Estimating topological properties of weighted networks from limited information

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A fundamental problem in studying and modeling economic and financial systems is represented by privacy issues, which put severe limitations on the amount of accessible information. Here we investigate a novel method to reconstruct the structural properties of complex weighted networks using only partial information: the total number of nodes and links, and the values of the strength for all nodes. The latter are used first as fitness to estimate the unknown node degrees through a standard configuration model; then, degrees and strengths are employed to calibrate an enhanced configuration model in order to generate ensembles of networks intended to represent the real system. The method, which is tested on the World Trade Web, while drastically reducing the amount of information needed to infer network properties, turns out to be remarkably effective—thus representing a valuable tool for gaining insights on privacy-protected socioeconomic networks.

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The reconstruction the statistical properties of a network when only partial information is available represents a largely unsolved problems in the field of statistical physics of networks [1, 2]. Yet, addressing this issue can bring to many concrete applications. A paramount example is provided by financial networks, where nodes represent financial institutions and edges stand for the various types of financial ties—such as loans or derivative contracts. These ties result in dependencies among institutions and constitute the ground for the propagation of financial distress across the network. However, due to confidentiality issues, the information that regulators are able to collect on mutual exposures is very limited [3], and this hinders the analysis of the system resilience to the default or distress on one or more institutions—which depends on the structure of the whole network [4, 5]. Typically, the analysis of systemic risk has been pursued by trying to reconstruct the unknown links of the network using Maximum Entropy algorithms [6, 7]. These approaches, also known as “dense reconstruction” methods, assume that the network is fully connected and estimate link weights via a maximum homogeneity principle, looking for the weighted adjacency matrix with minimal distance from the uniform matrix (i.e. with identical entries) that also satisfies the imposed constraints—represented for instance by the budget of individual banks. The strongest limitation of these algorithms lies in the hypothesis that the network is fully connected. In fact, not only empirical networks show a very heterogeneous distribution of the connectivity, but such dense reconstruction was shown to lead to systemic risk underestimation [8, 9]. More refined methods like “sparse reconstruction” algorithms [2] (that allow to obtain a matrix with an arbitrary level of heterogeneity) still leave open the question of what value of heterogeneity would be appropriate; moreover, even when link density is correctly recovered, systemic risk is still underestimated because of the homogeneity principle used to build the matrix. A more recent approach, named bootstrapping method (BS) [9, 10], instead uses the limited topological information available on the network to generate an ensemble of exponential random graphs (ERG) through the configuration model (CM) [11], where the Lagrange multipliers defining it are replaced by fitnesses [12], i.e., node-specific properties assumed to be known—in a way similar to fitness-dependent network models [13]. The estimation of network properties is then carried out within such fitness-induced ensemble. This method overcomes the limitations of its predecessors and turns out to be quite accurate in estimating network topological properties also when the available information is very limited. Yet, its main drawback consists in being applicable only to binary networks, whereas, the analysis of systemic risk is generally carried out within the weighted representation of the network system.

Here we aim at overcoming this limitation by extending the method to weighted networks, resorting on an amount of information comparable to what is used by the original BS: the total number of connections and the values of the strength for each node—playing the role of node fitness. In a nutshell, our method consists in estimating the number of connections for each node via the standard CM calibrated on the fitnesses, and then in using these values as well as node strengths to assess individual link weights through an enhanced configuration model (ECM) [14]. To validate our method, we use a real instance of an economic system: the World Trade Web (WTW) [15], i.e., the network whose N nodes represent countries and whose L links represent trade volumes among them [16]. In the dataset, the weight of the link between nodes i and j (wij) is the total monetary flux between countries i and j resulting from the
import/export between them; thus, the strength or total trade volume of country \( i \) is \( s_i = \sum_j w_{ij} \), while its degree or number of commercial partners is \( k_i = \sum_j a_{ij} \) (where \( a_{ij} := \lim_{\varepsilon \to 0} [1 + \varepsilon / w_{ij}]^{-1} \)). Since we have full information on this network, we will be able to assess unambiguously the accuracy of our method in estimating its topological properties.

As stated above, our method builds on two complementary network generation models. The CM \[14\] is a particular class of ERG model \[17\], which consists in generating an ensemble \( \Omega \) of networks which is maximally random—except for the ensemble average of the node degrees \( \{ \langle k_i \rangle_\Omega \}_{i=1}^N \) that are constrained to the observed values \( \{ k_i^* \}_{i=1}^N \). The probability distribution over \( \Omega \) is then defined via a set of Lagrange multipliers \( \{ x_i \}_{i=1}^N \) (one for each node), whose values can be set to satisfy the equivalence \( \langle k_i \rangle_\Omega = k_i^* \forall i \) \[18\]; in particular, the ensemble probability that any two nodes \( i \) and \( j \) are connected is given by:

\[
p_{ij} = \frac{x_i x_j}{1 + x_i x_j},
\]

so that \( x_i \) quantifies the ability of node \( i \) to create links with other nodes. The ECM \[14\] is instead obtained by specifying both the mean degree and strength sequences \( \{ k_i^* \}_{i=1}^N \) and \( \{ s_i^* \}_{i=1}^N \). In this case, two Lagrange multipliers \( \{ a_i, b_i \} \) are associated to each node \( i \), so that the ensemble probability \( q_{ij} \) that any two nodes \( i \) and \( j \) are connected and the ensemble average \( \langle w_{ij} \rangle \) for the weight of such link become \[19\]:

\[
q_{ij} = \frac{a_i a_j b_i b_j}{1 + a_i a_j b_i b_j - b_i b_j}, \quad \langle w_{ij} \rangle = \frac{q_{ij}}{1 - b_i b_j}.
\]

On the other hand, the fitness model \[12\] assumes the network topology to be determined by an intrinsic property (fitness) associated with each node. This approach has been successfully used in the past to model several economic networks, including the network of equity investments in the stock market \[20\], the Italian Interbank Market \[21\] and the WTW \[13\]. Note that fitnesses are often used within the ERG framework provided an assumed connection between them and the Lagrange multipliers. Our network reconstruction method will build exactly on such assumption.

Given these ingredients, we can now formulate the statistical procedure our method employs to find the most probable estimate for \( X(G_0) \), i.e., the value of a topological property \( X \) for the real network \( G_0 \) that we want to reconstruct. Such estimate has to rely on and be compatible with some constraints, given by the incomplete information we have on \( G_0 \): the total number of nodes \( N \) and links \( L \), and the whole strength sequence \( \{ s_i^* \}_{i=1}^N \).

We build on two important hypotheses:

1. \( G_0 \) can be seen as drawn from an appropriate ECM ensemble \( \Omega_{\text{ECM}} \), so that \( X(G_0) \) can be estimated as \( \langle X \rangle_{\Omega_{\text{ECM}}} \);

2. The strengths \( \{ s_i^* \}_{i=1}^N \) represent degree-induced node fitnesses, and are thus assumed to be proportional to the \( \{ x_i \}_{i=1}^N \) of the CM via a universal parameter \( z \): \( x_i = \sqrt{z s_i^*} \forall i \) \[22\].

The first hypothesis allows us to map the problem of evaluating \( X(G_0) \) into that of choosing the optimal ECM ensemble \( \Omega_{\text{ECM}} \) compatible with the known constraints on \( G_0 \). In other words, the question to address becomes: what ECM ensemble is the most appropriate to extract the real network \( G_0 \) from, given that we know only partial information? Then, once \( \Omega_{\text{ECM}} \) is determined, we can use the average \( \langle X \rangle_{\Omega_{\text{ECM}}} \) as a good estimation for \( X(G_0) \). However, in order to build an appropriate \( \Omega_{\text{ECM}} \), we need to know not only the strengths but also the degrees for all the nodes, and this is where hypothesis 2 comes in handy: the unknown degrees can be estimated within a CM ensemble \( \Omega_{\text{CM}} \) built using the strengths as degree-induced fitnesses (Fig.1). Our method thus proceeds in two steps. I) We first find the unknown parameter \( z \) that defines \( \Omega_{\text{CM}} \) through a maximum-likelihood argument \[12\], by comparing the average number of links of a network belonging to \( \Omega_{\text{CM}} \) with the (known) total number \( L(G_0) \) of links in \( G_0 \):

\[
\langle L \rangle_{\Omega_{\text{CM}}} = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{z s_i^* s_j^*}{1 + z s_i^* s_j^*} = L(G_0)
\]

(since \( \{ s_i^* \}_{i=1}^N \) are known, eq. 3 is an algebraic equation in \( z \)). We then use this \( z \) and hypothesis 2 to estimate the unknown degrees through eq. (4):

\[
\langle k_i \rangle_{\Omega_{\text{CM}}} = \sum_{j \neq i} p_{ij} = \sum_{j \neq i} \frac{z s_i^* s_j^*}{1 + z s_i^* s_j^*}.
\]

II) The degrees estimated in this way can then be used in
pute the linking probabilities 
\{\text{a proper ensemble to draw the real network } G \text{ needed to infer network properties is reduced to} \}

cally or numerically. At the end, the amount of quantities
\{\text{we first use the solutions} \}

ing a weighted network structure: the

\[ \Omega \text{ (instead of the original } 2^\text{N} \text{)} \]

The solution of this system of equations is the set of Lagrange multipliers \{a_i, b_i\}_{i=1}^N \text{ that define the ECM ensemble and allow to compute } \langle X \rangle_{\Omega_{\text{ECM}}}, \text{ either analyti-

cally or numerically. At the end, the amount of quantities
\{\text{needed to infer network properties is reduced to } N + 1 \}

t instead of the original } 2^N \text{).}

In order to check whether the ECM defined above is a proper ensemble to draw the real network \( G_0 \) from, we first use the solutions \{a_i, b_i\}_{i=1}^N \text{ of eq.(5) to compute the linking probabilities } \{q_{ij}\}_{i,j=1}^N \text{ and the average weights } \{\{w_{ij}\}_{i,j=1}^N \text{ of eq.(2): from these quantities we then obtain the average degrees } \langle k_i \rangle_{\Omega_{\text{ECM}}} = \sum_{j \neq i} q_{ij} \text{ and the average strengths } \langle s_i \rangle_{\Omega_{\text{ECM}}} = \sum_{j \neq i} w_{ij} \text{ of the ECM ensemble. Figure 2 shows the relation between node degree } k^*(G_0) \text{ of the real network and } \langle k \rangle_{\Omega_{\text{ECM}}}, \text{ as well as the consistency check between node strength } s^*(G_0) \text{ of the real network and } \langle s \rangle_{\Omega_{\text{ECM}}}. \text{ While the latter is, as it should, an identity (apart from small numerical errors), for the degrees we observe a scattered cloud around the identity, whose behavior reflects the noisy yet very high correlation between strengths and degrees—as we are not using the real } k^*(G_0) \text{ in eq.(5) but } \langle k \rangle_{\Omega_{\text{ECM}} \text{ obtained from the CM induced by node strengths. We move further and focus on the topological properties that are commonly regarded as the most significant for describing a weighted network structure: the average nearest neighbors strength:}

\[ s_i^{nn} := \frac{\sum_{j \neq i} a_{ij} s_j}{k_i} = \frac{\sum_{j \neq i} \sum_{k \neq i,j} a_{ij} w_{jk}}{\sum_{j \neq i} a_{ij}}, \]

and the weighted clustering coefficient:

\[ c_w := \frac{\sum_{j \neq i} \sum_{k \neq i,j} w_{ij} w_{ik} w_{jk}}{\sum_{j \neq i} \sum_{k \neq i,j} a_{ij} a_{ik}}. \]

For completeness, we also consider the binary version of these quantities: the average nearest neighbors degree:

\[ k_i^{nn} := \frac{\sum_{j \neq i} a_{ij} k_j}{k_i} = \frac{\sum_{j \neq i} \sum_{k \neq i,j} a_{ij} a_{jk}}{\sum_{j \neq i} a_{ij}}, \]

and the binary clustering coefficient:

\[ c_b := \frac{\sum_{j \neq i} \sum_{k \neq i,j} a_{ij} a_{ik} a_{jk}}{\sum_{j \neq i} \sum_{k \neq i,j} a_{ij} a_{ik}}. \]

Figure 3 shows a remarkable agreement between the values of these quantities computed on the real network \( G_0 \) and their ECM ensemble averages—obtained from eq.(4) [9] by replacing the binary adjacency matrix elements \( a_{ij} \) with the linking probabilities \( q_{ij} \), and the real link weights \( w_{ij} \) with their ensemble averages \( \langle w_{ij} \rangle \). Indeed, the ensemble averages \( \langle X \rangle_{\Omega_{\text{ECM}}} \) can be used as good estimates for the real quantities \( X(G_0) \), revealing the effectiveness of our method in reconstructing the topological properties of the real network.

It is important to remark that the applicability of our method strongly depends on the accuracy of hypothesis 2, i.e. on whether the CM induced by node strengths is able to provide good estimates for the unknown degrees. This is indeed the case of the WTW [13], but also of other economic and financial networks of different nature [20, 21]. Another important remark is that our method is based on a combination of CM and ECM rather than directly on the Weighted Configuration Model (WCM) [18], because the latter not only fails to reproduce the network topological properties (as shown by Figure 3), but also predicts a far denser network than observed. This happens not because strengths carry a “lower level” information than that of degrees—rather, they can be used to infer the degrees themselves, and this is what our method points out: the information on strength values should not be used to directly reconstruct the network, but to estimate the degree first, and only then to compute the quantities of interest. In this respect, note that using directly the knowledge of the strength sequence and number of links as fixed constraints to build a maximum-entropy ensemble would result in a different mathematical expressions. In particular, we would arrive at a variant of eq.(2) where \( a_i = a \ \forall i \). We have checked that, just like the WCM, this model gives a bad prediction of the network, leading to the conclusion that inferring the expected degrees through eq.(1) is a crucial step of the approach we are using here: the information on links presence is indispensable to achieve a faithful network reconstruction.

Further work is needed to address several issues that remain open, including testing the accuracy of our
method in estimating other higher-order topological properties. Possibly, for these cases the method could require a larger initial information to obtain the same effectiveness. Nevertheless, in its present version our method exploits a very limited information, which is indeed minimal but also often available for economic and financial systems: besides global statistics ($N$ and $L$), the strengths (that can be the operating revenue of firms, or the tier-1 capital of banks) are or should be accessible public data. In conclusion, our method is particularly useful to overcome the lack of topological information that often hampers systemic risk estimation in financial networks. More generally, our method can be applied to any network representing a set of dependencies among components in a complex system for which the available information is limited, and it is thus of general interest in the field of statistical physics of networks.

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[16] We use trade volume data for the year 2000, expressed in units of $10^8$. Original volumes were divided by 10 in order to keep the $\{b_i\}_{i=1}^N$ in eq. 5 away from 1 and thus avoid numerical instability. Additionally, link weights were rounded up to the closest integer to keep the whole network connected.
[22] The fitness model based on countries total trade volumes $\{s_i\}_{i=1}^N$ works well in modeling the WTW, and eq. 1 accurately describes the binary topology when $x_i \propto s_i \forall i \in [13]$. A valid alternative in this case would be the use of the GDP instead of the strength as fitness—yet, GDP and $s$ of a county are often highly correlated, thus the two approaches yield similar results.