Abstract. In this short note, we provide an inequality that holds in any finite group, only involving the orders of the elements; we prove that equality holds if and only if the group is nilpotent.

As pointed out by Marty Isaacs, there is a mistake in the proof of Theorem 2 (taken from the American Mathematical Monthly) on which we rely. Theorem 2 is still conjecturally true, but reduces our main theorem to a conjecture. We have therefore withdrawn our paper.

1. The statement

Let $G$ be a finite group of order $n$. For each $x \in G$, we write $o(x)$ for the order of $x$. For each natural number $k$, we write $\phi(k)$ for the Euler totient function of $k$, and $\sigma(k)$ for the sum of the divisors of $k$.

Theorem 1. Let $G$ be a finite group of order $n$. Then

$$
\sum_{x \in G} \frac{o(x)}{\phi(o(x))} \leq \sigma(n),
$$

and equality holds if and only if $G$ is nilpotent.

2. Motivation

It is an interesting question in general what properties of finite groups can be detected only using the orders of all elements of the group. It is obvious that cyclicity can be detected: the cyclic group of order $n$ is the unique group such that there are exactly $\phi(d)$ elements of order $d$, for each $d \mid n$. It is only a little bit harder to see that also nilpotency can be detected, using the fact that a finite group is nilpotent if and only if there is a unique Sylow $p$-subgroup for each prime $p$ dividing the order of the group.

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It is a famous open question, attributed to John Thompson (because of a letter that he wrote in 1987 to Wujie Shi) whether solvability can be detected in this fashion.

The property that we describe is another property detecting nilpotency, but the nice feature of it is that it can be rather elegantly expressed as an inequality that holds for every finite group, and for which equality holds if and only if the group is nilpotent. This result appeared as a conjecture in a previous paper \cite{1}, where we already noticed that equality holds for nilpotent groups; this observation was related to the study of the poset of cyclic subgroups of arbitrary finite groups.

The proof that we present is completely elementary, and is based on a beautiful (and equally elementary) result that we found in \cite{3}.

3. The proof

We begin by stating the result from \cite{3} that we will use.

**Theorem 2.** Let $G$ be a finite group of order $n$. Then there is a partition $(S_d(G))_{d | n}$ of $G$ such that for every divisor $d$ of $n$ we have:

(a) \( |S_d(G)| = \phi(d) \);
(b) \( x^d = 1 \) for all \( x \in S_d(G) \).

In particular, there is a bijection \( f : G \rightarrow \mathbb{Z}_n \) satisfying

\[
\text{o}(x) \mid \text{o}(f(x)), \quad \text{for all } x \in G.
\]

**Proof.** The “proof” which appeared in \cite{3} is incorrect. For a discussion, see \url{http://mathoverflow.net/questions/104183/}. \qed

We are now ready to prove our main theorem.

**Proof of Theorem 1.** Let

\[
\sigma(G) = \sum_{x \in G} \frac{\text{o}(x)}{\phi(\text{o}(x))},
\]

and for each $d \mid n$, we let $S_d(G)$ be as in Theorem 2 Then for each $x \in S_d(G)$ we have $\text{o}(x) \mid d$, and hence

\[
(*) \quad \frac{\text{o}(x)}{\phi(\text{o}(x))} \leq \frac{d}{\phi(d)}.
\]

It follows that

\[
\sigma(G) = \sum_{d \mid n} \sum_{x \in S_d(G)} \frac{\text{o}(x)}{\phi(\text{o}(x))} \leq \sum_{d \mid n} \sum_{x \in S_d(G)} \frac{d}{\phi(d)} = \sum_{d \mid n} d = \sigma(n).
\]
Assume now that $\sigma(G) = \sigma(n)$. Then for every divisor $d$ of $n$ and for every $x \in S_d(G)$, we have

$$
\frac{o(x)}{\phi(o(x))} = \frac{d}{\phi(d)};
$$

that is, $o(x)$ and $d$ have the same prime factors. In particular, for each $x \in S_d(G)$, the order of $x$ is a $p$-power if and only if $d$ is a $p$-power. This means that the number of $p$-elements in $G$ is equal to the number of $p$-elements in $\mathbb{Z}_n$. This implies that $G$ has a unique Sylow $p$-subgroup, for each prime $p$ dividing the order of $G$, and hence $G$ is nilpotent.

Conversely, if $G$ is nilpotent, then $\sigma(G) = \sigma(n)$ by [1, Theorem 5], completing the proof. □

### 4. A Generalization

We end by indicating a natural generalization of Theorem 1. For each $k \in \mathbb{N}^*$, we let

$$
\sigma_k(G) = \sum_{x \in G} \frac{o(x)^k}{\phi(o(x))} \quad \text{and} \quad \sigma_k(n) = \sum_{d \mid n} d^k.
$$

**Theorem 3.** Let $G$ be a finite group of order $n$. Then $\sigma_k(G) \leq \sigma_k(n)$ for all $k \in \mathbb{N}^*$, and we have equality if and only if $G$ is cyclic.

**Proof.** Again using [4], we obtain

$$
\sigma_k(G) = \sum_{d \mid n} \sum_{x \in S_d(G)} \frac{o(x)^k}{\phi(o(x))} \leq \sum_{d \mid n} \sum_{x \in S_d(G)} \frac{d^k}{\phi(d)} = \sum_{d \mid n} d^k = \sigma_k(n).
$$

Obviously, for $k \geq 2$ we have $\sigma_k(G) = \sigma_k(n)$ if and only if

$$
o(x) = d, \quad \text{for all } d \mid n \text{ and all } x \in S_d(G).
$$

This means that for every divisor $d$ of $n$, $G$ has exactly $\phi(d)$ elements of order $d$. This is equivalent to the fact that $G$ is cyclic, as desired. □

**Remark 4.** The expressions $\sigma_k(G)$ have a natural meaning in terms of the poset of cyclic subgroups of $G$. More precisely, we have

$$
\sigma_k(G) = \sum_{H \in C(G)} |H|^k;
$$

this can be shown in exactly the same way as for [1, Theorem 2].
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References

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