larger block–coding memory size. In consequence, the high–dimensional modulations have still a potential to improve error–rate performance with iterative decoding, in which we employ an appropriate outer code that is designed by matching the EXIT curves.

V. SUMMARY

We have presented an optimization method of multi–dimensional QAM signaling by adopting the best known densest lattices. The design method is assisted by the computationally efficient sphere detection to search for the best constellation points from infinite lattice points. We demonstrated that our optimized multi–dimensional modulations can offer excellent performance gains; for instance, the four–dimensional QAM exhibits approximately 3 dB gain over the single–dimensional one. For iterative decoding applications, we have further analyzed the impact of the increased dimensionality though the use of the EXIT chart. A multi–carrier transmission as well as multiple–antenna systems is one of potential applications for multi–dimensional signaling.

ACKNOWLEDGMENT

This work is partially supported by JSPS Postdoctoral Fellowships for Research Abroad, TAF, and SCAT Japan.

REFERENCES