Henstock’s Multiple Wiener Integral and Henstock’s Version of Hu-Meyer Theorem

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Abstract—Although it has been argued that the classical Riemann approach cannot be used to study stochastic integrals, it has been proved that the generalized Riemann approach (using nonuniform meshes) has been successful in defining stochastic integrals and even multiple Wiener integral in n-dimensional Euclidean space \( \mathbb{R}^n \). The multiple Wiener integral considers only the nondiagonal part of \( \mathbb{R}^n \). In this paper, we shall use generalized Riemann approach to study multiple Wiener integral on \( \mathbb{R}^n \), including both the diagonal and the nondiagonal part, and derive the classical Hu-Meyer Theorem. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Classically, it is well known that the Riemann approach cannot be used to define stochastic integrals. However, it has been proved that the generalized Riemann approach (with nonuniform meshes), also called the Henstock’s approach, can be used to study stochastic integrals. See details in [1–5]. The generalized Riemann approach has successfully been used to give an equivalent definition of the classical stochastic integral. While the classical stochastic integral is defined by a nonexplicit \( L^2 \) procedure, the Riemann approach is well known for its explicitness and directness in defining the stochastic integral.

The classical multiple Wiener integral was first studied by Wiener in 1938, see [6]. In the same fashion as the definition of one-dimension stochastic integrals, the multiple Wiener integral was later defined by Itô using the generalized \( L^2 \) procedure (see [7] for the detail of the procedure). The generalized Riemann approach has been extended to the study of the multiple Wiener integral, and can be similarly used to give an explicit definition of the classical Wiener integral, see [4]. The classical Wiener integral concerns itself with the integration over the nondiagonal part of the \( n \)-dimensional Euclidean space by letting the integrand vanish on the diagonal part.

In this paper, we shall apply the generalized Riemann approach to the integration of deterministic functions over both the diagonal and the nondiagonal part of the \( n \)-dimensional space. We shall derive the classical Hu-Meyer theorem using our approach. The classical treatment of the Hu-Meyer theorem can be found in [8].
2. SETTING AND DEFINITION OF THE INTEGRAL

In this section, we shall define the setting and the multiple stochastic integral using Riemann approach. Let \( T = [a, b] \subset [0, \infty) \) and \( T^m = [a, b]^m \).

In this paper, we shall bold the letter to denote intervals in \( T^m \). For example, \( I \subset T^m \) is an interval in \( T^m \), while \( I = \prod_{i=1}^m I_i \), where each \( I_i \) is a left-open interval in \( T \).

**Definition 2.1.** Let \((\Omega, P)\) be a probability space and \( W = \{W_t(\omega) : t \in [a, b]\} \) be a family of random variables on \((\Omega, P)\). Then, \( W \) is said to be a canonical Brownian motion if it satisfies the following properties,

1. it has normal increments, that is, \( W_t - W_s \) has a normal distribution with mean 0 and variance \( t - s \), for all \( t > s \) (which naturally implies that \( W_t \) has a normal distribution with mean 0 and variance \( t \));
2. it has independent increments, that is, \( W_t - W_s \) is independent of its past, that is, \( W_u, a \leq u < s < t \); and
3. its sample paths are continuous, i.e., for each \( \omega \in \Omega \), \( W_t(\omega) \) as a function of \( t \) is continuous on \([a, b]\).

**Definition 2.2.** Let \( \delta \) be a positive function defined on \( T^m \), \( \mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_m) \in T^m \) and \( I = \prod_{i=1}^m I_i \) be an interval of \( T^m \). An interval-point pair \((I, \mathbf{x})\) is said to be \( \delta \)-fine if \( I_k \subset [\xi_k - \delta(\xi), \xi_k + \delta(\xi)] \), for each \( k = 1, 2, 3, \ldots, m \).

Note that \( \xi_k \) may or may not be in \( I_k \) for each \( k = 1, 2, 3, \ldots, m \). A finite collection \( D \) of interval-point pairs \( \{(I^{(i)}, \mathbf{x}^{(i)}) : i = 1, 2, 3, \ldots, n\} \) is said to be a \( \delta \)-fine division of \( T^m \) if

1. \( I^{(i)}, i = 1, 2, 3, \ldots, n, \) are disjoint left-open intervals of \( T \);
2. \( \bigcup_{i=1}^n I^{(i)} = [a, b]^m \).

In this paper, a division \( D \) may be simply denoted by \( D = \{(I^{(i)}, \mathbf{x}^{(i)})\} \) or \( D = \{(I, \mathbf{x})\} \).

We remark that for any given positive function \( \delta \) on \( T^m \), a \( \delta \)-fine division of \( T^m \) exists. This can be proved directly by using continued bisection.

**Notation.** It could be seen that \( T^m \) consists of two parts, namely, the diagonal part of \( T^m \),

\[ D = \{(x_1, \ldots, x_m) \in T^m : x_i = x_j, \text{ for some } i \neq j\} \]

and

\[ D^c = \{(x_1, \ldots, x_m) \in T^m : x_i \neq x_j, \text{ for any } i \neq j\}, \]

which is the nondiagonal part of \( T^m \). The nondiagonal set \( D^c \) plays the basic role in the construction of the multiple Itô-Wiener integral, see [4].

The nondiagonal set can be decomposed to \( m! \) open connected sets in \( T^m \). For each \( \pi \in S_m \), the group of all permutations of \( m \) objects, we define

\[ G_\pi = \{(x_1, x_2, x_3, \ldots, x_m) \in T^m : x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)} < \cdots < x_{\pi(m)}\}, \]

and there are \( m! \) such sets. Each of these sets is said to be contiguous to the diagonal \( D \).

For any interval \( I = (u, v) \subset \mathbb{R} \) and \( k \in \mathbb{R} \), let \( W(I) \) and \( W^k(I) \) denote

\[ W(I) = W_v - W_u \quad \text{and} \quad W^k(I) = (W_v - W_u)^k, \]

respectively. Let \( f : T^m \to \mathbb{R} \) be a real-valued function and \( D = \{(I^{(i)}, \mathbf{x}^{(i)})\} \) a \( \delta \)-fine division of \( T^m \). Then, \( S(f, \delta, D) \) denotes the Riemann sum,

\[ S(f, \delta, D) = \sum f\left(\mathbf{x}^{(i)}\right) W\left(I^{(i)}\right), \]

where

\[ W\left(I^{(i)}\right) = \prod_{j=1}^m W\left(I_j^{(i)}\right), \]

if \( I^{(i)} = \prod_{j=1}^m I_j^{(i)} \) and each \( I_j^{(i)} \) is a left-open interval of \( T \).

Let \( L^2(\Omega) \) denote the space of all square integrable functions of \( \Omega \).
DEFINITION 2.3. A function $f : T^m \to \mathbb{R}$ is said to be multiple Wiener integrable to $M(f) \in L^2(\Omega)$ on $T^m$ if for every $\varepsilon > 0$, there exists a positive function $\delta$, such that
\[
E \left( |S(f, \delta, D) - M(f)|^2 \right) < \varepsilon,
\]
whenever $D = \{ (\Omega^{(i)}, x^{(i)}) : i = 1, 2, 3, \ldots, n \}$ is a $\delta$-fine division of $T^m$.

We remark that the definition of multiple Itô-Wiener integral using Riemann approach, given in [4, Definition 3.2], is similar to Definition 2.3 except that the integrand $f$ is replaced by $f_0$, where $f_0 = f$ on the nondiagonal part of $T^m$ while $f_0$ vanishes on the diagonal part of $T^m$.

**LEMMA 2.4.** (See [4, Lemma 3.3] for the proof.) Let $\delta$ be a positive function on $T^m$ and $\{D_k\}$ be a finite family of $\delta$-fine divisions of $T^m$. Then there exists a partition $\{A_1, A_2, \ldots, A_q\}$ of $[0, 1]$ and a finite family of $\delta$-fine divisions of $T^m$ denoted by $\{D_k'\}$, such that each interval of any $D_k'$ is of the form $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$ and each $D_k'$ is a refinement of $D_k$. Furthermore,
\[
S(f, \delta, D_k) = S(f, \delta, D_k'),
\]
for all $k$.

From Lemma 2.4, we have the following.

**LEMMA 2.5.** A function $f : T^m \to \mathbb{R}$ is multiple Wiener integrable to $M(f) \in L^2(\Omega)$ on $T^m$ if and only if for every $\varepsilon > 0$, there exists $\delta(x) > 0$, such that
\[
E \left( |S(f, \delta, D) - M(f)|^2 \right) < \varepsilon,
\]
whenever $D = \{ (\Omega^{(i)}, x^{(i)}) : i = 1, 2, 3, \ldots, n \}$ is a standard $\delta$-fine division of $T^m$.

**REMARK.** From standard properties of Brownian motion, we know that
(A) if $I_i = (u_i, v_i]$ and $I_j = (u_j, v_j]$ are disjoint, then
\[
E(W(I_i)W(I_j)) = 0,
\]
while
(B) if $I_i = I_j = (u, v]$, then $E(W(I_i)W(I_j)) = |v - u|$.

By using standard division (as in Definition 3.4) from $T^m$, we ensure that we have (A) in our subsequent computation.

### 3. PROPERTIES OF MULTIPLE WIENER INTEGRAL

The standard properties of the integral as in the integrability of the sum of integrable functions, integrability over subintervals, additivity of the integrals over intervals and the Cauchy criterion hold for the multiple Wiener integral. The proofs follow from the classical Henstock integration theory, see [9–13].

In this section, we shall state only one property of multiple Wiener integral, see Proposition 3.4.

**DEFINITION 3.1.** Let $f : T^m \to \mathbb{R}$ be a given function and $S_m$, the permutation group on the set $\{1, 2, 3, \ldots, m\}$. For each $\pi \in S_m$, let $f_\pi$ denote the permuted function of $f$ under $\pi$, which is the function,
\[
f_\pi(t_1, t_2, t_3, \ldots, t_m) = f(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(m)}),
\]
for each $(t_1, t_2, \ldots, t_m) \in T^m$.

**DEFINITION 3.2.** A function $f : T^m \to \mathbb{R}$ is said to be symmetric if $f_\pi(t) = f(t)$, for all $\pi \in S_m$ and all $t \in T^m$. 
DEFINITION 3.3. Let \( f : \mathbb{T}^m \to \mathbb{R} \) be a given function. The symmetrization of the function \( f \), denoted by \( \tilde{f} \), is the function \( \tilde{f} : \mathbb{T}^m \to \mathbb{R} \) defined as

\[
\tilde{f}(t_1, t_2, \ldots, t_m) = \frac{1}{m!} \sum_{\pi \in S_m} f(\pi(t_1, t_2, \ldots, t_m)),
\]

where the summation is over all \( \pi \in S_m \).

We have the following proposition involving the integral of the symmetrization of \( f \). The proof of the theorem is given in [4, Theorem 4.7].

PROPOSITION 3.4. Let \( f \) be multiple Wiener integrable with value \( M(f) \) and let \( \tilde{f} \) denote the symmetrization of the function \( f \). Then, \( \tilde{f} \) is also multiple Wiener integrable and

\[
M(f) = M(\tilde{f}).
\]

4. INTEGRATION ON THE DIAGONAL OF \( \mathbb{T}^m \)

The multiple (Itô-)Wiener integral over the nondiagonal part of \( \mathbb{T}^m \) has been considered in [4]. In this section, we shall consider the multiple Wiener integral over the diagonal part of \( \mathbb{T}^m \).

Let \( f : [a, b]^m \to \mathbb{R} \) be a function. Let \( \mathcal{N} \) be the nondiagonal of \( [a, b]^m \), that is, \( \mathcal{N} \) consists of all points \( (x_1, x_2, \ldots, x_m) \in \mathbb{T}^m \), such that \( x_i \neq x_j \) if \( i \neq j \). So, \( f \) can be written as

\[
f = f_0 + f_d,
\]

where \( f_0(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m) \) if \( (x_1, x_2, \ldots, x_m) \in \mathcal{N} \) and 0 otherwise, while

\[
f_d = f - f_0.
\]

DEFINITION 4.1. Let \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) be positive integers, such that

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_r = m.
\]

Consider the set of points \( \mathcal{D} \) on the diagonal of \( \mathbb{T}^m \), such that it has \( \alpha_1 \) equal components, \( \alpha_2 \) equal components, \( \alpha_3 \) equal components, \( \ldots \), \( \alpha_r \) equal components, none of which are equal across the different \( \alpha_i \) components, \( i = 1, 2, 3, \ldots, r \). Then, the order of the diagonal \( \mathcal{D} \) is defined as \( \max(\alpha_i, \ i = 1, 2, 3, \ldots, r) \).

REMARK 4.2.

1. From Proposition 3.4, we may consider the integrand to be a symmetric function. Hence, the positions of the \( \alpha_k \) equal components, \( k = 1, 2, 3, \ldots, r \), are not important.

2. It is known from the classical probability theory that if \( U \) is a normal random variable with mean 0 and variance \( h \), then

\[
E(U^p) = \langle c_{p/2} \rangle h^{p/2},
\]

where

\[
\langle c_{p/2} \rangle = \frac{p!}{2^{p/2}(p/2)!},
\]

if \( p \) is even and

\[
\langle c_{p/2} \rangle = 0,
\]

if \( p \) is an odd integer.

We shall next prove that the multiple Wiener integral, if it exists, on any diagonal of order three or more equals zero. We need Lemma 4.3 before we begin the proof.
LEMMA 4.3. Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) be as in the setting of Definition 4.1 and \( \{\beta_1, \beta_2, \ldots, \beta_s\} \) be a set of positive integers, such that \( \beta_1 + \beta_2 + \cdots + \beta_s = 2m \) and

\[
\{D_i : i = 1, 2, \ldots, n\} = \{\{I_{i1}, I_{i2}, \ldots, I_{in}\} : i = 1, 2, \ldots, n\}
\]

be a family of collections of subintervals of \([0, 1]\) where \( I_{ik} \cap I_{ij} = \phi \), for all \( k \neq l \) with \( k, l = 1, 2, \ldots, s \) and that \( I_{ik} \cap I_{ij} = \phi \) or \( I_{ik} = I_{ij} \), for all \( i \neq j \) with \( i, j = 1, 2, \ldots, n \) and all \( k \neq l \) where \( k, l = 1, 2, 3, \ldots, s \). Let \( 0 < \varepsilon \leq 1 \) be given and

\[
Q = \sum_{i=1}^{n} a_i a_i E \left[ W^{\beta_1} (I_{i1}) W^{\beta_2} (I_{i2}) \cdots W^{\beta_s} (I_{is}) \right].
\]

Then, \( |Q| < (c_2m)2^r \varepsilon \) if either of the following conditions hold true,

(a) there exist \( \beta_k, \beta_l \geq 4 \) with \( k \neq l \), such that

\[
|I_{ik}| < \frac{\sqrt{\varepsilon}}{|a_{i1}| + 1} \quad \text{and} \quad |I_{il}| < \frac{\sqrt{\varepsilon}}{|a_{il}| + 1},
\]

for all \( i = 1, 2, 3, \ldots, n \), or

(b) there exists \( \beta_k \geq 6 \), such that

\[
|I_{ik}| < \min \left\{ \frac{\sqrt{\varepsilon}}{|a_{i1}| + 1}, \frac{\sqrt{\varepsilon}}{|a_{il}| + 1} \right\},
\]

for all \( i = 1, 2, \ldots, n \).

PROOF. It is enough for either case to assume all \( \beta_k, i = 1, 2, 3, \ldots, s \) to be even. If \( A \in \mathbb{R} \), then \( |A| \) denotes the absolute value of \( A \) and if \( A \subset \mathbb{R} \), then \( |A| \) is the Lebesgue measure of \( A \); in particular, if \( A \) is an interval, then \( |A| \) is the length of the interval \( A \). Then,

\[
|Q| = \left| \sum_{i=1}^{n} a_i a_i E \left[ W^{\beta_1} (I_{i1}) W^{\beta_2} (I_{i2}) \cdots W^{\beta_s} (I_{is}) \right] \right| 
\leq (c_2m)^s \sum_{i=1}^{n} |a_{i1}| |a_{i2}| |I_{i1}|^{\beta_1/2} |I_{i2}|^{\beta_2/2} \cdots |I_{is}|^{\beta_s/2}.
\]

To prove Case (a), assume that \( \beta_1, \beta_2 \geq 4 \), so that we have \( (\beta_1/2) - 1 \geq 1 \) and \( (\beta_2/2) - 1 \geq 1 \) while \( (\beta_k/2) \geq 1 \) for \( k = 3, 4, \ldots, s \).

\[
|Q| \leq (c_2m)^s \sum_{i=1}^{n} |a_{i1}| |a_{i2}| \frac{\sqrt{\varepsilon}}{|a_{i1}| + 1} \frac{\sqrt{\varepsilon}}{|a_{i2}| + 1} |I_{i1}| |I_{i2}| \cdots |I_{is}|
\leq (c_2m)^s \varepsilon \sum_{i=1}^{n} |I_{i1}| |I_{i2}| \cdots |I_{is}|
\leq \varepsilon (c_2m)^s |[0, 1]^s| \leq (c_2m)^{2r} \varepsilon.
\]

To prove Case (b), assume that \( \beta_1 \geq 6 \) so that \( (\beta_1/2) - 2 \geq 1 \). Then,

\[
|Q| \leq (c_2m)^s \varepsilon \sum_{i=1}^{n} |I_{i1}| |I_{i2}| \cdots |I_{is}|
\leq (c_2m)^s |[0, 1]^s| \varepsilon \leq (c_2m)^{2r} \varepsilon,
\]

thereby completing the proof.

We are now ready to prove Theorem 4.4, the main result of this section.
Theorem 4.4. Let $f : T^m \to \mathbb{R}$ be a real-valued function and let $D$ be a diagonal of order $p \geq 3$ on $T^m$. If the multiple Wiener integral of $f$ exists on $D$, then its value equals zero.

Proof. Denote the integral by $M_m(f^D)$. Given $\varepsilon \in (0, 1)$ there exists a positive function $\delta$ on $T^m$, such that

$$E \left( \left| \sum_{i=1}^{n} f \left( x^{(i)} \right) 1_D \left( x^{(i)} \right) W \left( I^{(i)} \right) - M_m(f^D) \right|^2 \right) < \varepsilon,$$

for any standard $\delta$-fine division $D = \{(I^{(i)}, x^{(i)}) : i = 1, 2, \ldots, n\}$ of $T^m$. Choose the positive function $\delta$, such that the $\delta(x)$-fine interval with associated point $x \in G_\pi$ lies completely in $G_\pi$. This is possible since $G_\pi$ is open. Consider a special division of $T^m$ of the form that if $x \in D$, then the associated interval is of the form $\prod_{j=1}^{r_i} [j, j]$, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the equal components of the diagonals of $D$. Let the corresponding $\delta$-fine partial division be

$$P = \left\{ \left( \prod_{j=1}^{r_i} [j, j], x^i \right) : x^i \in D, \ i = 1, 2, 3, \ldots, n \right\}.$$

Then,

$$Q = E \left( \left( P \sum_{i=1}^{n} f \left( x^{(i)} \right) 1_D \left( x^{(i)} \right) \prod_{j=1}^{r_i} W^{\alpha_j} \left( I_{ij} \right) \right)^2 \right)$$

$$= E \left( \left( P \sum_{i=1}^{n} f^2 \left( x^{(i)} \right) \prod_{j=1}^{r_i} W^{2\alpha_j} \left( I_{ij} \right) \right) \right)$$

$$+ 2E \left( \left( P \sum_{i<j} f \left( x^{(i)} \right) f \left( x^{(j)} \right) \prod_{i=1}^{r_i} W^{\alpha_i} \left( I_{ij} \right) \prod_{j=1}^{r_j} W^{\alpha_j} \left( I_{ij} \right) \right) \right)$$

$$= Q_1 + Q_2,$$

where $Q_1 = \sum_{i=1}^{n} f^2(x^i)E[\prod_{j=1}^{r_i} W^{2\alpha_j}(I_{ij})]$. Assume that $\alpha_1 \geq 3$ and take $a_{i_1} = a_{i_2} = f(x^i)$. Also assume that $\delta(x^i) \leq \sqrt{\varepsilon}/(|f(x^i)| + 1)$ so that we have $Q_1 \leq (c_{2m})^{2r} \varepsilon$ by Lemma 4.3. Consider

$$Q_2 = \sum_{i<j} f \left( x^{(i)} \right) f \left( x^{(j)} \right) E \left[ \prod_{i=1}^{r_i} W^{\alpha_i} \left( I_{ij} \right) \prod_{i=1}^{r_j} W^{\alpha_j} \left( I_{ij} \right) \right]$$

$$= \sum_{i<j} f \left( x^{(i)} \right) f \left( x^{(j)} \right) E \left[ \prod_{k=1}^{s} W^{\beta_k} \left( I_{ij} \right) \right],$$

where $\beta_1 + \beta_2 + \cdots + \beta_s = 2m$. We only need to consider those pairs $\{i, j\}$ where all $\beta_k$ are even. We also assume that $\beta_1, \beta_2 \geq 4$. Write

$$Q_2 = \sum_{s=r+1}^{2r} \sum_{\{\beta_1, \beta_2, \ldots, \beta_s\}} \left( \sum_{i<j} f \left( x^{(i)} \right) f \left( x^{(j)} \right) E \left[ \prod_{k=1}^{s} W^{\beta_k} \left( I_{ij} \right) \right] \right),$$

where (i) for each $s = r + 1, r + 2, \ldots, 2r$, the number of even integer solutions for the equation,

$$\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_s = 2m,$$

does not exceed $\binom{m-1}{s-1}$, and (ii) for each product,

$$W^{\beta_1} \left( I_{ij} \right) W^{\beta_2} \left( I_{ij} \right) W^{\beta_3} \left( I_{ij} \right) \cdots W^{\beta_s} \left( I_{ij} \right),$$
can be the result of the product of two intervals of the form $I_{a_1}^{\alpha_1} \times I_{a_2}^{\alpha_2} \times \cdots \times I_{a_s}^{\alpha_s}$ and $I_{b_1}^{\beta_1-a_1} \times I_{b_2}^{\beta_2-a_2} \times \cdots \times I_{b_s}^{\beta_s-a_s}$ where $\alpha_1 + \alpha_2 + \cdots + \alpha_s = m$ and each $\alpha_i$ is a nonnegative integer. The number of such possible choices of $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ cannot exceed $\binom{m}{s-1}$ by simple counting. Consequently,

$$|Q_2| < \sum_{s=r+1}^{2r} \binom{m-1}{s-1} \binom{m+s-1}{s-1} \sum f(x') f(x) E \left( \prod_{k=1}^{s} W_{\beta_k} (I_{a_k}) \right),$$

where $\sum f(x') f(x) E (\prod_{k=1}^{s} W_{\beta_k} (I_{a_k}))$ denotes the summation over all the possible $I_{a_1} \times I_{a_2} \times \cdots \times I_{a_s}$ for each fixed set $\{\beta_1, \beta_2, \ldots, \beta_s\}$. We shall further assume that

$$\delta(x') \leq \frac{\sqrt{\varepsilon}}{|f(x')| + 1},$$

so that by Lemma 4.3(a),

$$|Q_2| \leq (2c_2m)^{2r} \left( \sum_{s=r+1}^{2r} \binom{m-1}{s-1} \binom{m+s-1}{s-1} \right) \varepsilon,$$

thereby showing that for given $\varepsilon > 0$, we have

$$E \left( \left( P \sum_{i=1}^{n} f(x') 1_D(x') \prod_{j=1}^{r} W_{\alpha_j} (I_{y_j}) \right)^2 \right) \leq K \varepsilon,$$

for any special division $P = \{(I, x)\}$ (chosen as in the proof above) and which is $\delta$-fine, where $\delta$ is appropriately chosen with the aid of Lemma 4.3. Therefore, $M_m(f 1_D)$, if it exists, equals zero.

5. HU-MEYER THEOREM

In this section, we shall derive the Hu-Meyer theorem. Theorem 4.4 in the previous section shows that the multiple Wiener integral over diagonals of order three or more are zero. When we consider the multiple Wiener integral on $T^m$, we only need to consider the integral over the nondiagonal part and the diagonals of order two.

REMARK 5.1. For each $j = 1, 2, 3, \ldots, \lfloor m/2 \rfloor$, let

$$C_j = \left\{ \left( \begin{array}{c} x_1, x_1, x_2, x_2, \ldots, x_j, x_j, x_{2j+1}, x_{2j+2}, \ldots, x_m \\ y_1, y_2, y_j, y_{j+1}, y_{j+2}, y_{m-j} \end{array} \right) \in T^m : \begin{array}{c} x_i \in [0, 1]; \ x_i \neq x_k, \ \text{if} \ i \neq k \end{array} \right\}$$

denote a diagonal of order two with exactly $j$ pairs of equal components (in fact, the $j$ pairs of equal ordinates may not be in that order. It is just for convenience of our presentation). Consider the projection of the points down to $m - j$ dimension as follows,

$$\left( \begin{array}{c} x_1, x_1, x_2, x_2, \ldots, x_j, x_j, x_{2j+1}, x_{2j+2}, \ldots, x_m \\ y_1, y_2, y_j, y_{j+1}, y_{j+2}, y_{m-j} \end{array} \right) \mapsto \left( y_1, y_2, y_3, \ldots, y_j, y_{j+1}, \ldots, y_{m-j} \right).$$

Thus, the diagonal in $T^m$ of order two with $j$ equal pairs can be projected down into the space $T^{m-j}$ by collapsing the $j$ pairs of equal components down to $j$ distinct coordinates.
PROPOSITION 4.2. Let \( j = 1, 2, 3, 4, \ldots, \lfloor m/2 \rfloor \). The number of distinct diagonals of \( T^m \) of order two with \( j \) pairs of equal ordinates can be expressed as

\[
\frac{m!}{2^j j! (m-2j)!}.
\]

PROOF. This is a direct consequence of elementary counting, since

\[
\frac{1}{j!} \binom{m}{2} \binom{m-2}{2} \binom{m-4}{2} \cdots \binom{m-(2j-2)}{2} = \frac{m!}{2^j j! (m-2j)!}.
\]

DEFINITION 4.3. Let \( f : T^m \to \mathbb{R} \) be a real-valued function. The projection of \( f \) on the diagonal in the above remark, denoted by \( f_{C_j} \), is the function \( f_{C_j} : T^{m-j} \to \mathbb{R} \) defined by

\[
f_{C_j}(y_1, y_2, \ldots, y_{m-j}) = f(x_1, x_1, \ldots, x_j, x_j, x_{2j+1}, \ldots, x_m),
\]
for all points \((x_1, x_2, \ldots, x_j, x_{2j+1}, x_{2j+2}, \ldots, x_m)\).

DEFINITION 4.4. Suppose that the function \( f_{C_j}(y_{j+1}, y_{j+2}, \ldots, y_{m-j}) \) is Lebesgue integrable on \( T^j \) for each point \((y_{j+1}, y_{j+2}, \ldots, y_{m-j}) \in T^{m-2j}\). Then, the function \( \text{tr}_{C_j}\{f\} : T^{m-2j} \to \mathbb{R} \) defined as

\[
\text{tr}_{C_j}\{f\}(y_{j+1}, \ldots, y_{m-j}) = (L) \int_{T^j} f_{C_j}(y_1, y_2, \ldots, y_j, y_{j+1}, \ldots, y_{m-j}) \, dy_1 dy_2 \ldots dy_j,
\]

where \((L) \int \) denotes the Lebesgue integral of \( f_{C_j} \) with respect to the first \( j \) components, is called the trace of \( f \) on the diagonal \( C_j \).

Note that by Proposition 4.2, there are \( m!/(2^j j! (m-2j)!) \) such diagonals \( C_j \), due to the different positions of the \( j \) pairs of ordinates. However, as we assume that \( f \) is symmetric, the function \( \text{tr}_{C_j}\{f\} \) is the same function for whichever diagonal \( C_j \) we choose.

We shall state an elementary proposition useful for our subsequent proofs.

PROPOSITION 4.5. Let \( J = \prod_{s=1}^j J_s^2 = J_1 \times J_1 \times J_2 \times J_2 \times \cdots \times J_j \times J_j \subset T^{2j} \), where each \( J_s \) is an interval from \( T = [a, b] \) and \( \{J_s : s = 1, 2, 3, \ldots, j\} \) are disjoint left-open intervals of \( T \). Let \( \lambda_p(J) \) denote \( \prod_{s=1}^j |J_s| \). Then,

(i) \( E[W(J)] = \lambda_p(J) \),

(ii) \( E[W^2(J)] = 3^j [\lambda_p(J)]^2 \).

LEMMA 4.6. Let \( g : T^j \to [0, \infty) \) be Lebesgue integrable on \( T^j \). Then, given \( \varepsilon > 0 \), there exists a positive function \( \delta \) on \( T^j \), such that whenever \( D = \{(J, x)\} \) is a \( \delta \)-fine division of \( T^j \), we have

\[
E \left[ \left( \left| \sum_{(J, x) \in D} g(x) (W^2(J) - \lambda(J))^2 \right| \right) \right] \leq \varepsilon.
\]

PROOF. Given \( \varepsilon > 0 \), choose a positive function \( \delta \) defined on \( T^m \) such that it also satisfies

\[
\delta(x) \leq \frac{\varepsilon}{2^j |T| (1 + \int_{T^j} g)}
\]

and that

\[
\left| \int_{T^j} g - \left( \sum_{(J, x) \in D} g(x) \lambda_p(J) \right) \right| \leq 1,
\]

whenever \( D = \{(J, x)\} \) is a \( \delta \)-fine division of \( T^j \).
For the chosen $\delta$, consider any $\delta$-fine division of $T$. Together with Proposition 4.5, we have

$$I = E \left( \left( (D) \sum g(x) (W^2(J) - \lambda(J))^2 \right) \right)$$
$$= \sum g(x) E (W^2(J) - \lambda(J))^2$$
$$\leq 2 \sum g(x) (E (W^4(J)) + \lambda(J)^2)$$
$$= 2^j \sum g(x) \lambda(J)^2$$
$$\leq 2^j \left( \frac{\varepsilon}{2(f_T, g + 1)} \right) \sum g(x) \lambda(J)$$
$$\leq \varepsilon,$$

thereby completing our proof.

LEMMA 4.7. Let $f : T^{2j} \times T^{m-2j} \to \mathbb{R}$ be a real valued function, such that $\text{tr}_{C_i} \{f\}$ exists. Furthermore, for each $y \in T^{m-2j}$, $f C_j (\cdot, y)$ is multiple Wiener integrable on $T^j$. Then, for given $\varepsilon > 0$ there exists a positive function $\delta$ on $T^j \times T^{m-2j}$ such that for any $\delta$-fine standard division $\mathcal{D} = \{(J_k \times I_m, (x_k, y_m))\}$, we have

$$E \left( \left( (D) \sum_k f_{C_j} (x_k, y_m) W (J_k \times I_m) - \text{tr}_{C_j} \{f\} (y_m) W (I_m) \right)^2 \right) \leq \varepsilon.$$

PROOF. Let $\varepsilon > 0$ be given. From Lemma 4.6, for each $y \in T^j$, choose a positive function $\delta_1 (\cdot, y)$ on $T^j$, such that

$$E \left( (D) \sum_k f_{C_j} (x_k, y_m) (W^2(J_k) - \lambda(J_k))^2 \right) \leq \frac{\varepsilon}{4 \lambda(T^j)}.$$

for any $\delta_1$-fine division $D(y) = \{(J_k, x_k)\}$ of $T^j$.

Let $\delta_1 (\cdot, y)$ be also chosen, such that

$$\left( (D) \sum_k f_{C_j} (x_k, y) \lambda(J) - \text{tr}_{C_j} \{f\} (y) \right) \leq \frac{\sqrt{\varepsilon}}{2 \sqrt{\lambda(T^j)}},$$

for any $\delta_1 (\cdot, y)$-fine division $D(y) = \{(J_k, x_k)\}$.

Choose a positive function $\delta_2$ on $T^{2j} \times T^{m-2j}$, such that, for all $(x, y) \in T^{2j} \times T^{m-2j} \setminus C_i$, $\delta(x, y)$ is defined such that the $\delta$-fine division with $(x, y)$ as the tag does not intersect with $C_i$. This is possible since $C_i$ is closed, hence, its complement is open. Thus, for any $(J_k \times I_m, (x_k, y_m))$ for which $f 1_B (x_k, y_m) W(J_k \times I_m)$ does not vanish occurs when $x_k$ is of the form with pairwise equal components.

Choose $\delta_2$ on $T^{2j} \times T^{m-2j}$, such that $\delta_2|_{C_i} \leq \delta_1$. Consider any $\delta_2$-fine division of $T^{2j} \times T^{m-2j}$, say $D_1 = \{(J_k \times I_m, (x_k, y_m))\}$.

$$E \left( \left( (D_1) \sum_m \left( \sum_k f_{1C_i} (x_k, y_m) W (J_k \times I_m) - \text{tr}_{C_i} \{f\} (y_m) W (I_m) \right) \right)^2 \right)$$
$$= E \left( \left( (D_1) \sum_m W (I_m) \left( \sum_k f_{1C_i} (x_k, y_m) W(J_k) - \text{tr}_{C_i} \{f\} (y_m) \right) \right)^2 \right)$$
$$\leq 2 \sum_m E (W^2(I_m)) E \left( \sum_k f_{1C_i} (x_k, y_m) \left[W^2(J_k) - \lambda(J_k)\right]^2 \right)$$
$$\leq 2 \sum_m E (W^2(I_m)) \left[ \sum_k f_{1C_i} (x_k, y_m) \lambda(J_k) - \text{tr}_{C_i} \{f\} (y_m) \right]^2$$
$$= 2I_1 + 2I_2.$$
By the previous lemma,

\[
I_1 = \sum_m E \left( W^2 (I_m) \right) E \left( \sum_k f^2 C_i (x_k, y_m) \left[ W^2 (J_k) - \lambda (J_k) \right]^2 \right) \\
\leq \frac{\varepsilon}{4 |T^2|} \sum_m E \left[ W^2 (I_m) \right] \\
\leq \frac{\varepsilon}{4},
\]

while

\[
I_2 = \sum_m E \left[ W^2 (I_m) \right] \left| \sum_k f^2 C_i (x_k, y_m) \lambda (J_k) - \text{tr} C_i \{ f \} (y_m) \right|^2 \\
\leq \sum_m E \left[ W^2 (I_m) \right] \frac{\varepsilon}{4 |T^2|} \\
\leq \frac{\varepsilon}{4}.
\]

Combining the above results, for any \( \delta \)-fine division of \( T^2 \times T \), with the \( \delta \) as chosen above, for any \( \delta \)-fine division of \( T^2 \times T \), we have

\[
E \left( \left( D \right) \sum_m \left\{ \sum_k f^1 C_i (x_k, y_m) W (J_k \times I_m) - \text{tr} C_i \{ f \} (y_m) W (I_m) \right\} \right)^2 \leq \varepsilon,
\]

thereby completing the proof.

**Lemma 4.8.** Let \( f : T^{2j} \times T^{m-2j} \to \mathbb{R} \) be such that \( \text{tr} C_i \{ f \} \) exists and that \( f^2 C_i (\cdot, y) \) is Lebesgue integrable on \( T^j \) for each \( y \in T^{m-2j} \). Suppose that \( \text{tr} C_i \{ f \} \) is multiple Wiener integrable on \( T^{m-2j} \), then

\[
M_m (f^{1B_j}) = M_{m-2j} (\text{tr} \{ f \}).
\]

**Proof.** Let \( \varepsilon > 0 \) be given. There exists \( \delta_1 \) on \( T^{m-2j} \), such that

\[
E \left( \left( D_1 \right) \sum \text{tr} C_i \{ f \} (y_m) W (I_m) - M (\text{tr} C_i \{ f \}) \right)^2 \leq \frac{\varepsilon}{4},
\]

for any \( \delta_1 \)-fine division \( D_1 = \{(I_m, y_m)\} \) of \( T^{m-2j} \).

By Lemma 4.7, there exists \( \delta_2 \) on \( T^{2j} \times T^{m-2j} \), such that

\[
E \left( \left( D \right) \sum_m \left\{ \sum_k f^1 C_i (x_k, y_m) W (J_k \times I_m) - \text{tr} C_i \{ f \} (y_m) W (I_m) \right\} \right)^2 \leq \frac{\varepsilon}{4},
\]

whenever \( \{(J_k \times I_m, (x_k, y_m))\} \) is a standard \( \delta_2 \)-fine division of \( T^{2j} \times T^{m-2j} \).

Choose \( \delta_2 \) on \( T^{2j} \times T^{m-2j} \), such that \( \delta_2 |C_i \leq \delta_1 (x_k) \), for all \( (x_k, y_m) \in C_i \). Then, for such a \( \delta_2 \)-fine division of \( T^{2j} \times T^{m-2j} \), we have

\[
E \left( \left( D \right) \sum_m \left\{ \sum_k f^1 C_i (x_k, y_m) W (J_k \times I_m) - M (\text{tr} C_i \{ f \}) \right\} \right)^2 \\
\leq 2E \left( \left( D \right) \sum_m \left\{ \sum_k f^1 C_i (x_k, y_m) W (J_k \times I_m) - \text{tr} C_i \{ f \} (y_m) W (I_m) \right\} \right)^2 \\
+ 2E \left( \left( D \right) \sum_m \text{tr} C_i \{ f \} (y_m) W (I_m) - M (\text{tr} C_i \{ f \}) \right)^2 \\
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
= \varepsilon,
\]

thereby completing the proof.

We are now ready to state the Hu-Meyer theorem.
Theorem 4.9. Hu-Meyer Theorem. Let $f \in L^2([0,1]^m)$ be a symmetric function. Suppose that

1. $f_0$ is multiple Wiener integrable on $T^{2j} \times T^{m-2j}$;
2. $f_{C_j}^2(y)$ is integrable on $T^j$ for each $y \in T^{m-2j}$;
3. for each $j = 1, 2, 3, \ldots, [m/2]$, the trace function $tr_{C_j} \{f\}$ (see Definition 4.4) exists.

Then, $f$ is multiple Wiener integrable and

$$M_m(f) = M_m(f_0) + \sum_{j=1}^{[m/2]} \frac{m!}{(m-2j)!j!2^j} M_{m-2j}(tr_j \{f\}).$$

Proof. The proof makes use of the above series of lemmas. We make the following remarks to complete the proof of this theorem. Since $C_j$ denote a diagonal with $j$ pairs of equal components, by Proposition 4.2, there are $m!/(2^j j!(m-2j)!)$ such sets due to the difference in the positions of the $j$ pairs of equal components. This explains the appearance of this term in the Hu-Meyer formula. Thus, the proof is complete.

5. Further Remarks

In the above section, we have used the generalized Riemann approach to derive the classical Hu-Meyer theorem by considering the integration along both the diagonal and the nondiagonal parts. By using the generalized Riemann approach, the classical Fubini's theorem involving the change of the order of integrals can be derived. This will appear as a paper elsewhere.

References