Complex and Hypercomplex Discrete Fourier Transforms Based on Matrix Exponential Form of Euler’s Formula

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Abstract—We show that the discrete complex, and numerous hypercomplex, Fourier transforms defined and used so far by a number of different researchers can be unified into a single theoretical framework based on a matrix exponential version of Euler’s formula $e^{j\theta} = \cos \theta + j \sin \theta$, and a matrix root of $-1$ isomorphic to the imaginary root $j$. The transforms thus defined can be computed numerically using standard matrix multiplications and additions with no hypercomplex code, the complex or hypercomplex algebra being represented by the form of the matrix root of $-1$, so that the matrix multiplications are equivalent to multiplications in the appropriate algebra. We present examples from the complex, quaternion and biquaternion algebras, and from Clifford algebras $Cl_{1,1}$ and $Cl_{2,0}$. The significance of this result is both in the theoretical unification achieved, and also in the scope it affords for insight into the structure of the various transforms, since the formulation is such a simple generalization of the classic complex case. It also shows that hypercomplex discrete Fourier transforms may be evaluated numerically using standard matrix arithmetic packages, which is of importance in providing a reference implementation for testing implementations based on hypercomplex libraries.

Index Terms—Discrete Fourier transform, hypercomplex algebra, quaternions, biquaternions, Clifford algebras.

I. INTRODUCTION

The discrete Fourier transform [1] is widely known and used in signal and image processing, and in many other fields where data is analyzed for frequency content. The discrete Fourier transform in one dimension is classically formulated as:

$$F[u] = S \sum_{m=0}^{M-1} f[m] \exp \left( -j2 \pi \frac{mu}{M} \right)$$

$$f[m] = T \sum_{u=0}^{M-1} F[u] \exp \left( j2 \pi \frac{mu}{M} \right)$$

(1)

where $j$ is the imaginary root of $-1$, $f[m]$ is real or complex with $M$ samples, $F[u]$ is complex, also with $M$ samples, and the product of the two scale factors $S$ and $T$ must be $\frac{1}{M}$. In this paper we show that the transform may be formulated using a matrix exponential form of Euler’s formula in which the imaginary square root of $-1$ is replaced by an isomorphic matrix root. The matrix exponential formulation is equivalent to all the known hypercomplex generalizations of the DFT known to the authors, based on quaternion, biquaternion or Clifford algebras, through a suitable choice of matrix root of $-1$, isomorphic to a root of $-1$ in the corresponding hypercomplex algebra. This result achieves a theoretical unification of diverse hypercomplex DFTs, and it also shows that computation of a hypercomplex DFT may be performed without a hypercomplex library using only standard real or complex matrix arithmetic.

Hypercomplex Fourier transforms were first developed in the early 1990s [2], [3] using two hypercomplex exponentials positioned either side of the signal function (since multiplication in a hypercomplex algebra is non-commutative, the ordering of terms within the summation is significant). This style of transform was followed by Chernov, Bülow and Sommer [4], [5], [6] and others since. In 1998 the present authors described a single-sided hypercomplex transform for the first time [7] exactly as in (1) except that $f$ and $F$ were quaternion-valued and $j$ was replaced by a general quaternion root of $-1$. Pei et al studied efficient implementation of quaternion FFTs and presented a transform based on commutative reduced biquaternions [8], [9]. Ebлинg and Scheuermann defined a Clifford Fourier transform [10, §5.2], but their transform used the pseudoscalar (one of the basis elements of the algebra) as the square root of $-1$. Thus, apart from the works by the present authors [7], [11], [12], the idea of using a root of $-1$ different to the basis elements of a hypercomplex algebra was not developed. This has changed since 2006, with the publication of a paper setting out the roots of $-1$ in biquaternions (a quaternion algebra with complex numbers as the components of the quaternions) [13]. This work prepared the ground for a biquaternion Fourier transform [14] based on the present authors’ one-sided quaternion transform. More recently, the idea of finding roots of $-1$ in other algebras has been advanced in Clifford algebras by Hitzer and Ablamowicz [15] with the express intent of using them in Clifford Fourier transforms, perhaps generalising the ideas of Ebлинg and Scheuermann [10]. Finally, in this very brief summary of prior work we mention that the idea of applying hypercomplex algebras in signal processing has been studied by other authors apart from those referenced above. For an overview see [16].

The final point we must make here is that associative hypercomplex algebras (and indeed the complex algebra) are known to be isomorphic to matrix algebras over the reals or the complex numbers. For example, Ward [17, §2.8] discusses
isomorphism between the quaternions and $4 \times 4$ real or $2 \times 2$ complex matrices so that quaternions can be replaced by matrices, the rules of matrix multiplication then being equivalent to the rules of quaternion multiplication by virtue of the pattern of the elements of the quaternion within the matrix. Also in the quaternion case, Ickes [18] wrote an important paper showing how multiplication of quaternions could be accomplished using a matrix-vector or vector-matrix product that could accommodate reversal of the product ordering by a partial transposition within the matrix. This paper, more than any other, led us to the idea presented here.

II. MATRIX FORMULATION OF THE DISCRETE FOURIER TRANSFORM

A. Matrix form of Euler’s formula

The transform presented in this paper depends on a generalization of Euler’s formula: $\exp j\theta = \cos \theta + j \sin \theta$, in which the imaginary root of $-1$ is replaced by a matrix root, that is, a matrix that squares to give a negated identity matrix. In the matrix generalization, the exponential must, of course, be a matrix exponential [19 §11.3]. No claim to originality is made for the following result, but it is essential to Theorem [1] and we have not been able to find it in the literature.

Lemma 1: Euler’s formula may be generalized as follows:

$$e^{J\theta} = I \cos \theta + J \sin \theta$$

where $I$ is an identity matrix, and $J^2 = -I$.

Proof: The result follows from the series expansions of the matrix exponential and the trigonometric functions. From the definition of the matrix exponential [19 §11.3]:

$$e^{J\theta} = \sum_{k=0}^{\infty} \frac{J^k \theta^k}{k!} = J^0 + J \theta + \frac{J^2 \theta^2}{2!} + \frac{J^3 \theta^3}{3!} + \frac{J^4 \theta^4}{4!} + \cdots$$

Noting that $J^0 = I$ (see [20] Index Laws), and separating the series into even and odd terms:

$$= I + \frac{I\theta^2}{2!} + \frac{I\theta^4}{4!} + \cdots + J \theta - \frac{J\theta^3}{3!} + \frac{J\theta^5}{5!} + \cdots$$

$$= I \cos \theta + J \sin \theta$$

Note that matrix versions of the trigonometric functions are not needed to compute the matrix exponential, because $\theta$ is a scalar. In fact, if the exponential is evaluated numerically using a matrix exponential algorithm or function, the trigonometric functions are not even explicitly evaluated [19 §11.3]. In practice, given that this is a special case of the matrix exponential, (because $J^2 = -I$), it is likely to be numerically preferable to evaluate the trigonometric functions and to sum scaled versions of $I$ and $J$.

Notice that the matrix $e^{J\theta}$ has a structure with the cosine of $\theta$ on the diagonal and the (scaled) sine of $\theta$ where there are non-zero elements of $J$.

B. Matrix form of DFT

The classic discrete Fourier transform of (1) may be generalized to a matrix form in which the signals are vector-valued with $N$ components each and the root of $-1$ is replaced by an $N \times N$ matrix root $J$ such that $J^2 = -I$. In this form, subject to choosing the correct representation for the matrix root of $-1$, we may represent a wide variety of complex and hypercomplex Fourier transforms.

Theorem 1: The following are a discrete Fourier transform pair:

$$F[:, \mu] = S \sum_{m=0}^{M-1} \exp \left( -J \frac{2 \pi \mu m}{M} \right) f[:, \mu] \quad (2)$$

$$f[:, \mu] = T \sum_{u=0}^{M-1} \exp \left( J \frac{2 \pi \mu u}{M} \right) F[:, \mu] \quad (3)$$

where $J$ is a $N \times N$ matrix root of $-1$, $f$ and $F$ are $N \times M$ matrices with one sample per column, and the two scale factors $S$ and $T$ multiply to give $1/M$.

Proof: The proof is based on substitution of the forward transform (2) into the inverse (3) followed by algebraic reduction to a result equal to the original signal $f$. We start by substituting (2) into (3), replacing $m$ by $M$ to keep the two indices distinct:

$$f[:, \mu] = \frac{1}{M} \sum_{u=0}^{M-1} e^{(J \frac{2 \pi \mu u}{M})} \sum_{M=0}^{M-1} e^{(-J \frac{2 \pi \mu (M-M)}{M})} f[:, \mu]$$

The scale factors can be moved outside both summations, and replaced with their product $1/M$; and the exponential of the outer summation can be moved inside the inner, because it is constant with respect to the summation index $M$:

$$f[:, \mu] = \frac{1}{M} \sum_{u=0}^{M-1} \sum_{M=0}^{M-1} e^{(J \frac{2 \pi \mu u}{M})} e^{(-J \frac{2 \pi \mu (M-M)}{M})} f[:, \mu]$$

The two exponentials have the same root of $-1$, namely $J$, and therefore they can be combined:

$$f[:, \mu] = \frac{1}{M} \sum_{u=0}^{M-1} \sum_{M=0}^{M-1} e^{(J \frac{2 \pi \mu (u-M)}{M})} f[:, \mu]$$

We now isolate out from the inner summation the case where $M = \mu$. In this case the exponential reduces to an identity matrix, and we have:

$$f[:, \mu] = \frac{1}{M} \sum_{u=0}^{M-1} f[:, \mu]$$

and

$$f[:, \mu] = \frac{1}{M} \sum_{u=0}^{M-1} \left[ \sum_{M=0}^{M-1} e^{(J \frac{2 \pi \mu (u-M)}{M})} \right] f[:, \mu]$$

The first line on the right sums to $f[:, \mu]$, which is the original signal, as required. To complete the proof, we have to show that the second line on the right reduces to zero. Taking the

$1$The colon notation used here will be familiar to users of MATLAB® (an explanation may be found in [19 §1.1.8]). Briefly, $f[:, \mu]$ means the $\mu^{th}$ column of the matrix $f$. 

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second line alone, and changing the order of summation, we obtain:

\[\sum_{M=0}^{M-1} \left[ \sum_{m=0}^{M-1} e^{j 2 \pi (m-M) u} \right] f[m,M]\]

Using Lemma 1, we now write the matrix exponential as the sum of a cosine and sine term.

\[\sum_{M=0}^{M-1} \left[ I \sum_{u=0}^{M-1} \cos \left( \frac{2 \pi (m-M) u}{M} \right) + J \sum_{u=0}^{M-1} \sin \left( \frac{2 \pi (m-M) u}{M} \right) \right] f[m,M]\]

Since both of the inner summations are sinusoids summed over an integral number of cycles, they vanish, and this completes the proof.

Notice that the requirement for \( J^2 = -I \) is the only constraint on \( J \).

It is not necessary to constrain elements of \( J \) to be real. Note that \( J^2 = -I \) implies that \( J^{-1} = -J \), hence the inverse transform is obtained by negating or inverting the matrix root of \(-1\) — the two operations are equivalent.

The matrices must be conformant according to the ordering inside the summation. As written above, for a complex transform represented in matrix form, \( f \) and \( F \) must have two rows and \( M \) columns. If the exponential were to be placed on the right, \( f \) and \( F \) would have to be transposed, with two columns and \( M \) rows.

It is important to realize that (2) is totally different to the classical matrix formulation of the discrete Fourier transform, as given for example by Golub and Van Loan [19, §4.6.4]. The classic DFT given in (1) can be formulated as a matrix equation in which a large \( M \times M \) Vandermonde matrix containing \( n \)th roots of unity multiplies the signal \( f \) expressed as a vector of real or complex values. Instead, in (2) each matrix exponential multiplies one column of \( f \), corresponding to one sample of the signal represented by \( f \), and the dimensions of the matrix exponential are set by the dimensionality of the algebra (2 for complex, 4 for quaternions, etc.). In (2) it is the multiplication of the exponential and the signal samples, dependent on the algebra involved, that is expressed in matrix form, not the structure of the transform itself.

Readers already familiar with hypercomplex Fourier transforms should note that the ordering of the exponential within the summation (2) is not related to the ordering within the hypercomplex formulation of the transform (which is significant because of non-commutative multiplication). The hypercomplex ordering can be accommodated within the framework presented here by changing the representation of the matrix root of \(-1\), in a non-trivial way, shown for the quaternion case by Ickes [18, Equation 10] and called transmutation. We have studied the generalization of Ickes’ transmutation to the case of Clifford algebras, and it appears that the operation should be more correctly described as negation of the off-diagonal elements of the lower-right sub-matrix, excluding the first row and column. We leave this for later work, as a full generalisation to Clifford algebras of arbitrary dimension requires further work, and is more appropriate to a mathematical paper on matrix representation of Clifford algebras.

III. EXAMPLES IN SPECIFIC ALGEBRAS

In this section we present the information necessary for \( \mathbb{C} \) and \( \mathbb{H} \) to be verified numerically. In each of the cases below, we present an example root of \(-1\) and a matrix representation (these are not unique – a transpose of the matrix, for example, is equally valid). We include in the Appendix a short MATLAB function for computing the transform in (2). The same code will compute the inverse by negating \( J \). This may be used to verify the results in the next section and to compare the results obtained with the classic complex FFT. In order to verify the quaternion or biquaternion results, the reader will need to install the QTFM toolbox [21], or use some other specialised software for computing with quaternions.

A. Complex algebra

The \( 2 \times 2 \) real matrix \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) can be easily verified by eye to be a square root of the negated identity matrix \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \), and it is easy to verify numerically that Euler’s formula gives the same numerical results in the classic complex case and in the matrix case for an arbitrary \( \theta \). This root of \(-1\) is based on the well-known isomorphism between a complex number \( a + j b \) and the matrix representation \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \). The structure of a matrix exponential \( e^{J \theta} \) using the above matrix for \( J \) is \( \begin{pmatrix} C & -S \sqrt{3} \\ S & C \end{pmatrix} \) where \( C = \cos \theta \) and \( S = \sin \theta \).

B. Quaternion algebra

The quaternion roots of \(-1\) were discovered by Hamilton [17, pp203, 209], and consist of all unit pure quaternions, that is quaternions of the form \( xi + yj + zk \) subject to the constraint \( x^2 + y^2 + z^2 = 1 \). A simple example is the quaternion \( \mu = (i + j + k)/\sqrt{3} \), which can be verified by hand to be a square root of \(-1\) using the usual rules for multiplying the quaternion basis elements \( (i^2 = j^2 = k^2 = ijk = -1) \). Using the isomorphism with \( 4 \times 4 \) matrices given by Ward [17, §2.8], between the quaternion \( w + xi + yj + zk \) and the matrix:

\[
\begin{pmatrix}
w & -x & -y & -z \\
x & w & -z & y \\
y & z & w & -x \\
z & -y & x & w
\end{pmatrix}
\]

we have the following matrix representation:

\[
\mu = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}
\]

Notice the structure that is apparent in this matrix: the \( 2 \times 2 \) blocks on the leading diagonal can be recognised as roots of \(-1\) in the complex algebra as shown in §III-A.

We have used the transpose of Ward’s representation for consistency with the quaternion and biquaternion representations in the two following sections.
Proposition 1: Any matrix of the form:
\[
\begin{pmatrix}
0 & -x & -y & -z \\
x & 0 & -z & y \\
y & z & 0 & -x \\
z & -y & x & 0
\end{pmatrix}
\]
with \(x^2 + y^2 + z^2 = 1\) is the square root of a negated \(4 \times 4\) identity matrix. Thus the matrix representations of the quaternion roots of \(-1\) are all roots of the negated \(4 \times 4\) identity matrix.

Proof: The matrix is anti-symmetric, and the inner product of the \(i^{th}\) row and \(j^{th}\) column is \(-x^2 - y^2 - z^2\), which is \(-1\) because of the stated constraint. Therefore the diagonal elements of the square of the matrix are \(-1\). Note that the rows of the matrix have one or three negative values, whereas the columns have zero or two. The product of the \(i^{th}\) row with the \(j^{th}\) column, \(i \neq j\), is the sum of two values of opposite sign and equal magnitude. Therefore all off-diagonal elements of the square of the matrix are zero.

The structure of a matrix exponential \(e^{J\theta}\) using a matrix as in Proposition 1 for \(J\) is:
\[
\begin{pmatrix}
C & -xS & -yS & -zS \\
xS & C & -zS & yS \\
yS & zS & C & -xS \\
zS & -yS & xS & C
\end{pmatrix}
\]
where, as before, \(C = \cos \theta\) and \(S = \sin \theta\).

C. Biquaternion algebra

The biquaternion algebra [17] Chapter 3 (quaternions with complex elements) can be handled exactly as in the previous section, except that the \(4 \times 4\) matrix representing the root of \(-1\) must be complex (and the signal matrix must have four complex elements per column). The set of square roots of \(-1\) in the biquaternion algebra is given in [13]. A simple example is \(i + j + k + I(j - k)\) (where \(I\) denotes the classical complex root of \(-1\), that is the biquaternion has real part \(i + j + k\) and imaginary part \(j - k\)). Again, this can be verified by hand to be a root of \(-1\) and its matrix representation is:
\[
\begin{pmatrix}
0 & -1 & -1 - I & -1 + I \\
1 & 0 & -1 + I & 1 + I \\
1 + I & 1 - I & 0 & -1 \\
1 - I & -1 - I & 1 & 0
\end{pmatrix}
\]
Again, sub-blocks of the matrix have recognizable structure. The diagonal \(2 \times 2\) blocks are roots of \(-1\), while the off-diagonal blocks are nilpotent – that is their square vanishes.

D. Clifford algebras

Recent work by Hitzer and Ablamowicz [15] has explored the roots of \(-1\) in Clifford algebras. Using their results, and by finding from first principles the layout of a real matrix isomorphic to a Clifford multivector in a given algebra, it has been possible to verify that the transform formulation presented in this paper is applicable to at least the lower order Clifford algebras. The quaternions and biquaternions are isomorphic to the Clifford algebras \(Cl_{0,2}\) and \(Cl_{3,0}\) respectively so this is not surprising. Nevertheless, it is an important finding, because until now quaternion and Clifford Fourier transforms were defined in different ways, using different terminology, and it was difficult to make comparisons between the two. Now, with the matrix exponential formulation, it is possible to handle quaternion and Clifford transforms (and indeed transforms in different Clifford algebras) within the same algebraic and/or numerical framework.

We present examples here from two 4-dimensional algebras, \(Cl_{1,1}\) and \(Cl_{2,0}\). These results have been verified against the CLICAL package [23] to ensure that the multiplication rules have been followed correctly and that the roots of \(-1\) found by Hitzer and Ablamowicz are correct.

Following the notation in [15], we write a multivector in \(Cl_{1,1}\) as \(\alpha + \beta e_1 + b_2 e_2 + \beta e_{12}\), where \(e_1^2 = +1, e_2^2 = -1, e_{12}^2 = +1\) and \(e_1 e_2 = e_{12}\). A possible real matrix representation is as follows:
\[
\begin{pmatrix}
\alpha & b_1 & -b_2 & \beta \\
b_1 & \alpha & -\beta & b_2 \\
b_2 & -\beta & \alpha & b_1 \\
\beta & b_2 & b_1 & \alpha
\end{pmatrix}
\]
In this algebra, the constraints on the coefficients of a multivector for it to be a root of \(-1\) are as follows: \(\alpha = 0\) and \(b_1^2 - b_2^2 + \beta^2 = -1\) [15 Table 1]. Choosing \(b_1 = \beta = 1\) gives \(b_2 = \sqrt{3}\) and thus \(e_1 + \sqrt{3} e_2 + e_{12}\) which can be verified algebraically or in CLICAL to be a root of \(-1\). The corresponding matrix is then:
\[
\begin{pmatrix}
0 & 1 & -\sqrt{3} & 1 \\
1 & 0 & -1 & \sqrt{3} \\
\sqrt{3} & -1 & 0 & 1 \\
1 & -\sqrt{3} & 1 & 0
\end{pmatrix}
\]
Following the same notation in algebra \(Cl_{2,0}\), in which \(e_1^2 = e_2^2 = +1, e_{12}^2 = -1\), a possible matrix representation is:
\[
\begin{pmatrix}
\alpha & b_1 & b_2 & -\beta \\
b_1 & \alpha & -\beta & b_2 \\
b_2 & -\beta & \alpha & b_1 \\
\beta & b_2 & b_1 & \alpha
\end{pmatrix}
\]
The constraints on the coefficients are \(\alpha = 0\) and \(b_1^2 + b_2^2 - \beta^2 = -1\), and choosing \(b_1 = b_2 = 1\) gives \(\beta = \sqrt{3}\) and thus \(e_1 + e_2 + \sqrt{3} e_{12}\) is a root of \(-1\). The corresponding matrix is then:
\[
\begin{pmatrix}
0 & 1 & 1 & -\sqrt{3} \\
1 & 0 & \sqrt{3} & -1 \\
1 & -\sqrt{3} & 0 & 1 \\
\sqrt{3} & -1 & 1 & 0
\end{pmatrix}
\]
Notice that in both of these algebras the matrix representation of a root of \(-1\) is very similar to that given for the quaternion case in Proposition 1 with zeros on the leading diagonal, an odd number of negative values in each row and an even number in each column. It is therefore simple to see that minor modifications to Proposition 1 would cover these algebras and the matrices presented above.

\footnote{We have re-arranged the constraint compared to [15] Table 1 to make the comparison with the quaternion case easier: we see that the signs of the squares of the coefficients match the signs of the squared basis elements.}
IV. AN EXAMPLE NOT BASED ON A SPECIFIC ALGEBRA

We show here using an arbitrary $2 \times 2$ matrix root of $-1$, that it is possible to define a Fourier transform without a specific algebra. Let an arbitrary real matrix be given as $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then by brute force expansion of $J^2 = -I$ we find the original four equations reduce to but two independent equations. Picking $(a, b)$ and solving for the remaining coefficients we find that any matrix of the form:

$$\begin{pmatrix} a & b \\ -(1 + a^2)/b & -a \end{pmatrix}$$

with finite $a$ and $b$, and $b \neq 0$, is a root of $-1$. Choosing instead $(a, c)$ we get the transpose form:

$$\begin{pmatrix} a & -(1 + a^2)/c \\ c & -a \end{pmatrix}$$

where $c \neq 0$. Choosing the cross-diagonal terms $(b, c)$ yields

$$\begin{pmatrix} \pm \sqrt{-1 - bc} & b \\ c & \mp \sqrt{-1 - bc} \end{pmatrix}$$

where $bc \leq -1$.

In all cases the resulting matrix has eigenvalues of $\lambda = \pm j$.

(VI. E.

where $\kappa = \sqrt{-1 - bc}$. Instead of mapping a real unit vector $(\frac{1}{\sqrt{2}})$ to a point on a circle, this matrix maps to an ellipse. Thus, we see that a transform based on a matrix such as that in (6) has basis functions that are projections of an elliptical, rather than a circular path in the complex plane, as in the classical complex Fourier transform. We refer the reader to a discussion on a similar point for the one-sided quaternion discrete Fourier transform in our own 2007 paper [12] §VI, in which we showed that the quaternion coefficients of the Fourier spectrum also represent elliptical paths through the space of the signal samples. Clearly there are interesting issues to be studied here, and further work to be done.

V. NON-EXISTENCE OF TRANSFORMS IN ALGEBRAS WITH ODD DIMENSION

In this section we show that there are no real matrix roots of $-1$ with odd dimension. This is not unexpected, since the existence of such roots would require the existence of a hypercomplex algebra of odd dimension. The significance of this result is to show that there is no discrete Fourier transform as formulated in Theorem 1 for an algebra of dimension 3, which is of importance for the processing of signals representing physical 3-space quantities, or the values of colour image pixels. We thus conclude that the choice of quaternion Fourier transforms or a Clifford Fourier transform of dimension 4 is inevitable in these cases. This is not an unexpected conclusion, nevertheless, in the experience of the authors, some researchers in signal and image processing hesitate to accept the idea of using four dimensions to handle three-dimensional samples or pixels. (This is despite the rather obvious parallel of needing two dimensions – complex numbers – to represent the Fourier coefficients of a real-valued signal or image.)

Theorem 2: There are no $N \times N$ matrices $J$ with real elements such that $J^2 = -I$ for odd values of $N$.

Proof: The determinant of a diagonal matrix is the product of its diagonal entries. Therefore $| -I | = -1$ for odd $N$. Since the product of two determinants is the determinant of the product, $| J^2 | = -1$ requires $| J |^2 = -1$, which cannot be satisfied if $J$ has real elements.

VI. EXTENSION TO TWO-SIDED DFTs

There have been various definitions of two sided hypercomplex Fourier transforms and DFTs. We consider here only one case to demonstrate that the approach presented in this paper is applicable to two-sided as well as one-sided transforms: this is a matrix exponential Fourier transform based on Ell’s original two-sided two-dimensional quaternion transform [2] Theorem 4.1. [3], [24]. A more general formulation is:

$$F[u, v] = S \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-J2\pi\frac{mv}{M}} f[m, n]e^{-K2\pi\frac{nv}{N}}$$

$$f[m, n] = T \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} e^{J2\pi\frac{vu}{M}} F[u, v]e^{K2\pi\frac{vu}{N}}$$

in which each element of the two-dimensional arrays $F$ and $f$ is a square matrix representing a complex or hypercomplex
number using a matrix isomorphism for the algebra in use, for example the representations already given in §III-B in the case of the quaternion algebra; the two scale factors multiply to give $1/MN$, and $J$ and $K$ are matrix representations of two arbitrary roots of $-1$ in the chosen algebra. (In Ell’s original formulation, the roots of $-1$ were $j$ and $k$, that is two of the orthogonal quaternion basis elements. The following theorem shows that there is no requirement for the two roots to be orthogonal in order for the transform to invert.)

**Theorem 3:** The transforms in (7) and (9) are a two-dimensional discrete Fourier transform pair, provided that $J^2 = K^2 = -I$.

**Proof:** The proof follows the same scheme as the proof of Theorem 1, but we adopt a more concise presentation to fit the available column space. We start by substituting (2) into (3), replacing $m$ and $n$ by $M$ and $N$ respectively to keep the indices distinct:

$$f[m, n] = T \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} e^{(J2\pi \frac{mu}{MN})} \times \left[ S \sum_{M=0}^{M-1} \sum_{N=0}^{N-1} e^{-J2\pi \frac{mu}{MN}} f[M, N] e^{(-K2\pi \frac{nv}{MN})} \right] e^{(K2\pi \frac{nu}{MN})}$$

The scale factors can be moved outside both summations, and replaced with their product $1/MN$; and the exponentials of the outer summations can be moved inside the inner, because they are constant with respect to the summation indices $M$ and $N$.

At the same time, adjacent exponentials with the same root of $-1$ can be merged. With these changes, the right-hand side of the equation becomes:

$$\frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \sum_{M=0}^{M-1} \sum_{N=0}^{N-1} e^{(J2\pi \frac{mu}{MN})} f[M, N] e^{(-K2\pi \frac{nu}{MN})}$$

We now isolate out from the inner pair of summations the case where $M = m$ and $N = n$. In this case the exponentials reduce to identity matrices, and we have:

$$\frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[m, n]$$

This sums to $f[m, n]$, which is the original two-dimensional signal, as required. To complete the proof we have to show that the rest of the summation, excluding the case $M = m$ and $N = n$, reduces to zero. Dropping the scale factor, and changing the order of summation, we have the following inner double summation:

$$\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} e^{(J2\pi \frac{mu}{MN})} f[M, N] e^{(K2\pi \frac{nu}{MN})}$$

Noting that the first exponential and $f$ are independent of the second summation index $v$, we can move them outside the second summation (we could do similarly with the exponential on the right and the first summation):

$$\sum_{u=0}^{M-1} e^{(J2\pi \frac{mu}{MN})} f[M, N] \sum_{v=0}^{N-1} e^{(K2\pi \frac{nu}{MN})}$$

and, as in Theorem 1, the summation on the right is over an integral number of cycles of cosine and sine, and therefore vanishes.

Notice that it was not necessary to assume that $J$ and $K$ were orthogonal: it is sufficient that each be a root of $-1$. This has been verified numerically using the two-dimensional code given in the Appendix.

**VII. DISCUSSION**

The discrete Fourier transform presented in this paper, based on matrix roots of $-1$, and a matrix generalisation of Euler’s formula, could be said to be a simple development. Nevertheless, to our knowledge it has not been reported before, as evidenced by the numerous definitions of hypercomplex Fourier transforms that have been presented in the literature, including works by the present authors. We have shown that any discrete Fourier transform in an algebra that has a matrix representation, can be formulated in the way shown here. This includes the complex, quaternion, biquaternion, and Clifford algebras (although we have demonstrated only certain cases of Clifford algebras, we believe the result holds in general).

Several immediate possibilities for further work, as well as ramifications, now suggest themselves. Firstly, the study of roots of $-1$ is accessible from the matrix representation as well as direct representation in whatever algebra is employed for the transform. All of the results obtained so far in hypercomplex algebras, and known to the authors [22, pp 203, 209], [13], [15], were achieved by working in the algebra in question, that is by algebraic manipulation of quaternion, biquaternion or Clifford multivector values. An alternative approach would be to work in the equivalent matrix algebra, but this seems difficult even for the lower order cases. Nevertheless, it merits further study because of the possibility of finding a systematic approach that would cover many algebras in one framework. Following the reasoning in §III it is possible to define matrix roots of $-1$ that appear not to be isomorphic to any Clifford or quaternion algebra, and these merit further study.

Secondly, the matrix formulation presented here lends itself to analysis of the structure of the transform, including possible factorizations for fast algorithms, as well as parallel or vectorized implementations for single-instruction, multiple-data (SIMD) processors, and of course, factorizations into multiple complex FFTs as has been done for quaternion FFTs (see for example [11]). In the case of matrix roots of $-1$ which do not correspond to Clifford or quaternion algebras, analysis of the structure of the transform may give insight into possible applications of transforms based on such roots.

Finally, at a practical level, hypercomplex transforms implemented in hypercomplex arithmetic are likely to be faster than any implementation based on matrices, but the simplicity of the matrix exponential formulation, and the fact that it can be computed using standard real or complex matrix arithmetic, means that the matrix exponential formulation provides a very simple reference implementation which can be used for verification of hypercomplex code.
APPENDIX

MATLAB® CODE

We include here two short MATLAB® functions for computing the forward transform given in (2), and (4), apart from the scale factors. The inverses can be computed simply by interchanging the input and output and negating the matrix roots of $-1$. Neither function is coded for speed, on the contrary the coding is intended to be simple and easily verified against the equations.

```matlab
function F = matdft(f, J)
M = size(f, 2);
F = zeros(size(f));
for u = 0:M-1
    for v = 0:N-1
        f (:, u + 1) = f (:, u + 1) + ... 
           expm(-J.*2.*pi.*m.*u./M) * ... 
           f (:, m + 1);
    end
end
end

function F = matdft2(f, J, K)
A = size(J, 1);
M = size(f, 1) ./ A;
N = size(f, 2) ./ A;
F = zeros(size(f));
for u = 0:M-1
    for v = 0:N-1
        for m = 0:M-1
            for n = 0:N-1
                f(A*u+1:A*u+M, A*v+1:A*v+N) = ... 
                expm(-J.*2.*pi.*m.*n.*u./M) * ... 
                f(A*m+1:A*m+M, A*n+1:A*n+N) * ...
            end
        end
    end
end
end
```

REFERENCES