Robust Power Allocation for Amplify-and-Forward Relay Networks

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Abstract— Relay power allocation has been shown to provide substantial performance gain in wireless relay networks when perfect global channel state information (CSI) is available. In this paper, we consider a more realistic scenario, where such global CSI is subject to uncertainty, and we aim to design robust power allocation protocols for both the coherent and noncoherent amplify-and-forward relay networks. The problem formulation is such that the output signal-to-noise ratio is maximized under both the aggregate and individual relay power constraints. Our previous results show that these optimization problems can be formulated as quasi-convex optimization problems, and are solved using the bisection method via a sequence of conic feasibility problems. We extend these results to the case of uncertain global CSI, and design robust relay power allocations using the robust optimization methodology. For simple ellipsoidal uncertainty sets, the robust counterparts of these optimization problems are semi-definite programs and can be solved efficiently via interior-point methods.

I. INTRODUCTION

To further increase the throughput and power efficiency of relay networks, it is important to study the fundamental performance gain promised by efficient resource allocation [1]–[4]. In particular, we look at the relay power allocation problem for amplify-and-forward (AF) relay networks [5]–[8]. However, all the above works assume that perfect global CSI is present. In practice, such an assumption may be too optimistic since the global channel state information (CSI) is always subject to uncertainty, which can arise as a consequence of imperfect channel estimation, quantization, synchronization errors, hardware limitations, implementation errors, or transmission errors in the feedback channels.1

An important question then arises: Is relay power allocation still beneficial compared to uniform power allocation when the global CSI is subject to uncertainty? That is, how much performance gain do we sacrifice at the expense of designing robust power allocation protocols for relay networks? To formulate our robust counterparts capturing different degrees of robustness, we adopt the worst-case approach [10]–[12]. In this approach, the uncertainty in the global CSI is modeled deterministically by assuming that the CSI is a deterministic variable within a known set of possible values. When the set is a singleton, the CSI is complete and perfect. The bigger the set is, the more uncertainty there is on the actual realization of CSI. The robust counterpart is formulated such that it immunizes the optimization problem against all parameter uncertainty.

In this paper, we consider the optimal relay power allocation problem under global CSI uncertainty for coherent and noncoherent AF relay networks. The problem formulation is such that the output signal-to-noise ratio (SNR) is maximized under both the aggregate and individual relay power constraints. Under perfect global CSI, our previous results show that these optimization problems can be formulated as quasi-convex optimization problems, and can be solved using the bisection method via a sequence of conic feasibility problems in the form of second-order cone programs (SOCPs) [13], [14]. Here, we show how we can apply the robust optimization methodology to these formulations in the case of uncertain global CSI. Under simple ellipsoidal uncertainty sets, we show that the robust counterparts of these problems are semi-definite programs (SDPs) and can be solved efficiently via interior-point methods [15]. Numerical results quantify the trade-off between degree of robustness and performance gain from relay power allocation in both the coherent and noncoherent AF relay networks.

The following notation is used. Boldface upper-case letters denote matrices, boldface lower-case letters denote column vectors, and plain lower-case letters denote scalars. The superscripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^†$ denote the transpose, the complex conjugate, and the transpose conjugate respectively. $I_n$ denotes the $n \times n$ identity matrix, $[B]_{ij}$ denotes the $i,j$th element of the matrix $B$. $\text{tr}(\cdot)$, $\parallel \cdot \parallel$, and $\parallel \cdot \parallel$ denote the trace operator, the absolute value, and the standard Euclidean norm, respectively. $\mathbb{R}_+^K$ and $\mathbb{R}_+^{K+}$ denote the nonnegative and positive orthants in Euclidean vector space of dimension $K$, respectively. $B \succeq 0$ and $B \succ 0$ denote that the matrix is positive semi-definite and positive definite, respectively.

II. PROBLEM FORMULATION

We consider a wireless relay network consisting of $K + 2$ single-antenna nodes: a designated source-destination node pair and $K$ relay nodes located randomly and independently in a domain of a fixed area. Throughout the paper, we assume

1Exactly how this global CSI can be obtained by the network manager is beyond the scope of this paper. Some work related to this can be found in [9].
that $K$ is finite. Furthermore, we assume that there is no direct link between the source and destination nodes. All nodes are in half-duplex mode, so transmission occurs over two time slots via two hops. We consider that all the relay nodes operate in a common frequency band.

In the first time slot, the relay nodes receive the signal transmitted by the source node. After processing the received signals, the relay nodes simultaneously transmit the processed data to the destination node during the second time slot while the source node remains silent. We assume perfect synchronization is attained at the destination node. The received signals at the relay and destination nodes can then be written

$$\begin{align*}
y_R &= h_B x_S + z_R, \\
y_D &= h_F^* y_R + z_D,
\end{align*}$$

(1) (2)

where $x_S$ is the transmitted signal from the source node to the relay nodes, $x_R$ is the $K \times 1$ transmitted signal vector from the relay nodes to the destination node, $y_R$ is the $K \times 1$ received signal vector at the relay nodes, $y_D$ is the received signal at the destination node, $z_R \sim \mathcal{CN}(0, \Sigma_R)$ is the $K \times 1$ circularly symmetric complex Gaussian noise vector, and $z_D \sim \mathcal{CN}(0, \sigma^2_D)$ is a circularly symmetric complex Gaussian random variable. Note that the different noise variances at the relay nodes are reflected in $\Sigma_R \triangleq \text{diag}(\sigma^2_{R,1}, \ldots, \sigma^2_{R,K})$. Moreover, $z_R$ and $z_D$ are independent and are mutually uncorrelated with $x_S$ and $x_R$. With perfect global CSI at the destination node, $h_B$ and $h_F$ are $K \times 1$ known deterministic channel vectors from the source node to the relay nodes and from the relay nodes to the destination node, respectively, where $h_B = [h_{B,1}, \ldots, h_{B,K}]^T \in \mathbb{C}^K$ and $h_F = [h_{F,1}, \ldots, h_{F,K}]^T \in \mathbb{C}^K$. For convenience, we shall refer to $h_B$ as the backward channel and $h_F$ as the forward channel. At the source node, we impose an individual source power constraint $P_S$, such that $\mathbb{E}\{|x_S|^2\} \leq P_S$. Similarly, at the relay nodes, we impose both an aggregate relay power constraint $P_R$ and an individual relay power constraint $P$, where $Q_R \triangleq \mathbb{E}\{x_R^* x_R\} h_B^*$, $\text{tr}(Q_R) \leq P_R$, and $P_k \triangleq \mathbb{E}\{|Q_R|^2\} \leq P$ for $k = 1, \ldots, K$. Note that $P_k$ is the power allocated to the $k$th relay.

Our goal is to improve system performance by optimally allocating power to the relays. In our model, we will adopt the SNR at the destination node as the quality-of-service metric and consider AF relaying protocol for each relay node. For the AF relaying protocol, the relay nodes simply transmit the exact signals they have received, scaled to meet the power constraints. In this case, $x_R$ in (2) is given by

$$x_R = G y_R,$$

(3)

where $G$ denotes the $K \times K$ diagonal relay gain matrix, and $Q_R$ becomes

$$Q_R = G \left( P_S h_B h_B^* + \Sigma_R \right) G^*.$$

(4)

The diagonal structure of $G$ implicitly assumes that each relay node only knows the signal it receives, and has no knowledge about the signals at the other relays. When each relay node has access to its own locally-bidirectionally CSI, it can perform distributed beamforming.\(^2\) As such, we denote this as coherent AF relaying, and the $k$th diagonal element of $G$ is given by [16]–[18]

$$g^{(k)}_{\text{coh}} = \sqrt{\beta_k} P_k \frac{h_{F,k}^*}{|h_{B,k}|},$$

(5)

where $\beta_k \triangleq 1/(P_S |h_{B,k}|^2 + \sigma^2_{R,k})$ is defined for notational simplicity. On the other hand, in the absence of forward CSI at each relay node, the relay node simply forwards a scaled version of its received signal without additional processing. In this case, this AF protocol is denoted as noncoherent AF relaying and the $k$th diagonal element of $G$ is given by [17], [19]

$$g^{(k)}_{\text{noncoh}} = \sqrt{\beta_k} P_k.$$

(6)

Using (1)–(3), the received signal at the destination node conditioned on $h_B$ and $h_F$ is given by

$$y_D = h_F^* G h_B x_S + z_D = h_F^* G h_B x_S + h_F^* G z_R + z_D,$$

(7)

where $z_D$ represents the effective noise vector at the destination node, and the instantaneous SNR at the destination node is defined as

$$\text{SNR} = \frac{P_k h_F^* G h_B h_B^* G h_F^*}{h_F^* G \Sigma_R G^* h_F^* + \sigma^2_D},$$

(8)

and we can formulate the relay power allocation problem mathematically as follows:

\[
\begin{align*}
\text{maximize} & \quad \text{SNR} \\
\text{subject to} & \quad \text{tr}(Q_R) \leq P_R, \\
& \quad 0 \leq |Q_{R,k}| \leq P, \quad \forall k \in \{1, \ldots, K\}.
\end{align*}
\]

(9)

III. OPTIMAL RELAY POWER ALLOCATION

In the section, we present some of our previous results without proof [13], [14].

**Proposition 1:** The coherent AF relay power allocation problem in (9) is a quasiconvex optimization problem with a nonempty and compact set of maxima. It can be equivalently written as

$$P_{\text{coh}} : \begin{align*}
\text{maximize} & \quad \epsilon(\zeta) \triangleq \frac{P_k}{\eta_p} \left( \sum_{k=1}^{K} \frac{\epsilon^* k}{\eta p^{1+T}} \right) \\
\text{subject to} & \quad \sum_{k=1}^{K} \zeta_k \leq 1, \\
& \quad 0 \leq \zeta_k \leq \sqrt{\eta_p}, \quad \forall k \in \{1, \ldots, K\},
\end{align*}$$

where $\zeta_k \triangleq \sqrt{\frac{P_k}{\eta_p}}$ is the decision variable of the optimization problem, $\eta_p = P/P_R$ denotes the ratio between the individual relay power constraint and the aggregate relay power constraint, and $0 < \eta_p \leq 1$. In addition, $\epsilon = [\epsilon_1, \ldots, \epsilon_K]^T \in \mathbb{R}^K$ and $A = \text{diag}(a_1, \ldots, a_K) \in \mathbb{R}_+^{K \times K}$ are defined for

\(^2\) Here, locally-bidirectional CSI refers to the knowledge of only $h_{B,k}$ and $h_{F,k}$ at the $k$th relay node.
notational convenience, where the $k$th element of $c$ and $A$ are given by
\begin{align}
c_k &= \sqrt{\beta_k P_{\text{r}}} \left| h_{B,k} \right| \left| h_{F,k} \right|, \\
a_k &= \frac{\sqrt{\beta_k P_{\text{r}}} \left| h_{F,k} \right| \sigma_{R,k}}{\sigma_D}.
\end{align}

Lemma 1: The program $\mathcal{P}_{\text{coh}}$ in Proposition 1 can be solved numerically by the bisection method via a sequence of convex feasibility programs in the form of an SOCP, given as follows:
\begin{equation}
\mathcal{P}_{\text{coh}}^{(\text{SOCP})} : \begin{cases}
\underset{\zeta}{\text{find}} & \zeta \\
\text{subject to} & \zeta \in \mathcal{S}_{\text{coh}}(t),
\end{cases}
\end{equation}
with the feasible set $\mathcal{S}_{\text{coh}}(t)$ given by
\begin{equation}
\mathcal{S}_{\text{coh}}(t) = \left\{ \zeta \in \mathbb{R}^K : \begin{bmatrix} \frac{c^T \zeta}{\sqrt{\frac{P_{\text{r}}}{P_h}}} & 1 \end{bmatrix} \succeq_k 0, \quad \begin{bmatrix} 1 \end{bmatrix} \succeq_k 0 , \quad \zeta_k \leq \sqrt{\frac{P_h}{P_{\text{r}}}}, \quad \forall k \in \{1, \ldots, K\} \right\}.
\end{equation}

Proposition 2: The noncoherent AF relay power allocation problem in (9) can be approximated as a quasiconvex optimization problem with a nonempty and compact set of maxima, and it can be equivalently written as
\begin{equation}
\mathcal{P}_{\text{noncoh}} : \begin{cases}
\text{maximize } c_{\text{noncoh}}(\zeta, L) = \frac{P_{\text{r}}}{\sigma_D} \left( \rho_L (c^T \zeta)^2 \right) & \\
\text{subject to} & \sum_{k=1}^K \zeta_k^2 \leq 1, \\
& 0 \leq \zeta_k \leq \sqrt{\frac{P_h}{P_{\text{r}}}}, \quad \forall k \in \{1, \ldots, K\},
\end{cases}
\end{equation}
where $\zeta_k = \sqrt{\frac{P_h}{P_{\text{r}}}}$ and $\rho_L (c^T \zeta)$ is the polyhedral approximation of $|c^T \zeta|$. In addition, we have defined $c = [c_1, \ldots, c_K]^T \in \mathbb{C}^K$, and $A = \text{diag}(a_1, \ldots, a_K) \in \mathbb{R}_+^{K \times K}$ for notational convenience, where the $k$th element of $c$ and $A$ are given by
\begin{align}
c_k &= \sqrt{\beta_k P_{\text{r}}} \left| h_{B,k} \right| \left| h_{F,k} \right|, \\
a_k &= \frac{\sqrt{\beta_k P_{\text{r}}} \left| h_{F,k} \right| \sigma_{R,k}}{\sigma_D}.
\end{align}

Lemma 2: The program $\mathcal{P}_{\text{noncoh}}$ in Proposition 2 can be decomposed into 2L subproblems, where each subproblem can be solved numerically by bisection method via a sequence of convex feasibility problems in the form of SOCP. The optimal solution $\zeta^*$ is then given by the subproblem that gives the maximum $c_{\text{noncoh}}(\zeta^*, L)$.

IV. ROBUST RELAY POWER ALLOCATION

To account for data uncertainty in our optimization problems, we adopt a robust optimization methodology rigorously formulated in [10], [11]. Specifically, we treat uncertainty as an uncertainty set containing the nominal data, which is assumed to be known completely, and a collection of data in neighboring regions. This methodology ensures that the constraints remain feasible for all realizations of the uncertainties within the deterministic uncertainty set.\(^3\) Without loss of generality, we follow the uncertainty set representation in [10], [11], and consider an ellipsoidal uncertainty set for simplicity.

A. Coherent AF

In order to incorporate uncertainties in $A$ and $c$ into Lemma 1, we formulate the robust counterpart of the feasibility program in Lemma 1 in the following theorem.

Theorem 1: The robust coherent AF relay power allocation problem conditioned on partial knowledge of the global CSI at the destination node can be solved numerically via the bisection method shown in Lemma 1, except that the feasibility program for a given $t$ is now replaced by its robust counterpart given by
\begin{equation}
\mathcal{P}_{\text{coh}}^{(\text{worst})} : \begin{cases}
\text{find } \zeta & \zeta \in \mathcal{W}_{\text{coh}}(t), \\
\text{subject to } & A \in \mathcal{U}_1, c \in \mathcal{U}_2,
\end{cases}
\end{equation}
where $A \in \mathbb{R}_+^{K \times K}$ and $c \in \mathbb{R}_+^K$ in Proposition 1 are assumed to subject to sidewise independent ellipsoidal uncertainty sets $\mathcal{U}_1$ and $\mathcal{U}_2$ given, as follows:
\begin{align}
\mathcal{U}_1 &= \left\{ A = A_0 + \sum_{j=1}^{N_A} z_j A_j : \| z \| \leq \rho_1 \right\}, \\
\mathcal{U}_2 &= \left\{ c = c_0 + \sum_{j=1}^{N_c} u_j c_j : \| u \| \leq \rho_2 \right\},
\end{align}
where $N_A$ and $N_c$ are the sizes of the primitive uncertainties in (16) and (17), respectively. Then, the optimal solution $\zeta_{\text{worst}}^*$ of the robust counterpart in (16) is also the optimal solution of the following SDP in the variables $\zeta \in \mathbb{R}_+^K, \tau \geq \sqrt{\frac{P_h}{P_{\text{r}}}}, \mu \in \mathbb{R}_+$:
\begin{equation}
\begin{aligned}
\text{find } \zeta & \zeta \in \mathcal{W}_{\text{coh}}(t), \\
\text{subject to } & \zeta \in \mathcal{W}_{\text{coh}}(t),
\end{aligned}
\end{equation}
such that the feasible set $\mathcal{W}_{\text{coh}}(t)$ is given by
\begin{equation}
\mathcal{W}_{\text{coh}}(t) = \left\{ \zeta \in \mathbb{R}_+^K : \begin{bmatrix} \tau & \mathbf{A}_0 \zeta & \mathbf{A} \\
\mathbf{A}^T & \tau \mu I_{N_A} & \mathbf{0}_{N_A} \\
\mathbf{0}_{N_A}^T & \mathbf{0}_{N_A} & \mathbf{0}_{N_A} \\
\right] \succeq 0, \\
\begin{bmatrix} (c_0 - \zeta)^T \zeta \\
\zeta \\
\sqrt{\frac{P_h}{P_{\text{r}}}} (\zeta^T c_k)^{\frac{1}{2}} \end{bmatrix} \succeq 0, \\
\begin{bmatrix} I_{N_c} \zeta \\
\zeta \\
\sqrt{\frac{P_h}{P_{\text{r}}}} (\zeta^T c_k)^{\frac{1}{2}} \end{bmatrix} \succeq 0, \\
\forall k \in \{1, \ldots, K\},
\end{aligned}
\end{equation}
where $\mathbf{A} = [A_1 \zeta, \ldots, A_{N_A} \zeta]$ and $\mathbf{c} = [c_1 \zeta, \ldots, c_{N_c} \zeta]$.\(^3\)

Proof: See Appendix I.\(^\square\)

\(^3\)As a result, this approach is conservative against data uncertainty and produces the worst-case solution.
The results follow straightforwardly from the analysis, where the perfect knowledge of subproblems using Lemma 2. Under the sidewise independent assumptions, we can still adopt the bisection method in Section III and IV. We assume that all plots are averaged over 10 independent Monte Carlo simulation runs.

Theorem 2: The robust noncoherent AF relay power allocation problem conditioned on partial knowledge of the global CSI at the destination node can be decomposed into 2L subproblems using Lemma 2. Under sidewise independent ellipsoidal uncertainty sets $U_1$ and $U_2$ given by (16) and (17), these subproblems can be solved numerically using the bisection method via a sequence of convex feasibility problems in the form of SDPs. Therefore, the robust optimal solution of $\mathcal{P}_{\text{normcoh}}$ with uncertainty in $\mathbf{A}$ and $\mathbf{c}$ can be obtained from solving these 2L subproblems, and it is equal to the feasible solution that maximizes $f_{\text{normcoh}}(\xi^*, L)$.

Proof: The results follow straightforwardly from the proofs for Lemma 2 and Theorem 1. As a result, we have omitted it due to space constraint.

V. NUMERICAL RESULTS

In this section, we illustrate the performance gain achieved by the relay power allocation for both the coherent and noncoherent AF relay networks using simple numerical results. Throughout the numerical results, we use the SeDuMi convex optimization MATLAB toolbox [20] to compute the optimal and robust relay power allocations for $\varepsilon = 0.001$ using the bisection method in Section III and IV. We assume that $\mathbf{h}_B$ and $\mathbf{h}_F$ are mutually independent circularly symmetric complex Gaussian noise vectors, such that $\mathbf{h}_{B,k} \sim \mathcal{CN}(0, 1)$ and $\mathbf{h}_{F,k} \sim \mathcal{CN}(0, 1)$ for all $k$. Without loss of generality, we consider a symmetric network topology and normalize the noise variances, such that $\sigma^2_{R,k} = 1$ and $\sigma^2_{D} = 1$. The uncertainty sets in Theorems 1 and 2 are chosen such that $N_A = 1$, $N_C = 1$, $\mathbf{A}_1 = \mathbf{A}_0$, and $\mathbf{c}_1 = \mathbf{c}_0$. Note that $\rho_1 = 0$ and $\rho_2 = 0$ corresponds to perfect knowledge of nominal data, and $\rho_1 = 1$ and $\rho_2 = 1$ corresponds to an uncertainty that is of the same size as the nominal data itself. In addition, we consider $\rho = \rho_1 = \rho_2$ in the following numerical results. Since we are only interested in relay power allocation, we fix $P_S/\sigma^2_D = 10\log 30$ dB. All plots are averaged over 10 independent Monte Carlo simulation runs.

In Figs. 1 and 2, we show that the effect of uncertain global CSI on the worst-case average output SNR with non-robust relay power allocation for both the coherent and noncoherent AF channels, respectively. By non-robust power allocation, we refer to optimization protocols in Section III that obtain relay power allocations only based on the knowledge of $\mathbf{A}_0$ and $\mathbf{c}_0$, and treating all uncertainty as if it is not present. Clearly, we see that simply ignoring CSI uncertainty in our designs can be detrimental when the size of the uncertainty is significant. In Fig. 1, we can observe that when the size of uncertainty is less than 10% of the nominal data, we can still adopt the conventional approach of ignoring CSI uncertainty. However, we pay a high price in performance as the size of uncertainty increases. For example, the average output SNR is 6 dB lower than that obtained from uniform power allocation when $\rho = 0.5$. Therefore, it is better to resort to uniform relay power allocation whenever the global CSI has too much uncertainty for the coherent AF relay network. On the other hand, we see that the non-robust relay power allocation can still achieve better performance than the uniform relay power allocation for noncoherent AF relay network. Nevertheless, performance degradation still exists as $\rho$ increases for the noncoherent case. In summary, we can conclude that relay power allocation is sensitive with respect to global CSI uncertainty, and the degree of performance degradation depends on the type of AF relaying protocol used.

In Figs. 3 and 4, we plot the worst-case average output SNR for both the coherent and noncoherent AF relay networks as a function of the size of the uncertainty set when $K = 20$ and $\eta_{\rho} = 0.2$, respectively. In Fig. 3, we can see that the robust design can provide some performance gain about 0.5 dB over uniform power allocation, provided that the size of the uncertainty set is not too large. When $\rho$ is too large, the robust
optimal value is reduced considerably so as to ensure that all realizations of the uncertainty set are included. In such cases, it is better to resort to uniform power allocation. In Fig. 4, we can see that robust design provides a more significant gain of about 7.5 dB over the uniform power allocation in noncoherent AF relay network. Therefore, it is more attractive to employ robust design in noncoherent AF relay network.

VI. CONCLUSIONS

In this paper, we obtained the robust relay power allocation that maximizes the output SNR of the coherent and noncoherent AF relay networks, where the global CSI is subject to uncertainty. For simple ellipsoidal uncertainty sets, we showed that the robust counterparts of our optimization problems are SDPs and can be solved efficiently via interior-point methods. Numerical results revealed that the gain achieved by relay power allocation for noncoherent AF relay network is much greater than that of coherent AF relay network. In addition, performance degradation in the presence of uncertain global CSIs can be significant, thus, motivating the use of robust power allocation designs. The trade-off between the degree of robustness and performance gain from power allocations for both the coherent and noncoherent AF relay networks are quantified.

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APPENDIX I

PROOF OF THEOREM 1

Since only the data \((A, c)\) in the first constraint of (13) is subject to uncertainty, we will focus on this constraint and build its robust counterpart given by

\[
    c^T \zeta \geq \sqrt{\frac{t \sigma_D^2}{P_S}} (1 + \|A \zeta\|^2), \quad \forall (A, c) \in \mathcal{U},
\]

where the data \((A, c)\) is known to belong to some uncertainty set \(\mathcal{U}\). Now, we assume that the uncertainty affecting (19) is sidewise, which implies that we can decouple the uncertainty set \(\mathcal{U}\) into two independent uncertainty sets \(\mathcal{U}_1\) and \(\mathcal{U}_2\). More specifically, we have the following fact that \(\zeta\) is robust feasible for (19) if and only if there exists \(\tau \in \mathbb{R}_+\) such that [21], [22]

\[
    \sqrt{\frac{t \sigma_D^2}{P_S}} (1 + \|A \zeta\|^2) \leq \tau, \quad \forall A \in \mathcal{U}_1 \quad \text{(20)}
\]

\[
    \tau \leq c^T \zeta, \quad \forall c \in \mathcal{U}_2, \quad \text{(21)}
\]

where \(\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2\) and the respective uncertainty sets are in the form of simple ellipsoidal uncertainty as given by (16) and (17). As a result, we are able to handle (19) by treating (20) and (21) separately.

Now, let us first consider (20) by rewriting it as follows:

\[
    \|A \zeta\| \leq \sqrt{\frac{P_S}{t \sigma_D^2}}, \quad \forall A \in \mathcal{U}_1. \quad \text{(22)}
\]

Thus, by letting \(\lambda = \tau \sqrt{\frac{P_S}{t \sigma_D^2}}\) and choosing \(\lambda\) such that it is always greater than one, we can then rewrite (22) as

\[
    -\|A_0 \zeta + A \bar{z}\| + \lambda \geq 0, \quad \forall \bar{z} \in \{z : \|z\| \leq \rho_1\}, \quad \lambda \geq 1, \quad \text{(23)}
\]
where we have substituted the ellipsoidal uncertainty set $\mathcal{U}_1$ defined in (16). Now, by expanding (23) in terms of a quadratic form of $z$, we have

$$q_0(z) = -z^T A^T A z - 2 \begin{bmatrix} A^T A \xi \end{bmatrix}^T z - \xi^T A_0^T A_0 \xi + \lambda^T \geq 0,$$

and $\lambda \geq 1$. By letting $q_1(z) = \rho_1^2 - z^T z$, we can obtain the robust counterpart in (22) by directly applying the S-lemma [23],\(^4\) Thus, it follows that (24) is satisfied if and only if there exists $\alpha \in \mathbb{R}_+$ such that

$$\begin{bmatrix} \lambda^2 - \xi^T A_0^T A_0 \xi & \begin{bmatrix} -A_0^T A_0 \xi \\ -A_0^T A \xi \\ -A^T A \end{bmatrix} \\ \begin{bmatrix} A_0^T A_0 \xi \\ A^T A \xi \\ A^T A \end{bmatrix} \end{bmatrix} - \alpha \begin{bmatrix} \rho_1^2 & 0^T_{N_A} \\ 0_{N_A} & -I_{N_A} \end{bmatrix} \succeq 0.$$  

(25)

To convert the above quadratic matrix inequality into linear matrix inequality, we first let $\alpha = \lambda \mu$ for some $\mu \in \mathbb{R}_+$. Indeed, this is trivially true for $\lambda > 1$. When $\lambda = 1$, we can easily see that $A_0 = 0$ and $A = 0$ from (24) and (25), which is the same condition in (25) for $\alpha = 0$. Rearranging (25), we have

$$\lambda^2 - \xi^T A_0^T A_0 \xi + \frac{\xi^T A_0^T A_0 \xi}{\mu I_{N_A}} - \frac{\xi^T A_0^T A_0 \xi}{\mu I_{N_A}} \succeq 0.$$  

(26)

To linearize (26), we use the Schur complement [23]. Therefore, it follows that under condition (22) and $\mathcal{U}_1$ defined in (16), a pair $(\xi, \tau)$ satisfies (22) if and only if there exists some $\mu \in \mathbb{R}_+$ and $\tau \geq \sqrt{\xi^T \frac{T_A}{T_S} / \mu T_S}$ such that the triple $(\xi, \tau, \mu)$ satisfies the following LMI:

$$\begin{bmatrix} I_K & \begin{bmatrix} A_0 \xi \\ A^T \xi \\ 0_{N_A} \end{bmatrix} & \begin{bmatrix} A \\ A^T \\ \mu I_{N_A} \end{bmatrix} \\ \begin{bmatrix} A_0 \xi \\ A^T \xi \\ 0_{N_A} \end{bmatrix} & \tau^2 \xi^T \frac{T_A}{T_S} \begin{bmatrix} A_0^T A_0 \\ A^T A \\ 0_{N_A} \end{bmatrix} & \begin{bmatrix} A_0^T A_0 \\ A^T A \\ 0_{N_A} \end{bmatrix} \end{bmatrix} \succeq 0.$$  

(27)

Now, we turn to the condition (21). By substituting $\mathcal{U}_2$ defined in (17) into (21), we have equivalently

$$c u - (c_0^T \xi - \tau), \quad \forall u \in \mathcal{U}_2 = \{ u \in \mathbb{R}^n : \| u \| \leq \rho_2 \}.$$  

(28)

Thus, the robust linear constraint in (28) can be expressed as

$$\min \{ c u : u \in \mathcal{U}_2 \} = -\rho_2 \| c \| \geq -\left( e_0^T \xi - \tau \right),$$  

(29)

which is evidently an SOC constraint. Therefore, it follows that under the condition (21) and $\mathcal{U}_2$ defined in (17), the equivalent robust counterpart of (21) can be represented in the form of LMI given by

$$\begin{bmatrix} (e_0^T \xi - \tau) / \rho_2 & e^T \xi \\ e & (e_0^T \xi - \tau) / \rho_2 \end{bmatrix} \succeq 0.$$  

(30)

Combining the above results, we obtain the desired results in (18).

\(^4\)Sometimes, S-lemma is also referred to as S-procedure, and it is frequently used in system theory to derive stability and performance results for nonlinear and uncertain systems.

REFERENCES


