Abstract

Varieties like groups, rings, or Boolean algebras have the property that, in any of their members, the lattice of congruences is isomorphic to a lattice of more manageable objects: e.g. normal subgroups of groups, two-sided ideals of rings, filters (or ideals) of Boolean algebras. Abstract algebraic logic can explain these phenomena at a rather satisfactory level of generality: in every member $A$ of a $\tau$-regular variety $V$, in fact, the lattice of congruences of $A$ is isomorphic to the lattice of deductive filters on $A$ of the $\tau$-assertional logic of $V$. Moreover, if $V$ has a constant 1 in its type and is 1-subtractive, the deductive filters on $A \in V$ of the 1-assertional logic of $V$ coincide with the $V$-ideals of $A$ in the sense of Gumm and Ursini, for which we have a manageable concept of ideal generation.

However, there are isomorphism theorems e.g. in the theories of residuated lattices, pseudointerior algebras and quasi-MV algebras that cannot be subsumed by these general results. The aim of the present paper is to appropriately generalise the concepts of subtractivity and $\tau$-regularity in such a way as to shed some light on the deep reason behind such theorems, as well as (possibly) many more. The tools and concepts we develop hereby provide a common umbrella for the algebraic investigation of several families of logics, including substructural logics, modal logics, quantum logics, logics of constructive mathematics.

1 Introduction

Universal algebraists have invariably been intrigued by varieties - such as groups, rings, or Boolean algebras - having the property that congruences on any algebra in the variety can be replaced by suitably chosen subsets of the universes of the algebras themselves: e.g. normal subgroups of groups, two-sided ideals of rings, filters (or ideals) of Boolean algebras. This property can take several different forms, although most typically it arises, in any member of the variety at issue, as a lattice isomorphism between the lattice of congruences and the lattice of these
"special" subsets. Being in a position to leave aside such irksome objects as congruences and work with more manageable ones is such an obvious advantage that several people bothered to look at the reason why such pleasant theorems occur. The received view has it that these phenomena can be explained at a rather satisfactory level of generality by the following facts:

- In every member $A$ of a $\tau$-regular variety $V$ there is a lattice isomorphism between the lattice of congruences of $A$ and the lattice of deductive filters on $A$ of the $\tau$-assertional logic of $V$. In fact, the demand that $V$ is $\tau$-regular, for a finite set $\tau$ of formal equations, is precisely the demand that $V$ arises as the equivalent algebraic semantics of an algebraisable deductive system, and this is precisely what forces such a lattice isomorphism. In particular, if $V$ has a constant $1$ in its type and is $1$-regular, in every member $A$ of $V$ we have an isomorphism between the lattice of congruences of $A$ and the lattice of deductive filters on $A$ of the $1$-assertional logic of $V$ ([7], [9]; [6]; [21, Section 4.4]; [17]).

- If, moreover, $V$ is also $1$-subtractive, then the deductive filters on $A \in V$ of the $1$-assertional logic of $V$ coincide with the $V$-ideals of $A$ in the sense of Gumm and Ursini [27], which is even better in that deductive filters might be very hard to describe in general, while for $V$-ideals this is less likely thanks to the availability of a manageable concept of ideal generation [9].

1-ideal determined varieties (that is, 1-subtractive and 1-regular varieties) are therefore, at least under this particular respect, the garden of Eden of the algebraist. However, upon a closer inspection it turns out that Eden is actually much larger than that. Even if we disregard the fact that the notion of ideal determined variety only makes sense for pointed varieties, results along the same lines abound in the literature that are not explained by what we said above. For example:

- The variety $\mathbb{PI}$ of pseudointerior algebras [8], introduced by Blok and Pigozzi as a tool for the investigation of varieties with a commutative regular TD term, is not 1-subtractive; however, it allows for a very manageable concept of open filter (provably distinct from the concept of $\mathbb{PI}$-ideal). It can be shown that in every pseudointerior algebra there is an isomorphism between the lattices of congruences and of open filters.

- The variety $\mathbb{RL}$ of residuated lattices [23] is 1-ideal determined and, in fact, it is well-known that in every residuated lattice the lattice of congruences is isomorphic to the lattice of $\mathbb{RL}$-ideals. It is likewise known that $\mathbb{RL}$-ideals on any residuated lattice coincide with convex normal subalgebras of such. There is a further isomorphism theorem, however (namely, between congruences and deductive filters in the sense of [23]), which cannot be explained by recourse to the general results we mentioned earlier.

- The variety $q\mathbb{MV}$ of quasi-MV algebras [35], generalisations of MV algebras introduced in the context of an investigation into the foundations of quan-
tum computing, is neither 1-subtractive nor 1-regular; still, in every quasi-MV algebra $A$ the lattice of $qMV$-congruences (where $\theta \in \text{Con}(A)$ is said to be $qMV$-iff $A/\theta$ is an MV algebra) is isomorphic to the lattice of the deductive filters on $A$ of infinite-valued Lukasiewicz logic.

The aim of the present paper is to appropriately generalise the concepts of subtractivity - and, in a final section, of $\tau$-regularity - in such a way as to shed some light on the deep reason behind such results, as well as (possibly) many more.

2 $\tau$-permutable varieties

Recall that a variety $V$ with (at least) a constant 1 in its type is 1-permutable iff for any algebra $A \in V$ and for any congruences $\theta, \varphi$ of $A$, $1^A/\theta \circ \varphi = 1^A/\varphi \circ \theta$. In their paper [27], Gumm and Ursini essentially observe that a variety $V$ with 1 is 1-permutable iff it is 1-subtractive (cf. also [19]).

In their article on assertionally equivalent quasivarieties [10], Blok and Raftery introduce a notion of $\tau$-congruence class which relativises the usual notion of congruence class to a given translation (i.e., to a set of equations which we assume henceforth to be finite: see [10] for the details). Although the paper at issue considers the more general setting of quasivarieties, in the present discussion we shall restrict ourselves to varieties: if $V$ is a variety, $A \in V$, $\theta$ is a congruence of $A$ and $\tau = \{ \delta_i(x) \approx \epsilon_i(x) : i \leq n \}$ is a translation in the similarity type of $V$, the $\tau$-congruence class of $\theta$ in $A$ - in symbols $\tau^A/\theta$ - is defined as

$$\tau^A/\theta = \{ a \in A : \delta_i^A(a) \theta \epsilon_i^A(a) \text{ for every } i \leq n \}.$$.

Blok and Raftery explicitly consider a property of $\tau$-permutability appropriately generalising the notion of 1-permutability to varieties which need not be pointed: a variety $V$ is $\tau$-permutable iff for any congruences $\theta, \varphi$ on any $A \in V$, $\tau^A/\theta \circ \varphi = \tau^A/\varphi \circ \theta$.

We prove the following result, which bears obvious connections to Proposition 14.2.6 in [10].

**Theorem 1** Let $\tau = \{ \delta_i(x) \approx \epsilon_i(x) : i \leq n \}$, where $\epsilon^A_i$ is a constant operation on every $A \in V$. Then $V$ is a $\tau$-permutable variety iff there exist $n$ binary terms, denoted by $\rightarrow_1, ..., \rightarrow_n$ and written in infix notation, such that $V$ satisfies the following equations for any $i \leq n$:

$$\delta_i(x) \rightarrow_i x \approx \epsilon_i(x)$$

$$\epsilon_i(x) \rightarrow_i x \approx \delta_i(x)$$

**Proof.** From right to left, take $a \in A \in V$ and let $a \in \tau^A/\theta \circ \varphi$. This means that there is some $b \in A$ s.t., for every $i \leq n$, $\delta_i^A(a) \varphi b$ and $b \theta \epsilon_i^A(a)$. By the above equations, then, $(\epsilon_i^A(a), b \rightarrow_i a) = (\delta_i^A(a) \rightarrow a, b \rightarrow_i a) \in \varphi$ and
\[ (b \rightarrow_i a, \delta_i^A(a)) = (b \rightarrow_i a, \epsilon_i^A(a) \rightarrow a) \in \theta \text{ for every } i \leq n, \text{ whence } a \in \tau^A / \varphi \circ \theta. \]

From left to right, let \( F \) be the free \( \mathcal{V} \)-algebra with free generators \( x, y \). Let \( \theta_i = \theta_F(\epsilon_i(y), x), \varphi_i = \theta_F(x, \delta_i(y)) \). Then \( y \in \tau^F / \theta_i \circ \varphi_i \) and, since \( \mathcal{V} \) is \( \tau \)-permutable, \( y \) also belongs to \( \tau^F / \varphi_i \circ \theta_i \); said otherwise, there is \( z \) s.t. \((z, \delta_i(y)) \in \theta_i \) and \((\epsilon_i(y), z) \in \varphi_i \). Now, recall that \( F \) is the free \( \mathcal{V} \)-algebra with free generators \( x, y \), whence \( z \) is simply a binary term which we may denote by \( x \rightarrow_i y \). Thus, we can rephrase the information collected so far as follows:

\[
(\epsilon_i(y), x \rightarrow_i y) \in \theta_F(x, \delta_i(y)) \quad (x \rightarrow_i y, \delta_i(y)) \in \theta_F(\epsilon_i(y), x)
\]

By properties of free algebras and our assumption concerning \( \epsilon^F_i \), then, \( \mathcal{V} \) satisfies the above equations. \( \blacksquare \)

### 3 Quasi-subtractive varieties

#### 3.1 Definition and examples

We now want to focus on varieties whose type \( \nu \) includes a constant 1 and a unary term \( \Box \), and in particular on \( \tau \)-permutable varieties where \( \tau \) is the singleton \( \{\Box x \approx 1\} \). We call these varieties \( (\Box x, 1) \)-permutable. By Theorem 1, every such variety satisfies the equations

\[
\Box x \rightarrow x \approx 1 \\
1 \rightarrow x \approx \Box x
\]

for some binary implication term \( \rightarrow \). Within this class, we single out by means of the next definition a subclass which still counts as a generalisation of subtractive varieties.

**Definition 2** A variety \( \mathcal{V} \) whose type \( \nu \) includes a nullary term 1 and a unary term \( \Box \) is called *quasi-subtractive* w.r.t. 1 and \( \Box \) iff there is a binary term \( \rightarrow (x, y) \) (hereafter written in infix notation) of type \( \nu \) s.t. \( \mathcal{V} \) satisfies the equations

Q1 \( \Box x \rightarrow x \approx 1 \)
Q2 \( 1 \rightarrow x \approx \Box x \)
Q3 \( \Box (x \rightarrow y) \approx x \rightarrow y \)
Q4 \( \Box (x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1 \)

Whenever no risk of confusion is present, reference to 1 and \( \Box \) will be left implicit. When discussing particular examples, we will sometimes say that "the terms \( \rightarrow, \Box \) and 1 witness quasi-subtractivity for \( \mathcal{V} \)". meaning that \( \mathcal{V} \) is quasi-subtractive w.r.t. 1 and \( \Box \) and that \( \rightarrow \) is the binary term referred to in Definition
2. On other occasions, we will instead say that \( e \rightarrow \) witnesses quasi-subtractivity w.r.t. 1 and \( \Box \) for \( \forall^n \). These expressions, as well as slight stylistical variants thereof, should be understood as synonymous. Members of quasi-subtractive varieties will be called, by extension, quasi-subtractive as well.

**Lemma 3** Every quasi-subtractive variety satisfies the equations: (i) \( \Box 1 \approx 1 \); (ii) \( \Box x \approx \Box x \).

**Proof.** Let \( a \in A \in V \). (i) By Q1 and Q3,

\[
\Box 1 = \Box (\Box a \rightarrow a) = \Box a \rightarrow a = 1.
\]

(ii) By Q2 and Q3,

\[
\Box a = 1 \rightarrow a = \Box (1 \rightarrow a) = \Box a.
\]

We now list some examples of quasi-subtractive varieties.

**Example 4** (Subtractive varieties). Every 1-subtractive variety \( V \) ([42], [2], [3], [4]) is quasi-subtractive: it suffices to take as arrow the term witnessing 1-subtractivity for \( V \), and as box the identity term.

**Example 5** (Pointed varieties). Let \( V \) be any pointed variety, i.e., a variety whose type includes a constant 1. Defining \( \Box x = 1 = x \rightarrow y \) it is immediately verified that \( V \) is quasi-subtractive with the above witness terms.

This last example may be uncharitably interpreted as showing that the theory we develop is trivial, but we prefer to think of it as showing that quasi-subtractivity is an essentially relative notion. Non-trivial cases arise only with non-trivial choices of witness terms.

**Example 6** (Nilpotent shifts of subtractive varieties). Let \( \nu \) be a similarity type. If \( t, s \) are terms of type \( \nu \), the equation \( t \approx s \) is called normal iff either (i) \( t \) and \( s \) are the same variable; or else (ii) neither \( t \) nor \( s \) is a variable. If \( V \) is a variety of type \( \nu \), the nilpotent shift of \( V \) (denoted as \( N(V) \)) ([37], [25], [13], [14]) is the variety satisfying exactly the normal equations satisfied by \( V \).

If \( V \) is a 1-subtractive variety, its nilpotent shift \( N(V) \) is a quasi-subtractive variety: it suffices to take as arrow the term witnessing 1-subtractivity for \( V \), and as box the assigned term of \( V \) (in the sense of [14]).

**Example 7** (Residuated lattices). Residuated lattices (see e.g. [23]) are the equivalent algebraic semantics of the 0-free fragment \( RL \) of the substructural logic \( FL \), and arise in several different areas of mathematics. The variety \( RL \) of residuated lattices is both 1-subtractive and 1-regular; however: (i) 1-subtractivity and 1-regularity are witnessed by terms bearing no connection whatsoever to each other; (ii) the \( RL \) ideals of a residuated lattice \( A \) are its
convex normal subalgebras, which are not the deductive filters on $A$ of the substructural logic $RL$. The latter coincide with the upward closures (w.r.t. the residuated lattice order) of convex normal subalgebras.

The variety $RL$ is quasi-subtractive, witness the terms $(x \setminus y) \land 1, x \land 1 \text{ and } 1$. Remark that the symmetrisation of the term $(x \setminus y) \land 1$ also witnesses point regularity for $RL$.

**Example 8** (Subresiduated lattices). A subresiduated lattice [20] is a pair $(A, Q)$, where $A$ is a bounded distributive lattice (with least element $0$ and greatest element $1$) and $Q$ is a sublattice of $A$ containing $0, 1$ such that for each $a, b \in A$ there is an element $c \in Q$ with the property that for all $q \in Q, a \land q \leq b$ iff $q \leq c$. This $c$ is denoted by $a \Rightarrow b$. The class $SRL$ of all subresiduated lattices is equationally definable and quasi-subtractive, witness the terms $x \setminus y, 1 \setminus x$ and $1$.

**Example 9** (Quasi-MV algebras). Quasi-MV algebras are a generalisation of Chang’s MV algebras motivated by an investigation into quantum computational logics (see e.g. [35], [38], [11]). The variety of quasi-MV algebras is neither $1$-subtractive nor $1$-regular, and it differs from the nilpotent shift of the variety of MV algebras [16]. However, it is quasi-subtractive, witness the terms $x' \oplus y, x \oplus 0 \text{ and } 1$.

**Example 10** (Basic algebras). Basic algebras are a generalisation of Heyting algebras motivated by a philosophical investigation into constructive mathematics (see e.g. [5]). The variety $BAL$ of basic algebras is neither $1$-subtractive nor $1$-regular. However, its subvariety $HA \_ V$ - i.e. the join (computed in the lattice of subvarieties of $BAL$) of the variety $HA$ of Heyting algebras and of the variety $V (\mathcal{Z})$ of distributive lattices expanded by a constant implication - is quasi-subtractive, witness the terms $x \rightarrow y, 1 \rightarrow x \text{ and } 1$.

**Example 11** (Congruence lattices). Let $V$ be a congruence modular variety. Recall that, for $A \in V$ and $\theta \in \text{Con}(A)$ the iterated commutator $[\theta]^k$ is defined inductively as follows: $[\theta]^0 = \theta, [\theta]^{k+1} = [[\theta]^k, [\theta]^k]$. Also, recall that the congruence lattice $\text{Con}(A)$ of any $A \in V$ can be expanded to a bounded residuated $\ell$-groupoid $G_{\text{Con}(A)}$ with the commutator $[\theta, \varphi]$ as multiplication and the centraliser $(\theta : \varphi)$ as its residual [22]. Now, suppose that $E_V = \{G_{\text{Con}(A)}; A \in V\}$ satisfies the following commutator identities, viewed as identities in the language of bounded residuated $\ell$-groupoids:

\[
(x : x^1)^1 \approx 1^1 \\
(x : 1^1)^1 \approx x^1 \\
((y : x)^1)^1 \approx (y : x)^1 \\
((y^1 : x^1)^1 : ((y : x)^1)^1)^1 \approx 1^1
\]
These identities can be shown to hold in all varieties satisfying the commutator identity \([x, y] = x \land y \land [1, 1]\), which is trivially satisfied by Abelian and congruence distributive varieties, but also [29] by any modular variety with the congruence extension property. Then every \(G \in \mathbb{K}_V\) is a quasi-subtractive algebra, and this property is witnessed by:

\[
\begin{align*}
   x \rightarrow y &= [(y : x), (y : x)] \\
   \Box x &= [x, x] \\
   1 &= [1, 1]
\end{align*}
\]

The commutative ternary deductive term is an important generalisation of the ternary discriminator. Unlike discriminator varieties, varieties with a commutative TD term need not be semisimple or congruence permutable. Varieties with a commutative and regular TD term include most of the varieties of traditional algebraic logic. In the above-referenced paper [8], Blok and Pigozzi introduce certain hybrids of (topological) interior algebras and residuated partially ordered monoids, called pseudointerior algebras, and show that a variety has a commutative, regular TD term if and only if it is term equivalent to a variety of pseudointerior algebras with compatible operations.

Let \(V\) be a variety with a commutative TD term \(p(x, y, z)\) and let 1 be a constant term of \(V\). If we define:

\[
\begin{align*}
   x \rightarrow y &= p(x, 1, 1) \\
   x \rightarrow y &= p(x, p(x, y, x), 1)
\end{align*}
\]

then \(V\) is quasi-subtractive with the above witness terms. The following examples are noteworthy applications of this observation.

**Example 12 (Pseudointerior algebras).** The variety of pseudointerior algebras is not 1-subtractive; however, it is quasi-subtractive with open left residuation as arrow and the pseudointerior operation as box.

**Example 13 (Boolean algebras with operators).** Recall that Boolean algebras with operators (BAOs) are algebras \(A = (A, \land, \lor, \neg, f_i (i \in I), 0, 1)\) such that the reduct \((A, \land, \lor, \neg, 0, 1)\) is a Boolean algebra and the operations \(f_i (i \in I)\) distribute over join and preserve 0 in each argument. If \(V\) is a variety of BAOs of finite type, we can define a unary term \(q\) as follows. First, for each \(i \in I\), put \(f_i^k x = f_i(1, \ldots, 1, x, 1, \ldots, 1)\), with \(x\) in the \(k\)-th place. Then, let \(F_i x = \bigvee \{f_i^k x : 0 \leq k \leq m - 1\}\), with \(m\) the arity of \(f_i\). Finally, put \(qx = x \lor \bigvee \{F_i x : i \in I\}\). It is easy to see that \(q\) is itself an operator, i.e., \(q(x \lor y) = qx \lor qy\) and \(q0 = 0\). Moreover, \(qx \geq x\) holds. We define, inductively, \(q^0 x = x\), and \(q^{n+1} x = q(q^n x)\) and say that \(V\) is \(k\)-potent, if \(V \models q^{k+1} x \approx q^k x\). It is well-known that a variety of BAOs of finite type has EDPC if and only if it is \(k\)-potent for some \(k\). In [32] it is shown that all semisimple varieties of BAOs of finite type are \(k\)-potent for some \(k\), and hence are discriminator varieties.

Define \(dx = -q_x x\). Then, \(d1 = 1\), \(d(x \land y) = dx \land dy\), and \(dx \leq x\) holds. Moreover, if \(V\) is \(k\)-potent, we have \(d^{k+1} x = d^{k} x\). For a \(k\)-potent \(V\), define
\( \square x = d^k x \) and \( x \rightarrow y = \square (\neg x \lor y) \). It is not difficult to verify that these terms witness quasi-subtractivity of \( V \).

**Example 14 (Interior algebras).** Interior algebras are the equivalent algebraic semantics of the modal deductive system S4. Blok and Pigozzi [8] show that every interior algebra is a pseudointerior algebra with open left residuation defined by

\[
x \rightarrow y = \square (\neg x \lor \square ((\neg x \lor y) \land (\neg y \lor x))).
\]

Therefore, any interior algebra is quasi-subtractive according to Example 12. However, interior algebras have a Boolean algebra reduct, whence material implication \( \neg x \lor y \) clearly witnesses 1-subtractivity (and then, a fortiori, quasi-subtractivity w.r.t. 1 and the identity term, according to Example 4). Finally, it should be observed that according to Example 13 quasi-subtractivity of interior algebras is also witnessed by strict implication \( \square (\neg x \lor y) \), together with \( \square \) and 1.

**Example 15 (Integral \( k \)-potent residuated lattices and Nelson algebras).** A residuated lattice is \( k \)-potent if it satisfies the identity \( x^k = x^{k+1} \) for some \( k \in \omega \). In integral RLs, \( k \)-potency implies that \( x^n \approx x^k \) holds for any \( n \geq k \). It is well-known that \( k \)-potent integral commutative residuated lattices have EDPC, and it can be proved that all semisimple varieties of integral commutative residuated lattices are \( k \)-potent, and hence are discriminator varieties [31].

Let \( \mathbb{V} \) be a variety of \( k \)-potent integral RLs. Define \( \square x = x^k \) and \( x \rightarrow y = (x \setminus y)^k \). These terms witness quasi-subtractivity of \( \mathbb{V} \). In particular, the variety \( \mathbb{P}_k \mathbb{I} \mathbb{R} \mathbb{L} \) of all \( k \)-potent integral RLs is quasi-subtractive with the above witness terms, and it again emphasises the fact that quasi-subtractivity is a relative notion. Namely, the terms \( x \land 1 \) and \( (x \setminus y) \land 1 \), witnessing quasi-subtractivity of \( \mathbb{R} \mathbb{L} \), here reduce to the identity and \( x \setminus y \) respectively, so \( \mathbb{P}_k \mathbb{I} \mathbb{R} \mathbb{L} \) is subtractive with these witness terms.

A particular case of the above is given by Nelson algebras, the equivalent algebraic semantics of constructive logic with strong negation (see e.g. [43]). In [40], [41], in fact, it is shown that the variety of Nelson algebras is term equivalent to a particular variety \( \mathbb{N} \mathbb{R} \mathbb{L} \) of 3-potent FL\(_{ew}\)-algebras (commutative, integral and double-pointed residuated lattices).

### 3.2 Open filters

In many of the motivating examples of the concept of quasi-subtractive variety, a distinction is made between a notion of filter (or ideal) and a notion of open filter (or open ideal) of an algebra. Normally, open filters are closed, unlike filters, under a two-way necessitation rule (\( a \) is in the open filter \( F \) iff \( \square a \) is). For example:

- in pseudointerior algebras, open filters of an algebra \( A \) are defined as subsets of \( A \) containing 1 and closed w.r.t. modus ponens and necessitation [8]. Filters, on the other hand, need not be closed w.r.t. necessitation [30];
• in interior algebras, one usually distinguishes between filters of the Boolean reduct and congruence filters; only the latter need be closed under necessitation (see e.g. [39]);

• in quasi-MV algebras, there is a distinction between weak ideals and ideals [35]; ideals are exactly the closures of weak ideals under the two-way necessitation rule [33].

• in any residuated lattice \( A \), filters correspond to convex normal subalgebras, while open filters are the so-called deductive filters (namely, the already mentioned RL-filters on \( A \)) which coincide with closures of convex normal subalgebras under two-way necessitation [23].

This distinction between open filters and filters partly, but not exactly, overlaps with the distinction between open filters and ideals in the sense of Gumm-Ursini. While in quasi-MV algebras and residuated lattices the second members of each pair coincide, in pseudointerior algebras filters (as defined in [30]) do not necessarily coincide with \( \mathsf{PI} \)-ideals.

Abstracting away from these particular examples, our next goal is to formalize a general concept of open filter which is as central for the investigation of quasi-subtractive varieties as the Gumm-Ursini concept of ideal is for the investigation of subtractive varieties. We start therefore with the following

**Definition 16** Let \( V \) be a variety whose type \( \nu \) includes a nullary term \( 1 \) and a unary term \( \Box \). A \( V \)-open filter term in the variables \( \overrightarrow{x} \) is an \( n + m \)-ary term \( p(\overrightarrow{x}, \overrightarrow{y}) \) of type \( \nu \) s.t.

\[
\{ \Box x_i \approx 1 : 1 \leq i \leq n \} \vdash_{E_{q(V)}} \Box p(\overrightarrow{x}, \overrightarrow{y}) \approx 1.
\]

The wording "\( V \)-open filter term" will be simplified to "open filter term" whenever this replacement produces a decrease in clumsiness with no increase in ambiguity. The same applies to "\( V \)-open filter" below.

**Lemma 17** Composition of open filter terms yields an open filter term: more precisely, if \( q_1(\overrightarrow{x}, \overrightarrow{y}), \ldots, q_m(\overrightarrow{x}, \overrightarrow{y}) \) are open filter terms in \( \overrightarrow{x} \), and \( p(\overrightarrow{v}, \overrightarrow{z}) \) is an open filter term in \( \overrightarrow{v} \), the term

\[
p(q_1(\overrightarrow{x}, \overrightarrow{y}), \ldots, q_m(\overrightarrow{x}, \overrightarrow{y}), \overrightarrow{z})
\]

is an open filter term in \( \overrightarrow{x} \).

**Proof.** The assumption that \( q_1(\overrightarrow{x}, \overrightarrow{y}), \ldots, q_m(\overrightarrow{x}, \overrightarrow{y}) \) are open filter terms in \( \overrightarrow{x} \) can be rephrased as

\[
\{ \Box x_i \approx 1 : 1 \leq i \leq n \} \vdash_{E_{q(V)}} \Box q_j(\overrightarrow{x}, \overrightarrow{y}) \approx 1 \quad \text{for every} \ j \leq m.
\]

We must show that

\[
\{ \Box x_i \approx 1 : 1 \leq i \leq n \} \vdash_{E_{q(V)}} \Box p(q_1(\overrightarrow{x}, \overrightarrow{y}), \ldots, q_m(\overrightarrow{x}, \overrightarrow{y}), \overrightarrow{z}) \approx 1.
\]
Suppose that $\square x_i \approx 1$ for $i \leq n$. Then $\square q_j (\overrightarrow{x}, \overrightarrow{y}) \approx 1$ for every $j \leq m$, but since $p (\overrightarrow{v}, \overrightarrow{z})$ is an open filter term in $\overrightarrow{v}$, our conclusion is automatically satisfied. □

**Definition 18** Let $\mathbb{V}$ be as in Definition 16. A $\mathbb{V}$-open filter of $A \in \mathbb{V}$ is a subset $F \subseteq A$ with the following properties:

i) it is closed w.r.t. all $\mathbb{V}$-open filter terms $p$: whenever $a_1, ..., a_n \in F, b_1, ..., b_m \in A$, $p (\overrightarrow{a}, \overrightarrow{b}) \in F$;

ii) for every $a \in A$, we have that $a \in F$ iff $\square a \in F$.

A classical result of the theory of subtractive varieties states that every subtractive variety $\mathbb{V}$ has normal ideals: every $\mathbb{V}$-ideal of any member $A$ of a subtractive variety is a coset of some $\theta \in \text{Con}(A)$. We will be after an analogous result: in any member $A$ of a quasi-subtractive variety $\mathbb{V}$, $\mathbb{V}$-open filters coincide with $\tau$-congruence classes of some $\theta \in \text{Con}(A)$, where $\tau$ is the singleton $\{ \square x \approx 1 \}$. We call these $\tau$-congruence classes $(\square x, 1)$-classes for short. The right-to-left inclusion is easy to show:

**Lemma 19** In every member $A$ of a quasi-subtractive variety $\mathbb{V}$, every $(\square x, 1)$-class of some $\theta \in \text{Con}(A)$ is a $\mathbb{V}$-open filter of $A$.

**Proof.** Suppose that $\overrightarrow{a} \in (\square x, 1)^A / \theta$, $\overrightarrow{b} \in A$ and $p$ is an open filter term. This last assumption means that every $B$ in $\mathbb{V}$ satisfies the quasiequation

\[
\text{if } \square x_i \approx 1 \text{ for all } i \leq n, \text{ then } \square p (\overrightarrow{x}, \overrightarrow{y}) \approx 1.
\]

Choose $B = A / \theta$. Since the antecedent of the above conditional is true in $A / \theta$ when each $x_i$ is assigned the value $a_i / \theta$, we conclude that $\square p^B (\overrightarrow{a}/\theta, \overrightarrow{b}/\theta) = 1/\theta$, which means that $\square p^A (\overrightarrow{a}, \overrightarrow{b}) \theta 1$, whence $p^A (\overrightarrow{a}, \overrightarrow{b}) \in (\square x, 1)^A / \theta$.

As regards the remaining condition, simply observe that $a \in (\square x, 1)^A / \theta$ iff $\square a \theta 1$ iff $\square \theta a 1$ iff $\square a \in (\square x, 1)^A / \theta$. □

For the other inclusion, let us first introduce a definition:

**Definition 20** Let $\mathbb{V}$ be as in Definition 16. $\mathbb{V}$ has normal open filters iff for all $A \in \mathbb{V}$, every $\mathbb{V}$-open filter of $A$ is a $(\square x, 1)$-class of some $\theta \in \text{Con}(A)$.

Next, we need the following lemma, which fits to our needs the well-known Mal’cev criterion for being a congruence class [36]:

**Lemma 21** Let $\mathbb{V}$ be as in Definition 16, let $A \in \mathbb{V}$ and let $F \subseteq A$ be such that: i) $1 \in F$; ii) for every $a \in A$, $a \in F$ iff $\square a \in F$. Then $F = (\square x, 1)^A / \theta$ for some $\theta \in \text{Con}(A)$ iff, for every unary polynomial $g(x, \overrightarrow{a}')$ of type $\nu$, if $b, c \in F$ and $g^A (\overrightarrow{b}, \overrightarrow{a}) \in F$, then $g^A (\overrightarrow{c}, \overrightarrow{a}) \in F$. 

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**Lemma 23**

Let \( g \) be a unary polynomial, \( b, c \in F \), and \( g^A(\Box b, \Box c) \in F \). Suppose that \( g^A(\Box b, \Box c) \in F \) means \( \Box a \theta_1 \Box c \), whence \( g^A(\Box b, \Box c) \in F \) implies that \( \Box g^A(\Box b, \Box c) \theta_1 \). However, \( g^A(\Box b, \Box c) \in F \) implies that \( \Box g^A(\Box b, \Box c) \theta_1 \) and thus \( g^A(\Box b, \Box c) \in F \).

From right to left, suppose \( F \) satisfies the assumptions. Consider the congruence \( \theta = Cg^A((\Box u, \Box w): u, w \in F) \). We will show that \( \Box a \theta_1 \) implies \( a \in F \). Suppose \( \Box a \theta_1 \). By the congruence generation theorem, there exist elements \( s_0, \ldots, s_k \in A \), and polynomials \( g_1, \ldots, g_k \in \text{Pol}_1 A \) such that (omitting superscripts from the very beginning):

\[
\begin{align*}
\{ g_1(\Box u_1), g_1(\Box w_1) \} &= \{ s_0, s_1 \} \\
\{ g_2(\Box u_2), g_2(\Box w_2) \} &= \{ s_1, s_2 \} \\
&\vdots \quad \vdots \\
\{ g_k(\Box u_k), g_k(\Box w_k) \} &= \{ s_{k-1}, s_k \}
\end{align*}
\]

where \( u_1, \ldots, u_k, w_1, \ldots, w_k \in F \), \( s_0 = 1 \) and \( s_k = \Box a \). Without loss of generality, we can assume \( g_1(\Box u_1) = s_0 = 1 \). Then, \( g_1(\Box u_1) = s_1 \in F \). Again, without loss of generality, we can assume \( s_1 = g_2(\Box u_2) \). Repeating this argument \( k \) times, we get that \( g_k(\Box u_k) \in F \) and \( g_k(\Box w_k) \in F \). Thus, \( s_k = \Box a \in F \), and therefore \( a \in F \), as claimed.

The ternary term \( t(x, y, z) = (\Box x \to (\Box y \to \Box z)) \to z \) will be of some use in this and in the next section. The lemma below states two of its most useful properties. For the next series of lemmas, we assume that \( A \) is a generic member of an arbitrary quasi-subtractive variety \( \forall \).

**Lemma 22** Let \( t(x, y, z) \) be the term defined above. Then:

1. \( t(x, y, z) \) is an open filter term in \( x, y \);

2. If \( a, b \in A \), \( t(a \to b, a, b) = \Box b \).

**Proof.**

1. Suppose \( \Box a, \Box b = 1 \). Then

\[
\begin{align*}
\Box ((\Box a \to (\Box b \to \Box c)) \to c) &= (\Box a \to (\Box b \to \Box c)) \to c \\
&= (1 \to (1 \to \Box c)) \to c \\
&= \Box \Box \Box c \to c \\
&= \Box c \to c = 1.
\end{align*}
\]

2. \( t(a \to b, a, b) = (\Box (a \to b) \to (\Box a \to \Box b)) \to b \\
= 1 \to b = \Box b. \)

**Lemma 23** Let \( S \subseteq A \) be closed under open filter terms. Then,

\( a \in S \) and \( a \to b \in S \) imply \( \Box b \in S \).
If, moreover, \( S \) is an open filter, then
\[ a \in S \text{ and } a \to b \in S \text{ imply } b \in S. \]

**Proof.** By Lemma 22 and the assumption of \( S \) being closed under open filter terms, we get \( \Box b \in S \). If \( S \) is an open filter, then \( b \in S \) as well. ■

**Lemma 24** Let \( S \subseteq A \) be closed under open filter terms, \( a, b \in S \), and \( g \in \text{Pol}_1 A \). Then,

1. \( \Box g(\Box a) \in S \) iff \( \Box g(\Box b) \in S \),
2. \( g(\Box a) \in S \) implies \( \Box g(\Box b) \in S \).

Moreover, if \( S \) is an open filter, then

3. \( g(\Box a) \in S \) iff \( g(\Box b) \in S \).

**Proof.** Observe that \( \Box g(\Box x) \to g(\Box y) \) is an open filter term in \( x, y \). Since \( S \) is closed under open filter terms, we have \( \Box g(\Box a) \to g(\Box b) \in S \). Suppose \( \Box g(\Box a) \in S \). By Lemma 23 then, we obtain \( \Box g(\Box b) \in S \). This proves one direction of (1) and the other follows by symmetry. For (2), notice that \( g(\Box a) \in S \) implies \( \Box g(\Box a) \in S \), because \( \Box x \) is an open filter term in \( x \). If \( S \) is an open filter, then (3) follows from (1). ■

**Theorem 25** Every quasi-subtractive variety \( V \) has normal open filters.

**Proof.** Suppose \( F \) is an open filter of \( A \in V \), \( b, c \in F \), and \( g \in \text{Pol}_1 A \). By Lemma 24, we have \( g(\Box b) \in F \) iff \( g(\Box c) \in F \), and therefore, by Lemma 21, \( F = (\Box x, 1) / \theta \) for some congruence \( \theta \). As \( F \) and \( A \) were arbitrary, this establishes that \( V \) has normal open filters. ■

According to established notational practice (see e.g. [10]), the \( \tau \)-assertional logic of a variety \( V \) is referred to as \( S(\mathbb{V}, \tau) \). Here, however, we only work with the particular translation \( \tau = \{ \Box x \approx 1 \} \), a circumstance which gives us the freedom of simplifying our notation to \( S(\mathbb{V}) \). The next lemma pins down the logical import of open filters in a variety with normal open filters.

**Lemma 26** If \( \mathbb{V} \) is a variety with normal open filters and \( A \in \mathbb{V}, \mathbb{V} \)-open filters of \( A \) coincide with deductive filters on \( A \) of \( S(\mathbb{V}) \).

**Proof.** Let \( A \) be as in the statement of our lemma. We first show that its \( \mathbb{V} \)-open filters are deductive filters on \( A \) of \( S(\mathbb{V}) \). Our initial assumption implies that every open filter of \( A \) has the form \( (\Box x, 1)^A / \theta \) for some \( \theta \in \text{Con}(A) \). We must prove that if \( \Gamma \vdash_{S(\mathbb{V})} p \) and \( s^A(\vec{a}) \in (\Box x, 1)^A / \theta \) for all \( s \in \Gamma \), then \( p^A(\vec{a}) \in (\Box x, 1)^A / \theta \). So, suppose that the antecedent of the last conditional holds true. By definition of \( \tau \)-congruence class, it follows that
\[ \{ p^A(\vec{a}) : s \in \Gamma \} \subseteq 1^A / \theta. \]
Unwinding the other half of the assumption, we get

$$\{\Box s(\overline{x}) \approx 1 : s \in \Gamma\} \vdash_{\text{Eq}(\mathbb{V})} \Box \overline{p}(\overline{x}) \approx 1,$$

whence $\square \overline{p}^{A}(\overline{a}) \in 1^{A}/\theta$ because the equational consequence relation $\text{Eq}(\mathbb{V})$ preserves congruences. But this simply means that $\overline{p}^{A}(\overline{a}) \in (\Box \overline{x}, 1)^{A}/\theta$.

Conversely, observe that $\Box \overline{x} \vdash_{S(\mathbb{V})} \overline{x}$ by Lemma 3.(ii); moreover, $p$ is a $\mathbb{V}$-open filter term iff

$$\overline{x} \vdash_{S(\mathbb{V})} p(\overline{x}, \overline{y}).$$

It follows that every deductive filter on $A$ of $S(\mathbb{V})$ is in particular a $\mathbb{V}$-open filter of $A$. ■

As a consequence, we obtain:

**Theorem 27** If $\mathbb{V}$ is a quasi-subtractive variety and $A \in \mathbb{V}$, $\mathbb{V}$-open filters of $A$ coincide with deductive filters on $A$ of $S(\mathbb{V})$.

We conclude this subsection by showing that

**Lemma 28** In every member $A$ of a quasi-subtractive variety $\mathbb{V}$, if the $\mathbb{V}$-open filter $F \subseteq A$ contains $(\Box \overline{x}, 1)^{A}/\theta$, for $\theta \in \text{Con}(A)$, then $F$ is a union of $\theta$-blocks.

**Proof.** Suppose $(\Box \overline{x}, 1)^{A}/\theta \subseteq F$, for $F$ a $\mathbb{V}$-open filter of $A$. This means that for every $a \in A$, if $\square a \theta 1$, then $a \in F$. Suppose also that $b \in F$ and $b \theta c$. From $b \in F$ we get $\Box b \in F$. From $b \theta c$ we get $1 = \Box b \rightarrow b \theta \Box b \rightarrow c$, whence $\Box b \rightarrow c \in F$. Since $F$ is closed under detachment, $c \in F$. ■

### 3.3 Open filter generation

As already observed, ideals are amenable to a nice description of ideal generation and, in particular, joins of ideals can be appropriately characterised. We present here an analogous result for open filters. If $A$ is any algebra and we define for $X \subseteq A$:

$$\uparrow X = X \cup \{a : \Box a \in X\};$$

$$\Gamma X = \left\{\overline{p}^{A}(\overline{a}, \overline{b}) : \overline{a} \in X, \overline{b} \in A, p \text{ an open filter term}\right\},$$

we can show that the open filter $[X]$ generated by $X \subseteq A \in \mathbb{V}$, where $\mathbb{V}$ is quasi-subtractive, equals $\uparrow \Gamma X$. We assume throughout the next sequence of lemmas that $\mathbb{V}$ is a quasi-subtractive variety and $A$ an algebra from $\mathbb{V}$.

**Lemma 29** Let $S \subseteq A$ be closed under open filter terms, and $p(\overline{x}, \overline{y})$ be an open filter term in $\overline{z}$. For any $\overline{a} \in S$ and $\overline{b} \in A$ we have $\Box p(\overline{a}, \overline{b}) \in S.$
Proof. Suppose $\bar{\alpha} \in S$ and let $\theta$ be the smallest congruence for which $\bar{\alpha}$ and 1 are all in the same $\theta$-coset. As $p$ is an open filter term and $(\square a_1, 1), \ldots, (\square a_n, 1) \in \theta$, we have $\square p^A(\bar{a}, \bar{b})/\theta = 1/\theta$, i.e., $(\square p^A(\bar{a}, \bar{b}), 1) \in \theta$. By the congruence generation theorem, there exist elements $s_0, \ldots, s_k \in A$ and polynomials $g_1, \ldots, g_k \in \text{Pol}_1 A$ such that (omitting once again all superscripts for the sake of simplicity)

\[
\begin{align*}
\{g_1(\square u_1), g_1(\square w_1)\} &= \{s_0, s_1\} \\
\{g_2(\square u_2), g_2(\square w_2)\} &= \{s_1, s_2\} \\
& \vdots \\
\{g_k(\square u_k), g_k(\square w_k)\} &= \{s_{k-1}, s_k\}
\end{align*}
\]

where $u_1, \ldots, u_k, w_1, \ldots, w_k \in \bar{a} \cup \{1\}$, $s_0 = \square a_i$ for some $a_i \in \bar{a} \cup \{1\}$, and $s_k = \square p(\bar{a}, \bar{b})$. It follows that

\[
\begin{align*}
\{\square g_1(\square u_1), \square g_1(\square w_1)\} &= \{\square s_0, \square s_1\} \\
\{\square g_2(\square u_2), \square g_2(\square w_2)\} &= \{\square s_1, \square s_2\} \\
& \vdots \\
\{\square g_k(\square u_k), \square g_k(\square w_k)\} &= \{\square s_{k-1}, \square s_k\}
\end{align*}
\]

hold as well. Moreover, $\square s_0 = s_0 = \square a_i$ and $\square s_k = s_k = \square p(\bar{a}, \bar{b})$. Observe that $u_1, \ldots, u_k, w_1, \ldots, w_k \in S$ and $s_0 \in S$. It follows that at least one of $\square g_1(\square u_1), \square g_1(\square w_1)$ belongs to $S$. Without loss of generality, we can assume $\square g_1(\square u_1) \in S$. By Lemma 24 we then get $\square g_1(\square w_1) \in S$, so $\square s_1 \in S$ and therefore at least one of $\square g_2(\square u_2), \square g_2(\square w_2)$ belongs to $S$. Applying this reasoning successively to all $g_i$, we finally arrive at $\square s_k = \square p(\bar{a}, \bar{b})$, which is the desired conclusion.

Theorem 30 Let $V$ be a quasi-subtractive variety, $A \in V$ and $X \subseteq A$. The $V$-open filter $[X]$ generated by $X$ is precisely $\uparrow \chi X$.

Proof. It suffices to show that $\Gamma \uparrow \chi X \subseteq \chi X$. Let $p(\bar{x}, \bar{y})$ be an open filter term in $\bar{x}$, and $\bar{a}$ be a sequence of members of $\uparrow \chi X$. Suppose first that $\bar{a} \in \chi X$. As $\chi X$ is closed under open filter terms, Lemma 29 applies, yielding $\square p^A(\bar{a}, \bar{b}) \in \chi X$, for any sequence $b$ of elements of $A$. Therefore $p^A(\bar{a}, \bar{b}) \in \uparrow \chi X$, proving the claim. The same conclusion obviously holds if $\bar{a} \in \chi X$.

Theorem 31 If $A \in V$, where $V$ is a quasi-subtractive variety, and $F, G$ are $\forall$-open filters of $A$, then

$$ F \lor G = \uparrow \{t^A(a, b, c) : a \in F, b \in G, c \in A \}. $$

Proof. We first show that, if $f \in \{t^A(a, b, c) : a \in F, b \in G, c \in A \}$, then for some $g \in G$ we have $g \to f \in F$ or $g \to \square f \in F$. So, let $f \in \{t^A(a, b, c) : a \in F, b \in G, c \in A \} = \uparrow \Gamma \{t^A(a, b, c) : a \in F, b \in G, c \in A \}.$
This means that either \( f \) or \( \Box f \) coincides with \( p^A (\overrightarrow{m}, \overrightarrow{c}) \), where \( p \) is an open filter term, \( m_i = t^A (u_i, v_i, d_i) \) and \( u_i \in F, v_i \in G, d_i \in A \). Looking at the definition of the term \( t \), it is easily seen that \( m_i = t^A (\Box u_i, v_i, d_i) \). Now define
\[
g = \Box p^A (t^A (1, v_1, d_1), \ldots, t^A (1, v_k, d_k), \overrightarrow{c}).
\]

Since \( v_1, \ldots, v_k, 1 \in G \), by Lemma 17 \( g \in G \). Now,
\[
\Box p (t (1, y_1, z_1), \ldots, t (1, y_k, z_k), \overrightarrow{x}) \rightarrow p (t (\Box w_1, y_1, z_1), \ldots, t (\Box w_k, y_k, z_k), \overrightarrow{x})
\]
is an open filter term in \( \overrightarrow{w} \), so
\[
\Box p^A (t^A (1, v_1, d_1), \ldots, t^A (1, v_k, d_k), \overrightarrow{c}) \rightarrow p^A (t^A (\Box u_1, v_1, d_1), \ldots, t^A (\Box u_k, v_k, d_k), \overrightarrow{c}),
\]
which is either \( g \rightarrow f \) or \( g \rightarrow \Box f \), is a member of \( F \).

Now, to establish our conclusion it suffices to show that
\[
\{ t^A (a, b, c) : a \in F, b \in G, c \in A \} = \uparrow \{ t^A (a, b, c) : a \in F, b \in G, c \in A \}.
\]
The right-to-left inclusion is clear in the light of Theorem 30. From left to right, suppose that \( f \in \{ t^A (a, b, c) : a \in F, b \in G, c \in A \} \). Then for some \( g \in G \), either \( g \rightarrow f \in F \) or \( g \rightarrow \Box f \in F \). However, \( t^A (g \rightarrow f, g, f) = \Box f \) and \( t^A (g \rightarrow \Box f, g, \Box f) = \Box \Box f = \Box f \). So, in both cases,
\[
\Box f \in \{ t^A (a, b, c) : a \in F, b \in G, c \in A \}
\]
which means
\[
f \in \uparrow \{ t^A (a, b, c) : a \in F, b \in G, c \in A \}.
\]

Lemma 32 Let \( \mathbb{V} \) be a quasi-subtractive variety, \( A \in \mathbb{V} \), and \( F, G \) be open filters on \( A \). Then
\[
F \lor G = \{ f \in A : g \rightarrow f \in F \text{ or } g \rightarrow \Box f \in F \text{ for some } g \in G \}
\]

Proof. Inclusion from left to right has already been proved, so we need to show the converse. Suppose \( g \in G \) and \( g \rightarrow f \in F \). Then, \( g, g \rightarrow f \in F \lor G \), and thus \( f \in F \lor G \) by closure under detachment. Now, suppose \( g \in G \) and \( g \rightarrow \Box f \in F \). Then, arguing as above we get \( \Box f \in F \lor G \), and thus \( f \in F \lor G \), because \( F \lor G \) is an open filter. \( \blacksquare \)

Lemma 33 Let \( \mathbb{V} \) be a quasi-subtractive variety. Then the lattice of open filters of any \( A \in \mathbb{V} \) is modular.

Proof. We will show that the lattice of open filters of \( A \) cannot have a sublattice isomorphic to the pentagon. Let \( F, G, H \) be open filters of \( A \) such that \( F \subseteq G, F \cap H = G \cap H \), and \( F \lor H = G \lor H \). We will show that \( F = G \). Take an element \( a \in G \). Then \( a \in G \lor H = F \lor H \). So, for some \( b \in F \) we have \( b \rightarrow a \in H \).
or \( b \to \Box a \in H \). But also \( b \in G \), and since \( \Box x \to \Box y \) is an open filter term in \( x, y \), we have \( \Box b \to \Box a \in G \). Suppose first that \( b \to a \in H \). Then, by Q3, Q4 and closure under detachment, \( \Box b \to \Box a \in G \cap H = F \cap H \), so in particular \( \Box b \to \Box a \in F \). Since \( b \in F \), \( \Box b \in F \) and thus \( \Box a \in F \), whence \( a \in F \). If \( b \to \Box a \in H \), by a similar argument we attain the same conclusion. Therefore, in both cases \( a \in F \), and since \( a \) was arbitrary, \( G = F \) as required. \( \blacksquare \)

### 3.4 Open filters and congruences

We use the label \( \text{Opfil} \) (\( A \)) ambiguously, to denote from time to time either the lattice of open filters of \( A \) or its universe. Consider the following mappings from \( \text{Con} \) (\( A \)) to \( \text{Opfil} \) (\( A \)) or conversely:

\[
\begin{align*}
 f(\theta) &= (\Box x, 1)^A / \theta \\
g(F) &= \bigwedge \{ \theta \in \text{Con} (A) : \forall a, b (a, b \in F \Rightarrow \Box a \theta \Box b) \} \\
 F^\delta &= \bigwedge \{ \theta \in \text{Con} (A) : f(\theta) = F \} \\
 F^e &= \bigvee \{ \theta \in \text{Con} (A) : f(\theta) = F \}
\end{align*}
\]

The first mapping is well-defined in every quasi-subtractive algebra by Lemma 19. The other mappings are well-defined in every algebra whose type contains a unary \( \Box \) and a nullary 1. Here are a few easy facts concerning these mappings:

**Lemma 34** In every member \( A \) of a quasi-subtractive variety, \( f \) is a lattice homomorphism from \( \text{Con} (A) \) to \( \text{Opfil} (A) \).

**Proof.** It suffices to prove that \( f(\theta \lor \varphi) \subseteq f(\theta) \lor f(\varphi) \), the remaining claims being evident. Let \( a \in (\Box x, 1)^A / \theta \lor \varphi \), which means that there is a natural number \( n \) s.t.

\[ a \subseteq a \theta a_1 \varphi a_2 \ldots a_{n} \varphi 1 \text{ or } \Box a \varphi a_1 \theta a_2 \ldots \varphi a_n \theta 1. \]

We prove our claim by induction on \( n \). If \( n = 0 \), then \( \Box a \theta 1 \) or \( \Box a \varphi 1 \), whence \( a \in f(\theta) \lor f(\varphi) \). Now, assume \( \Box a \theta a_1 \varphi a_2 \ldots \varphi a_n \theta 1 \), whence \( 1 = \Box a \to a \theta a_1 \to a \varphi a_2 \to a \varphi a_3 \to \ldots a \varphi a_n \to a \). By IH \( a_m \to a \in f(\theta) \lor f(\varphi) \). Since \( a_m \theta 1 \), it also holds that \( \Box a_m \theta 1 \), i.e. \( a_m \in f(\theta) \subseteq f(\theta) \lor f(\varphi) \). But \( f(\theta) \lor f(\varphi) \), being an open filter, is closed w.r.t. detachment, whereby \( a \in f(\theta) \lor f(\varphi) \). \( \blacksquare \)

**Lemma 35** In every member \( A \) of a variety with normal open filters, \( f(F^\delta) = f(F^e) = F \).

**Proof.** We first observe that if \( a \in F = f(\theta) \), then we have that \( \Box a \theta \Box b \) iff \( b \in F \). From left to right, if \( a \in F \), then by our hypothesis \( \Box a \theta 1 \), whence \( \Box b \theta 1 \), i.e. \( b \in F \). Conversely, if \( a, b \in F \), then \( \Box a \theta 1 \Box b \). The equality \( f(F^\delta) = F \) follows upon checking that

\[ f(F^\delta) = \{ a : \Box a F^\delta 1 \} = \{ a : (\Box a, 1) \in \bigwedge \{ \theta \in \text{Con} (A) : f(\theta) = F \} \} = F. \]
Suppose finally that \( a \in f(F^\ast) \). Then \((\Box a, 1) \in \bigvee \{ \varphi \in \text{Con}(A) : f(\varphi) = F \} \),
which means that for some \( \theta_1, \ldots, \theta_n \) s.t. \( f(\theta_i) = F \) (for \( i \leq n \)),
\[
1 = b_1 \theta_1 b_2 \cdots b_n = \Box a.
\]

Now, \( 1 \theta_1 b_2 \) implies \( 1 \theta_1 \Box b_2 \), so \( b_2 \in F \). \( b_2 \theta_2 b_3 \) implies \( \Box b_2 \theta_2 \Box b_3 \), whence
by the previous observation \( b_3 \in F \). Carrying over the reasoning, we finally
conclude that \( \Box a \in F \). Consequently, \( a \in F \) and we are through. \( \blacksquare \)

The previous lemma implies that \([F^\delta, F^\ast]\) is an interval in \( \text{Con}(A) \), for \( A \)
in a variety with normal open filters.

**Lemma 36** In every member \( A \) of a quasi-subtractive variety, \( f(\theta) = f(g(f(\theta))) \).

**Proof.** For the left-to-right inclusion, suppose \( a \in f(\theta) \), i.e. \( \Box a \theta 1 \). We must
prove that \((\Box a, 1) \in g(f(\theta)) \), or else that \((\Box a, 1) \) belongs to every congruence
\( \varphi \) with the property that \( \varphi \) contains \((\Box b, \Box c) \) whenever \( \theta \) contains \((\Box b, 1) \) and
\((\Box c, 1) \). So, let \( \varphi \) have the property at issue. Choosing \( b = a, c = 1 \), clearly
\( \theta \) contains the pairs \((\Box a, 1) \) and \((\Box 1, 1) \), whence \((\Box a, 1) \) belongs to \( \varphi \).
For the other inclusion, let our previous thesis be an assumption. Choose \( \varphi = \theta \). The
antecedent of the assumption now becomes: \( \theta \) contains \((\Box b, \Box c) \) whenever \( \theta \) contains
\((\Box b, 1) \) and \((\Box c, 1) \), which is clearly true. Therefore, so is its consequent, whence \( \Box a \theta 1 \). \( \blacksquare \)

**Lemma 37** In every member \( A \) of a quasi-subtractive variety, \( g(f(\theta)) = f(\theta)^\delta \).

**Proof.** Recall that, by definition, \( g(f(\theta)) \) is the congruence \( \varphi \) with the following two properties:

\[
\forall a, b \ (\Box a \theta 1 \& \Box b \theta 1 \Rightarrow \Box a \varphi \Box b)
\]

\[
\forall \psi \ (\forall a, b \ (\Box a \theta 1 \& \Box b \theta 1 \Rightarrow \Box a \psi \Box b) \Rightarrow \varphi \subseteq \psi)
\]

First, we have to prove that \((\Box x, 1)^A / \theta = (\Box x, 1)^A / \varphi \). The left-to-right
inclusion is immediate if we choose \( b = 1 \) in the first property. From the second
property, however, we conclude \( \varphi \subseteq \theta \), which yields the missing half. Next,
we must show that \( \varphi \) is contained in every congruence \( \psi \) s.t. \((\Box x, 1)^A / \varphi = (\Box x, 1)^A / \psi \). However, if \( \psi \) is such that for every \( a, \Box a \theta 1 \) iff \( \Box a \psi 1 \), then the
antecedent of the conditional in the second property is true and consequently
\( \varphi \subseteq \psi \), which is what we were committed to show. \( \blacksquare \)

Since quasi-subtractive algebras have normal open filters, the mappings \( g \)
and \( (\cdot)^\delta \) are necessarily coincident. Remark, however, that this need not be true
of an arbitrary algebra whose type includes \( \Box \) and \( 1 \).

**Corollary 38** In every member \( A \) of a quasi-subtractive variety, \( f(\theta) = f \left( f(\theta)^\delta \right) \).

A more detailed investigation of these operators will be left for a sequel to the present paper.
4 Open and flat subvarieties

Let $\mathbb{V}$ be a quasi-subtractive variety, with witness terms $\rightarrow, \square$ and $1$. If we add the equation $\square x \approx x$ to an equational basis for $\mathbb{V}$, we obtain, of course, a 1-subtractive variety $\mathbb{V}'$. $\mathbb{V}'$ may not be the largest 1-subtractive subvariety of $\mathbb{V}$ (for all we know, $\mathbb{V}$ itself might be 1-subtractive), but it is the largest 1-subtractive subvariety of $\mathbb{V}$ where 1-subtractivity is witnessed by $\rightarrow$. We call open any subvariety of $\mathbb{V}$ which satisfies $\square x \approx x$. We extend this label also to members of $\mathbb{V}$.

At the opposite end of the spectrum, in the lattice of subvarieties of a quasi-subtractive variety $\mathbb{V}$, we find varieties $\mathbb{V}''$ that are trivialised by adding the equation $\square x \approx x$ to an equational basis for $\mathbb{V}''$. We call such subvarieties (and their members) flat. Clearly, an open and a flat subvariety are always disjoint. It is of some interest to determine conditions under which every member of a quasi-subtractive variety can be (directly or at least subdirectly) decomposed into the product of an open algebra and a flat algebra.

In [34], two of the present authors generalised the well-known notion of independence of varieties, due to Grätzer, Lakser and Plonka [26], in order to investigate a number of sufficient conditions under which two disjoint similar varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ are such that every member of their varietal join $\mathbb{V}_1 \vee \mathbb{V}_2$ is subdirectly embeddable into a product $A_1 \times A_2$, with $A_1 \in \mathbb{V}_1$ and $A_2 \in \mathbb{V}_2$. We will put this machinery to good use in providing sufficient conditions under which every member of a given quasi-subtractive variety is subdirectly (or directly) embeddable into the product of an open and a flat algebra. The results proved hereafter subsume, as we will see, several direct or subdirect decomposition results known in the literature. Now for a résumé of the relevant definitions.

**Definition 39** If $\mathbb{V}$ is a variety of type $\nu$, an at most unary term $t$ of the same type is a $\mathbb{V}$-compatible term iff the operation $t^A$ is an endomorphism on every algebra $A \in \mathbb{V}$. Moreover, $t$ is a $\mathbb{V}$-idempotent term iff $\mathbb{V}$ satisfies the equation $t (t (x)) \approx t (x)$.

**Definition 40** Two similar varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ are quasi-independent if there are at most unary $\mathbb{V}_1 \vee \mathbb{V}_2$-compatible and $\mathbb{V}_1 \vee \mathbb{V}_2$-idempotent terms $t_1 (x)$ and $t_2 (x)$, and an at most binary term $x \ast y$, such that the following conditions are satisfied:

1. $\mathbb{V}_1 \models x \ast y \approx t_1 (x), t_2 (x) \approx x$
2. $\mathbb{V}_2 \models x \ast y \approx t_2 (y), t_1 (x) \approx x$

This definition generalises Grätzer’s independence relation, which coincides with the special case where $t_1, t_2$ are the identity.
Definition 41 Two quasi-independent varieties \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are orthogonal if their varietal join \( \mathcal{V}_1 \cup \mathcal{V}_2 \) satisfies the following quasiequations:

\[
\begin{align*}
(i) & \quad t_1(x) \approx t_2(y) \& t_1(y) \approx t_2(x) \Rightarrow x \approx y \\
(ii) & \quad t_1(x) \approx t_1(y) \& t_2(y) \approx t_2(x) \Rightarrow x \approx y
\end{align*}
\]

We finally recall that the direct product of two similar varieties \( \mathcal{V}_1, \mathcal{V}_2 \) is defined as

\[
\mathcal{V}_1 \times \mathcal{V}_2 = \{ A_1 \times A_2 : A_1 \in \mathcal{V}_1, A_2 \in \mathcal{V}_2 \}.
\]

In the same vein, we can define the subdirect product of two similar varieties \( \mathcal{V}_1, \mathcal{V}_2 \) as the class \( \mathcal{V}_1 \times_s \mathcal{V}_2 \) of all algebras that can be subdirectly embedded into a product \( A_1 \times A_2 \), for some \( A_1 \in \mathcal{V}_1 \) and \( A_2 \in \mathcal{V}_2 \).

In [34] it is proved that:

Theorem 42 If \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are quasi-independent and orthogonal varieties, then \( \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_1 \times_s \mathcal{V}_2 \).

This result generalises the celebrated theorem of [26] according to which if \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are independent varieties, then \( \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_1 \times \mathcal{V}_2 \).

A further piece of terminology is now needed. For any algebra \( A \) and an operation \( f \) on \( A \), we say that \( f \) preserves a subset \( S \) of \( A \), if \( S \) is closed under \( f \). Now, let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be similar varieties whose common type includes a constant 1, and \( t \) be a unary term of the same type such that \( \mathcal{V}_1 \models t(x) \approx x \) and \( \mathcal{V}_2 \models t(x) \approx 1 \).

Lemma 43 If on any \( A \in \mathcal{V}_1 \cup \mathcal{V}_2 \) the operation \( t^A \) commutes with all operations that do not preserve \( \{1\} \), then \( t \) is \( \mathcal{V}_1 \cup \mathcal{V}_2 \)-compatible. In particular, if \( \{1\} \) is a subalgebra of every algebra in \( \mathcal{V}_1 \cup \mathcal{V}_2 \), then \( t \) is \( \mathcal{V}_1 \cup \mathcal{V}_2 \)-compatible.

Proof. It follows immediately from the relevant definitions that \( t \) is \( \mathcal{V}_1 \cup \mathcal{V}_2 \)-compatible iff for any basic operation symbol \( b \) the equation

\[
t(b(x_1, \ldots, x_n) \approx b(t(x_1), \ldots, t(x_n))
\]

holds in \( \mathcal{V}_1 \cup \mathcal{V}_2 \). Thus, it suffices to prove that the equation above holds in every algebra from \( \mathcal{V}_1 \cup \mathcal{V}_2 \). Since \( t \) is the identity on every algebra from \( \mathcal{V}_1 \), indeed it suffices to prove it for every algebra from \( \mathcal{V}_2 \). Let \( A \) be such an algebra. Then, if \( b^A \) preserves \( \{1\} \), for any \( a_1, \ldots, a_n \in A \) we get

\[
t^A(b^A(a_1, \ldots, a_n)) = 1 = b^A(1, \ldots, 1) = b^A(t^A(a_1), \ldots, t^A(a_n))
\]

and if \( b^A \) does not preserve \( \{1\} \), then \((C)\) holds by assumption.

Let now \( \mathcal{V} \) be a quasi-subtractive variety. Define \( \mathcal{V}_O = \mathcal{V} \cap \text{Mod}\{\Box x = x\} \) and \( \mathcal{V}_S = \mathcal{V} \cap \text{Mod}\{\Box x = 1\} \). In other words, \( \mathcal{V}_O \) is the largest open subvariety of \( \mathcal{V} \), while \( \mathcal{V}_S \) is a flat subvariety of \( \mathcal{V} \).

Theorem 44 If \( \Box \) commutes with all operations not preserving \( \{1\} \) on all algebras in \( \mathcal{V}_O \cup \mathcal{V}_S \), then \( \mathcal{V}_O \) and \( \mathcal{V}_S \) are quasi-independent and orthogonal.
Proof. Define $x * y = \square x$, $t_1(x) = x$, and $t_2(x) = \square x$. If $a, b \in A \in \mathbb{V}_O$, then $a * b = \square a = a = t_1(a)$ and $t_2(a) = \square a = a$, while if $a, b \in A \in \mathbb{V}_S$, then $a * b = \square a = 1 = \square b = t_2(b)$ and $t_1(a) = a$. Since $t_1$ is the identity and $t_2$ commutes with all operations not preserving $\{1\}$, by Lemma 43 we get that $t_1^A$ and $t_2^A$ are endomorphisms on any member $A$ of $\mathbb{V}_O \vee \mathbb{V}_S$. As $\mathbb{V}_O \vee \mathbb{V}_S$-idempotency of $t_2$ follows from Lemma 3.(ii), we conclude that $\mathbb{V}_O$ and $\mathbb{V}_S$ are quasi-independent. For the nontrivial part of the orthogonality condition, we need to show that the quasi-identity

$$x \approx \square y \& y \approx \square x \Rightarrow x \approx y$$

holds in $\mathbb{V}_O \vee \mathbb{V}_S$. Suppose therefore that for some $a, b \in A$, with $A \in \mathbb{V}_O \vee \mathbb{V}_S$, we have $a = \square b$ and $b = \square a$. But then we also have $b = \square a = \square \square b = \square b = a$ as required. ■

Corollary 45 If $\square$ commutes with all operations not preserving $\{1\}$ on all algebras in $\mathbb{V}_O \vee \mathbb{V}_S$, then $\mathbb{V}_O \vee \mathbb{V}_S = \mathbb{V}_O \times_s \mathbb{V}_S$.

Corollary 45 subsumes a number of decomposition results known in the literature on individual quasi-subtractive varieties: e.g. Theorem 55 of [35], according to which every quasi-MV algebra is a subdirect product of an MV algebra and a flat quasi-MV algebra, or a theorem according to which every member of $\mathcal{N}(\mathcal{V})$ (the nilpotent shift of a variety $\mathcal{V}$) is a subdirect product of a member of $\mathcal{V}$ and a member of $\mathcal{N}(\mathcal{V})$ which satisfies all and only the normal equations in the language of $\mathcal{V}$ (see e.g. [37]). However, many interesting direct decomposition theorems still do not fall under this common umbrella. For example, Galatos and Tsinakis introduced in [24] a noncommutative and nonintegral generalisation of MV algebras, called generalised MV algebras, and showed that each member of this variety, henceforth called GMV, decomposes as a direct product of an $\ell$-group and an integral residuated lattice. Actually, this property holds not only for GMV, but for the join $\mathbb{L} \mathbb{G} \vee \mathbb{I} \mathbb{R} \mathbb{L}$ of the last two varieties in the subvariety lattice of $\mathbb{R} \mathbb{L}$ (see e.g. [28]). It is easy to see that $\mathbb{I} \mathbb{R} \mathbb{L} \supseteq \mathbb{G} \mathbb{M} \mathbb{V} \mathbb{O}$ and that $\mathbb{L} \mathbb{G}$ is a flat subvariety of $\mathbb{G} \mathbb{M} \mathbb{V} \mathbb{O}$. We now aim at obtaining an abstract companion of this theorem.

From now on, let $\mathbb{V}_F$ be an arbitrary flat subvariety of a quasi-subtractive variety $\mathcal{V}$.

Lemma 46 There is a unary term $\exists x$ such that $\mathbb{V}_O \models \exists x \approx x$ and $\mathbb{V}_F \models \square x \approx 1$. The terms $\rightarrow', \square$ and 1, where $x \rightarrow' y = \square (x \rightarrow y)$, witness quasi-subtractivity for $\mathbb{V}_O \vee \mathbb{V}_F$.

Proof. (sketch) We work in the free $\mathbb{V}_O \vee \mathbb{V}_F$-algebra $\mathbb{F}$ on two generators $x, y$. Consider the congruences $\theta_O$ and $\theta_F$, respectively corresponding to the equational theories of $\mathbb{V}_O$ and $\mathbb{V}_F$. Since $\theta_O \vee \theta_F$ is the full congruence, it follows that $x \rightarrow y \theta_O \vee \theta_F 1$, whence an easy inductive argument resting on $(\exists x, 1)$-permutability ensures that $x \rightarrow y \theta_O \theta_F 1$ holds as well. Therefore, there exists an at most binary term $g(x, y)$ such that $x \rightarrow y \theta_O g(x, y) \theta_F 1$ hold. Since
θ₀ and θ₁ are both fully invariant, we conclude that □x = 1 \rightarrow x θ₀ g(1, x) θ₁ 1 hold as well, and thus, putting □x = g(1, x) we obtain

\[ ∇_O \models □x \approx x \approx □x \]

and

\[ ∇_F \models □x \approx 1 \]

proving our first claim. As to the other one, observe that since □x is the identity in ∇₀, the quasi-subtractivity equations are trivially satisfied therein, and that the same happens for ∇₁ since □x is constant.

**Theorem 47** If □ commutes with all operations not preserving {1} on all algebras in ∇₀ ∨ ∇₁, then ∇₀ and ∇₁ are quasi-independent and orthogonal (whence ∇₀ ∨ ∇₁ = ∇₀ ×, ∇₁).

**Proof.** The proof of Theorem 44 works with □ replaced by □x. ■

The decomposition theorems of Galatos-Jónsson-Tsinakis are immediate consequences of Theorem 47, because {1} is a subalgebra of every residuated lattice and thus the assumption of the theorem is automatically satisfied. Actually, it is easily seen that in such a case the definition

\[ □x = x (x^\prime 1) \]

does the job. The only remaining drawback is that such a result only ensures subdirect, rather than direct, decomposition. We now make up for this flaw.

Recall that a unital groupoid is an algebra A = ⟨A, o, 1⟩ of type ⟨2, 0⟩ satisfying the identities x o 1 ≈ x ≈ 1 o x. We can prove:

**Theorem 48** Let V be a quasi-subtractive variety s.t. every member of V has a unital groupoid term reduct, and let ∇₀ and ∇₁ be as above. ∇₀ and ∇₁ are independent if and only if there exists a unary term □ satisfying the following conditions:

1. ∇₀ \models □x \approx 1,
2. ∇₁ \models □x \approx x.

**Proof.** From right to left, define x * y as □x o □y. This binary term is well-defined in virtue of Lemma 46. It is readily verified that ∇₀ \models x * y = x and ∇₁ \models x * y = y, showing that ∇₀ and ∇₁ are independent. Conversely, if ∇₀ and ∇₁ are independent, we define □x = 1 * x. It is again easy to verify that ∇₀ \models □x \approx 1 and ∇₁ \models □x \approx x. ■

In particular, if V is 1-subtractive and every member of V has a unital groupoid term reduct, the unary term of Theorem 48 necessarily exists (see e.g. [34]) and therefore the conclusion of the same theorem necessarily holds. Since the variety of residuated lattices satisfies precisely these conditions, we get the decomposition results in [24] and [28] as immediate corollaries. In this case we can choose

\[ □x = (x^\prime 1) \setminus 1. \]
5 Open contractions

Let $\mathcal{V}$ be a quasi-subtractive variety and $A \in \mathcal{V}$. For any term $t(x_1, \ldots, x_k)$ in its type, we define its open translation $t^{\square}$ inductively as follows:

- $x^{\square} = x$, for a variable $x$,
- $o^{\square}(t_1, \ldots, t_k) = \Box o(t_1^{\square}, \ldots, t_k^{\square})$, for a $k$-ary basic operation $o$ and terms $t_1, \ldots, t_k$.

Let further $A^{\square}$ be the set $\{a \in A: \Box a = a\}$, i.e., the set of open elements of $A$. Since $\Box$ is idempotent, $A^{\square}$ is closed under $o^{\square}$ for every basic operation $o$. Define $A^{\square}$ to be the algebra $\langle A^{\square}, (o^{\square})_{o \in O} \rangle$, where $O$ is the set of all basic operations in the type. We will call $A^{\square}$ an open contraction of $A$. Clearly, $A^{\square}$ is an algebra similar to $A$; moreover, the open contraction construction is functorial, a fact which the lemma below spells out.

Lemma 49 Let $h: A \rightarrow B$ be a homomorphism. Then, $h|_{A^{\square}}: A^{\square} \rightarrow B^{\square}$ is a homomorphism, and the diagram

![Diagram](image)

commutes. In particular, if $\theta$ is a congruence on $A$, then $\theta|_{A^{\square}}$ is a congruence on $A^{\square}$.

Proof. Immediate from the definition of open translation and the fact that $\Box$ is a term operation and so $h$ commutes with it.

Observe that by Theorem 25 the correspondence above extends to $\mathcal{V}$-open filters. Although these are well-defined in all cases, $A^{\square}$ may not itself belong to $\mathcal{V}$. It is therefore natural to consider the class $\mathcal{V}^{\square} = \{A^{\square}: A \in \mathcal{V}\}$, but at this level of generality not much can be said. Things begin to improve if the open translation preserves some structure. A modest requirement is the following. An open contraction $A^{\square}$ is smooth, if $\Box^{\square}$ and $\rightarrow^{\square}$ coincide on $A^{\square}$ with $\Box$ and $\rightarrow$. Notice that if $\Box$ and $\rightarrow$ are basic operations in the type, then the open translation is smooth by their definitional properties. However, if $\Box$ or $\rightarrow$ are compound terms, smoothness may fail, as the next example shows.

Example 50 Let $A = \langle\{0,1,2\}, \rightarrow, f, g\rangle$, with the operations given by the tables below.

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

22
Upon defining $\square x = f(g(x))$, it is easy to verify that $V = \forall(A)$ is quasi-subtractive, with witness terms $\rightarrow, \square$ and 1. The open contraction $A^{\square}$ of $A$ is term-equivalent to $\mathbb{Z}_2$, but the open translation is not smooth, as we have $\square 0 = 1 \neq 0 = \square 0$. Observe that this is the smallest possible example, since in every two-element quasi-subtractive algebra $\square$ is either the identity or the constant 1, and the open translation is smooth in either case.

**Lemma 51** Let $A$ be quasi-subtractive with witness terms $\rightarrow, \square$ and 1. If $A^{\square}$ is smooth, then $A^{\square}$ is 1-subtractive with witness term $\rightarrow^{\square}$.

**Proof.** Since $\rightarrow^{\square}$ coincides with $\rightarrow$ on $A^{\square}$, we only need to show that $\square a = a$, for every $a \in A^{\square}$. But $\square a = \square a$ by assumption, and $\square a = a$ since $a \in A^{\square}$, so the claim holds. ■

If $A^{\square}$ is smooth for all $A \in V$, we call $V$ smooth as well. Smoothness obviously carries over to subvarieties, and is a rather weak property of $V$. Indeed all examples of quasi-subtractive varieties we gave earlier are smooth. A more substantial requirement on $V$ is that it be closed under open contractions, i.e., $A \in V$ implies $A^{\square} \in V$. If $V$ has this property we will call it contractive. Being contractive is a fairly strong property.

**Lemma 52** Let $V$ be smooth and contractive. Then, the class $V^{\square}$ is a variety and it coincides with $V_O$, the open subvariety of $V$.

**Proof.** Since $(A^{\square})^{\square} = A^{\square}$ by smoothness, and $A^{\square} \in V$ by contractivity, by Lemma 49 $V^{\square}$ is closed under direct products, homomorphic images and subalgebras. Each member of $V^{\square}$ satisfies the equation $\square x \approx x$, so we have $V^{\square} \subseteq V_O$. Conversely, since any algebra $A \in V_O$ satisfies $\square x \approx x$, we obtain $A^{\square} = A$ and therefore $A \in V^{\square}$, proving $V_O \subseteq V^{\square}$. ■

A subvariety $\mathbb{W}$ of a contractive variety $V$ may fail to be contractive, and there are many natural examples of non-contractive varieties (see Section 5.2 below). However, each quasi-subtractive variety is a subvariety of a contractive one, as we will now show. Namely, given a quasi-subtractive variety $\mathbb{W}$, we define $\mathbb{W}_C$ to be the class of models of the part of the equational theory of $\mathbb{W}$ that is preserved under taking open contractions. More precisely, $\mathbb{W}_C = \text{Mod}(\text{Eq}(\mathbb{W}) \cap \text{Eq}(\mathbb{W}^{\square}))$.

**Lemma 53** For any quasi-subtractive smooth variety $\mathbb{W}$, we have $(\mathbb{W}_C)^{\square} = \mathbb{W}^{\square}$. Thus, the variety $\mathbb{W}_C$ is contractive.

**Proof.** It suffices to show that for any $A \in \mathbb{W}_C$ we have $A^{\square} \in \mathbb{W}^{\square}$. By definition, $\mathbb{W}_C = V(\mathbb{W} \cup \mathbb{W}^{\square})$, so $A \in HS(C \times D^{\square})$, for some $C, D \in \mathbb{W}$. Thus, we have $C \times D^{\square} \supseteq E \rightarrow A$ for some $E$. By Lemma 49 we get $C^{\square} \times (D^{\square})^{\square} = (C \times D^{\square})^{\square} \supseteq E^{\square} \rightarrow A^{\square}$.

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and smoothness yields \((D^\Box)^\Box = D^\Box\). We obtain
\[
C^\Box \times D^\Box \supseteq E^\Box \to A^\Box
\]
proving the claim. ■

**Lemma 54** Let \(\forall\) be smooth and contractive. Suppose \(A \in \forall\). If \(F\) is a \(\forall\)-open filter on \(A\), then \(F|_{A^\Box}\) is a \(\forall\)-open filter on \(A^\Box\).

**Proof.** By Theorem 25, \(F = (\Box x, 1)/\theta\) for some congruence \(\theta\). If \(a \in F|_{A^\Box}\), then \(\Box a = a\), and thus \((a, 1) \in \theta\). Therefore, \((a, 1) \in \theta|_{A^\Box}\) is a congruence on \(A^\Box\), the claim is proved. ■

A form of converse also holds.

**Lemma 55** Let \(\forall\) be smooth and contractive. Suppose \(A \in \forall\). If \(F \subseteq A^\Box\) is closed under \(\forall\)-open filter terms taken in \(A\), then \(\uparrow F\) is a \(\forall\)-open filter on \(A\).

**Proof.** By Theorem 30, the open filter generated by \(F\) is \(\uparrow F\). Since \(F\) is closed under \(\forall\)-open filter terms, we have \(\Gamma F = F\), so \(\uparrow \Gamma F = \uparrow F\). ■

If a variety \(\forall\) is contractive, then, obviously, all identities satisfied by \(\forall\) are also satisfied by \(\forall^\Box\). This observation can be made into a characterisation of contractive varieties. It should be particularly useful if \(\forall\) has a manageable equational basis.

An identity \(t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)\) will be called \(\forall\)-stable if

- \(\forall \models t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)\), and
- \(\forall \models t^\Box(\Box x_1, \ldots, \Box x_n) \approx s^\Box(\Box x_1, \ldots, \Box x_n)\),

where \(t^\Box\) and \(s^\Box\) are the respective open translations of \(t\) and \(s\).

**Theorem 56** Let \(\forall\) be quasi-subtractive. Then the following are equivalent:

1. \(\forall\) is contractive;
2. \(\forall\) has a basis of \(\forall\)-stable identities;
3. every basis of \(\forall\) consists of \(\forall\)-stable identities;
4. the equational theory of \(\forall\) consists of \(\forall\)-stable identities.

**Proof.** Clearly (4) \(\Rightarrow\) (3) \(\Rightarrow\) (2), so we only need to prove (2) \(\Rightarrow\) (1) \(\Rightarrow\) (4).

First, for any term \(t(x_1, \ldots, x_n)\), it is easily shown by induction on the complexity of \(t\) that (i) \(t^\Box(\Box x_1, \ldots, \Box x_k)\) evaluated in \(A\) on elements \(a_1, \ldots, a_n \in A\) coincides with \(t(x_1, \ldots, x_k)\) evaluated in \(A^\Box\) on the elements \(\Box a_1, \ldots, \Box x_n\), and (ii) \(t^\Box(\Box x_1, \ldots, \Box x_k)\) evaluated in \(A\) on elements \(a_1, \ldots, a_n \in A^\Box\) coincides with \(t(x_1, \ldots, x_k)\) evaluated in \(A^\Box\) on the same elements.

Now, to prove (2) \(\Rightarrow\) (1), suppose \(\Sigma\) is a set of \(\forall\)-stable identities and \(\forall = \text{Mod}(\Sigma)\). Let \(t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)\) be an identity from \(\Sigma\), and \(A \in \forall\). Since \(t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)\) is \(\forall\)-stable, \(A \models t^\Box(\Box x_1, \ldots, \Box x_n) \approx s^\Box(\Box x_1, \ldots, \Box x_n)\) in \(A^\Box\).
Consider \( \text{by smoothness,} \) Proof. \( \text{Lemma 57} \)

Let \( A \) of \( \varnothing \) on \( A \). By invertibility, \( \phi \) on \( \varnothing \) there is a congruence \( \theta \) on \( A \) such that \( \phi = \theta |_{A^\varnothing} \). Invertibility can be viewed as a rather weak form of congruence extension property, except that \( A^\varnothing \) is in general not a subalgebra of \( A \).

Lemma 57 Let \( \varnothing \) be smooth and invertible and \( A \in \varnothing \). Suppose \( F \) is a \( \varnothing \)-open filter on \( A^\varnothing \). Then, \( \uparrow F \) is a \( \varnothing \)-open filter on \( A \).

Proof. By smoothness, \( \varnothing \)- is subtractive. Therefore, \( F = 1/\phi \) for some congruence \( \phi \) on \( A^\varnothing \). By invertibility, there is a congruence \( \theta \) on \( A \), with \( \theta |_{A^\varnothing} = \phi \). Consider \( (\sqcap x, 1)^A/\theta \). By Lemma 19 this is an open filter on \( A \), and since \( A^\varnothing \models \sqcap x \approx x \), the restriction of \( (\sqcap x, 1)^A/\theta \) to \( A^\varnothing \) is precisely \( F \). But then \( \uparrow F = F \cup \{ a \in A : \sqcap a \in F \} = 1/\phi \cup (\sqcap x, 1)^A/\theta \), and

\[
1/\phi \subseteq 1/\theta \subseteq (\sqcap x, 1)^A/\theta,
\]

so \( \uparrow F = (\sqcap x, 1)^A/\theta \), as needed. \( \blacksquare \)

Notice that the assumptions of Lemma 57 did not include contractivity, suggesting that invertibility and contractivity are independent. This is indeed the case: the variety of residuated lattices is contractive, but not invertible; the variety of lattice-ordered groups is invertible, but not contractive.

5.2 Examples

5.2.1 Residuated lattices

Let \( \text{RL} \) be the variety of residuated lattices. As we know from Example 7, the terms \( \sqcap x = x \land 1 \) and \( x \to y = x \land y \land 1 \) (together with the constant 1) witness quasi-subtractivity of \( \text{RL} \). Then, \( \text{RL}^\varnothing \) is the variety \( \text{IRL} \) of integral residuated lattices, a subvariety of \( \text{RL} \), of course, so \( \text{RL} \) is contractive. The
open contraction above is better known as negative cone contraction [23], a particular case of kernel contraction. Negative cone contractions preserve many important structural properties of residuated lattices, but congruences on the negative cone do not extend to congruences on the whole algebra, so \( \mathbb{RL} \) is not invertible. However, the commutative subvariety of \( \mathbb{RL} \) is invertible ([23], Lemma 3.57).

5.2.2 Lattice-ordered groups

The variety \( \mathbb{LG} \) of lattice-ordered groups is a subvariety of \( \mathbb{RL} \), but \( \mathbb{LG} \) is not contractive, as \( \mathbb{LG}^{-} \) is the class \( \mathbb{LG}^{-} \) of negative cones of \( \ell \)-groups, which in this case happens to be a variety, but not a subvariety of \( \mathbb{LG} \). We have \( \mathbb{LG}_\mathbb{O} = V(\mathbb{LG} \cup \mathbb{LG}^{-}) = \mathbb{LG} \vee \mathbb{LG}^{-} = \mathbb{LG} \times \mathbb{LG}^{-} \) in this case. Although \( \ell \)-groups are not commutative, they are invertible (cf. Theorem 7.9 in [18]), showing that commutativity is sufficient but not necessary for invertibility in the context of \( \mathbb{RL} \).

5.2.3 Interior algebras

We know from Example 14 that, upon defining \( x \rightarrow y = \Box(\neg x \vee y) \), interior algebras are quasi-subtractive with witness terms \( \rightarrow, \Box \) and 1. Essentially by Gödel’s translation, open contractions of interior algebras are Heyting algebras, and conversely, every Heyting algebra is an open contraction of some interior algebra. This shows that the variety of interior algebras is not contractive. Moreover, every filter \( F \) on a Heyting algebra \( A^{-} \) extends to an open filter \( G \) of \( A \) such that \( G = \uparrow F \). Therefore, the variety of interior algebras is invertible. Remark, on the other hand, that \( \mathbb{V}_O \) is the variety of Boolean algebras.

5.2.4 Algebraic models of linear logic

Although algebraic semantics for linear logic is by far less popular than phase space semantics, or other semantics in the category theory vein, it is rather nicely developed in [1] in the form of the variety of girales. These algebras are term equivalent to bounded involutive \( \text{FL}_e \)-algebras with an additional unary operation \(!\), satisfying the identities

1. \(!1 = 1\)
2. \(!x \leq x \land 1\)
3. \(!x \cdot !y = !\langle x \land y \rangle\)
4. \(!!x = !x\)

We now define \( \square x = !x \) and \( x \rightarrow y = !(x \rightarrow y) \), where we let \( x \rightarrow y \) stand for \( x \setminus y \) to remain faithful to the usual linear logic notation. Using the results in [1] it is reasonably straightforward to prove that these terms witness quasi-subtractivity of the variety of girales. It then follows by a simple restatement
of results in the same paper that for any girale $A$ its open contraction $A \Box$ is (term-equivalent to) a Heyting algebra, and, moreover, any filter $F$ on $A \Box$ extends to an open filter $G$ on $A$, with $G = \uparrow F$. So, the variety of girales fails to be contractive but is invertible.

5.2.5 Heyting algebras

Let $A$ be a Heyting algebra. Define $\Box x = \neg \neg x$, and $x \to y = \neg \neg (x \supset y)$, where $\supset$ stands for Heyting implication. We have $1 \to x = \neg \neg(1 \supset x) = \neg \neg x = \Box x$, $\Box x \to x = \neg \neg(\neg \neg x \supset x) = 1$ by Glivenko's Theorem, $\Box(x \to y) = \neg \neg \neg(x \supset y) = \neg \neg(x \supset y) = x \to y$, and $\Box(x \to y) \to (\Box x \to \Box y) = \neg \neg(\neg \neg(x \supset y) \supset \neg \neg(\neg \neg x \supset \neg \neg y)) = 1$ again by Glivenko's Theorem. So, Heyting algebras form a quasi-subtractive variety which is smooth and contractive.

5.2.6 Varieties of FL-algebras with (left) Glivenko property

Let $V$ and $W$ be varieties of FL algebras. Following [23, Chapter 8], we say that $V$ has the left Glivenko property relative to $W$ if for every term $t$ the following holds

$$W \models 1 \leq t \iff V \models 1 \leq \neg \neg t.$$ 

By Proposition 8.8, Lemma 8.12 and Proposition 8.13 in [23], this is equivalent to the conjunction of

$$W \models 1 \leq \neg \neg t \text{ implies } W \models 1 \leq t$$

for any term $t$ (a variety $W$ with this property is called left Glivenko involutive) and any of the mutually equivalent properties below:

- $W \models \neg s \approx \neg t \iff V \models \neg s \approx \neg t$
- $W \models -s \approx -t \iff V \models -s \approx -t$
- $W \models \neg s \leq \neg t \iff V \models \neg s \leq \neg t$
- $W \models -s \leq -t \iff V \models -s \leq -t$

for any terms $s, t$ (varieties $V, W$ with this property are called Glivenko equivalent). Suppose $V$ has the left Glivenko property relative to $W$, and moreover $W$ is left involutive, that is, $W \models \neg \neg x \leq x$. Let $\Box x = \neg \neg x \land 1$ and $x \to y = \neg \neg(x \setminus y) \land 1$. It can be shown that $V$ is quasi-subtractive with the above witness terms. It can also be shown that $V \Box = W$ (so in particular $V \Box$ is a variety) and that the open translation is smooth, if $V \models \neg \neg \neg x \leq \neg \neg x$. We do not know whether that (or indeed any) assumption is necessary.
6 Weakly \( \tau \)-regular varieties

As we already recalled, in every algebra \( A \) belonging to an ideal determined variety \( V \) the lattice of congruences on \( A \) is isomorphic to the lattice of \( V \)-ideals of \( A \). We also mentioned in the introduction that some isomorphism results along the same lines do not lend themselves to be viewed as special cases of this theorem, either because the variety at issue fails to be ideal determined, or else because the lattice proved isomorphic to \( \text{Con}(A) \) is not the lattice of \( V \)-ideals of \( A \). With an eye to subsuming under a common umbrella all these theorems, we start by introducing a suitably weakened version of the property of \( \tau \)-regularity [10].

**Definition 58** A variety \( V \) is called weakly \( \tau \)-regular iff the \( \tau \)-assertional logic \( S(V, \tau) \) of \( V \) is strongly and finitely algebraisable.

Observe that a weakly \( \tau \)-regular variety \( V \) is \( \tau \)-regular iff, in addition, \( V \) is the equivalent algebraic semantics of \( S(V, \tau) \). It is well possible to be weakly \( \tau \)-regular without being \( \tau \)-regular: for example, the variety \( qMV \) of quasi-MV algebras is weakly \((x \oplus 0, 1)\)-regular, but it fails to be \((x \oplus 0, 1)\)-regular because the equivalent algebraic semantics of \( S(V, \{x \oplus 0 \approx 1\}) \) (namely, of infinite-valued Łukasiewicz logic) is the variety \( MV \) of MV algebras.

The next Lemma is implicit in Theorem 5.1 of [7], although it is explicitly stated only for the special case where \( V \) coincides with \( V' \):

**Lemma 59** \( V \) is weakly \( \tau \)-regular iff, given \( A \) in \( V \) and given any two \( V \)-\( V' \) congruences \( \theta, \varphi \) on \( A \) (where \( V' \) is the equivalent algebraic semantics of \( S(V, \tau) \)), we have that \( \tau^A/\theta = \tau^A/\varphi \) implies \( \theta = \varphi \).

We now leave again the general scenario of an arbitrary translation \( \tau \) to focus once more on translations of the form \( \Box x \approx 1 \); recall that \( S(V) \) is short for \( S(V, \Box x \approx 1) \). The next Theorem is also more or less contained in Blok’s and Pigozzi’s monograph - nonetheless, we include a detailed proof since Blok and Pigozzi do not explicitly consider the more general case where \( V \) may fail to coincide with \( V' \).

**Theorem 60** If \( V \) is weakly \((\Box x, 1)\)-regular and \( V' \) is the equivalent algebraic semantics of \( S(V) \), then in any \( A \in V \) there is a lattice isomorphism between the lattice of \( V \)-\( V' \) congruences on \( A \) and the lattice of deductive filters on \( A \) of \( S(V) \).

**Proof.** Suppose \( A \in V \). We want to show that the mapping \( f(\theta) \), defined in Subsection 3.4, is the required isomorphism. By the proof of Lemma 26, such a mapping is well-defined, i.e. \((\Box x, 1)^A/\theta \) is a deductive filter. Clearly, \( f \) preserves binary meets and joins. By Lemma 59, it is also one-one. It remains to be shown that it is onto, i.e. that every deductive filter on \( A \) is the \((\Box x, 1)^A\)-class of some congruence \( \theta \).
So, let $F$ be a deductive filter on $A$, and let
\[ \Omega^A_\rho (F) = \{(a,b) : \rho^A (a,b) \subseteq F\}, \]
where $\rho$ is a set of equivalence terms for $S(\mathcal{V})$ (observe that this set must perforce exist, as $\mathcal{V}$ is weakly $\Box x, 1$-regular). We will prove that
\[ F = f (\Omega^A_\rho (F)). \]
Since $S(\mathcal{V})$ is algebraisable, we are in a position to resort to Theorem 4.7 in [7], which guarantees that $\Omega^A_\rho (F)$ is a congruence, and a $\mathcal{V}$-$\mathcal{V}'$ congruence at that. The same result also implies that for every term $t$ we have
\[ t \vdash_{S(\mathcal{V})} \rho (\Box t, 1). \]
Therefore, for every $a \in F$, $\rho^A (\Box a, 1) \subseteq F$ and conversely, by the definition of deductive filter. Hence
\[
\begin{align*}
a \in F & \iff \rho^A (\Box a, 1) \subseteq F \\
& \iff (\Box a, 1) \in \Omega^A_\rho (F) \\
& \iff a \in f (\Omega^A_\rho (F)).
\end{align*}
\]

\textbf{Corollary 61} If $\mathcal{V}$ is quasi-subtractive and weakly $\Box x, 1$-regular and $\mathcal{V}'$ is the equivalent algebraic semantics of $S(\mathcal{V})$, then in any $A \in \mathcal{V}$ there is a lattice isomorphism between the lattice of $\mathcal{V}$-$\mathcal{V}'$ congruences on $A$ and the lattice of $\mathcal{V}$-$\mathcal{V}'$ open filters on $A$.

\textbf{Proof.} By Theorems 27 and 60. □

Corollary 61 subsumes many lattice isomorphism theorems known in the literature. In fact:

\begin{itemize}
  \item If we let $\Box$ be the identity term therein, then $\mathcal{V}$ is 1-subtractive and weakly 1-regular. However, for a variety $\mathcal{V}$, being weakly 1-regular means nothing else than that the 1-assertional logic of $\mathcal{V}$ is finitely and regularly algebraisable with $\mathcal{V}$ as equivalent algebraic semantics. So $\mathcal{V} = \mathcal{V}'$ and $\mathcal{V}$ is after all 1-subtractive and 1-regular, i.e. 1-ideal determined. In this particular case, therefore, we get as an instance of Corollary 61 the isomorphism between the lattice of congruences and the lattice of $\mathcal{V}$-ideals in members of ideal determined varieties.

  \item The variety $\mathbb{RL}$ of residuated lattices is quasi-subtractive and $(x \land 1, 1)$-regular: as already mentioned, its $(x \land 1, 1)$-assertional logic is nothing but the 0-free fragment $\mathbb{RL}$ of the substructural logic $\mathbb{FL}$ (see [23]), which is algebraisable with $\mathbb{RL}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any residuated lattice $A$ there is an isomorphism between the lattice of congruences on $A$ and the lattice of $\mathbb{RL}$-open filters on $A$, which are its deductive filters in the sense of [23]. This isomorphism theorem is exactly the content of Theorem 3.47 in [23].
\end{itemize}
• The variety $\mathbb{P}$ of pseudointerior algebras is quasi-subtractive and $(x^\oplus, 1)$-regular: its $(x^\oplus, 1)$-assertional logic is algebraisable with $\mathbb{P}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any pseudointerior algebra $A$ there is an isomorphism between the lattice of congruences on $A$ and the lattice of $\mathbb{P}$-open filters on $A$, which are its open filters in the sense of [7]. This isomorphism theorem is exactly the content of Theorem 2.16 in [7].

• The variety $q\text{MV}$ of quasi-MV algebras is quasi-subtractive and $(x \oplus 0, 1)$-regular: its $(x \oplus 0, 1)$-assertional logic is nothing but infinite-valued Lukasiewicz logic (see [33]), which is algebraisable with the variety $\text{MV}$ of MV algebras as equivalent algebraic semantics. According to Corollary 61, therefore, in any quasi-MV algebra $A$ there is an isomorphism between the lattice of $q\text{MV}$-MV congruences on $A$ and the lattice of $q\text{MV}$-open filters on $A$, which are (the dualisation of) ideals in the sense of [35]. This isomorphism theorem is exactly the content of Theorem 45 in [35].

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