Sampling Bessel functions and Bessel sampling

Dragana Jankov Maširević*, Tibor K. Pogány†‡, Árpád Baricz§∥ and Aurél Galántai‡

* University of Osijek/Department of Mathematics, Osijek, Croatia
† University of Rijeka/Faculty of Maritime Studies, Rijeka, Croatia
‡ Babeș–Bolyai University/Department of Economics, Cluj–Napoca, Romania
§ Babes–Bolyai University/John von Neumann Faculty of Informatics, Budapest, Hungary
∥ e-mails: djankov@mathos.hr (D. Jankov Maširević); tkpogany@gmail.com (T. K. Pogány);
bariczocsi@yahoo.com (Á. Baricz) and galantai.aurel@nik.uni-obuda.hu (A. Galántai)

Abstract—The main aim of this article is to establish summation formulae in form of sampling expansion series for Bessel functions \( Y_\nu, I_\nu \) and \( K_\nu \), and obtain sharp truncation error upper bounds occurring in the \( Y–Bessel \) sampling series approximation.

The principal derivation tools are the famous sampling theorem by Kramer and various properties of Bessel and modified Bessel functions which lead to the so-called Bessel sampling when the sampling nodes of the initial signal function coincide with a set of zeros of different cylinder functions.

Index Terms—Kramer’s sampling theorem, Bessel functions of the first and second kind \( J_\nu, Y_\nu \), modified Bessel functions of the first and second kind \( I_\nu, K_\nu \), sampling series expansions, \( Y–Bessel \) sampling, sampling series truncation error upper bound.

I. INTRODUCTION

Development of sampling theory has been rapid and continuous since the middle of the 20th century [1]. It is one of the most important mathematical techniques used in communication engineering and information theory, and it is also widely represented in many branches of physics and engineering, such as signal analysis, image processing, physical chemistry etc. [2], [3]. Generally speaking, sampling theory can be used in any discipline where functions need to be reconstructed from sampled data, usually from the values of the functions and/or their derivatives at certain points.

In this article we are interested in sampling of Bessel functions motivated by the immense research on sampling theory of special functions, especially given by Zayed (see e.g. [3], [4], [5], [6]). The article is organized as follows. In the next section we present some new summation formulae for Bessel functions \( Y_\nu, I_\nu \) and \( K_\nu \), which we derived by using Kramer’s sampling theorem and some Zayed’s results. The \( J–Bessel \) sampling method was known already by J. Whittaker [7], Helms and Thomas [8] and Yao [9]. However, we derive general \( Y–Bessel \) sampling approximation result, using Bessel function of the second kind \( Y_\nu \) as the building block of the kernel function, see Theorem 4.

Truncation error upper bound approach in finite uniform and nonuniform sampling sum approximation, when the input signal function has negative power polynomial decay rate, was intensively studied e.g. by Li [10] and by Olenko and Pogány [11], [12], [13]. Helms and Thomas [8] and Jerri and Joslin [14] reported on truncation error upper bounds considered for \( J–Bessel \) sampling for band–limited Hankel transform exclusively. Applying the \( Y–Bessel \) sampling result derived in Theorem 4 of the previous section, we establish sharp truncation error upper bounds for polynomially decaying input signal functions, compare Theorem 5.

II. SUMMATION FORMULAE FOR \( Y_\nu, I_\nu \) AND \( K_\nu \)

In this section, we recall two theorems which will help us to derive our first set of summation formulae for Bessel functions \( Y_\nu, I_\nu \) and \( K_\nu \).

Theorem A [17]: Let \( K(x,t) \) be in \( L^2(I) \) as a function of \( x \) for each real number \( t \), where \( I = [a,b] \) is some finite closed interval, and let \( E = \{t_k\}_{k \in \mathbb{Z}} \) be a countable set of real numbers such that \( \{K(x,t_k)\}_{k \in \mathbb{Z}} \) is a complete orthogonal family of functions in \( L^2(I) \). If

\[
 f(t) = \int_a^b g(x) K(x,t) \, dx,
\]

for some \( g \in L^2[a,b] \), then \( f \) admits the sampling expansion

\[
 f(t) = \sum_{k \in \mathbb{Z}} f(t_k) S_k(t),
\]

where

\[
 S_k(t) = \frac{\int_a^b K(x,t) R(x,t_k) \, dx}{\int_a^b |K(x,t_k)|^2 \, dx}.
\]

Remark 1: The points \( \{t_k\}_{k \in \mathbb{Z}} \), which are for practical reasons preferred to be real, can also be a complex numbers [18, p. 25].

Let us also mention that it is usual, in the sampling literature to say that a function \( f \), having integral representation property (1) is band–limited on \([a,b]\) or has a band–region contained in \([a,b]\). We next recall a summation formula given by Zayed.
Theorem B [4, p. 701]: If for some \( g \in L^2(0, a) \)
\[
f(t) = \int_0^a g(x) \cos(ax) \, dx,
\]
then
\[
f(t) = \sum_{k \in \mathbb{Z}} f \left( k + \frac{1}{2} - \frac{1}{2} \right) \frac{\sin(at - (k + \frac{1}{2}) \pi)}{at - (k + \frac{1}{2}) \pi},
\]
uniformly on compact \( t \)-subsets of \( \mathbb{R} \).

At this point we introduce the normalized sinc function:
\[
sinc(x) := \begin{cases} 
\frac{\sin(\pi x)}{\pi x} & x \neq 0 \\
1 & x = 0.
\end{cases}
\]

In [4] Zayed also proved a new summation formula involving
the Bessel function of the first kind \( J_\nu \), of order \( \nu \), using the previous theorem. Precisely, he used an integral representation
[19, p. 716, Eq. 6.681.1]
\[
\int_0^\pi J_{2\nu} \left( 2b \cos \frac{x}{2} \right) \cos(tx) \, dx = \pi J_{\nu+\frac{1}{2}}(b)J_{\nu-\frac{1}{2}}(b),
\]
where \( \Re\{\nu\} > -\frac{1}{2} \) and (2) to derive the following summation formula [4, p. 703, Eq. 2.8]
\[
J_{\nu+\frac{1}{2}}(b)J_{\nu-\frac{1}{2}}(b) = \sum_{k \in \mathbb{Z}} J_{\nu+k+\frac{1}{2}}(b)J_{\nu-k-\frac{1}{2}}(b) \sin(t - k - \frac{1}{2})
\]
valid for all \( \nu > -\frac{1}{2} \).

For a real number \( \nu \), the functions \( J_{\nu}(z) \) and \( Y_{\nu}(z) \) each have an infinite number of real zeros, all of which are simple
with possible exception of \( z = 0 \) [20, p. 370]. For non-negative \( \nu \) the \( k \)th positive zero of these functions are
denoted by \( j_{\nu,k}, y_{\nu,k} \), respectively.

Theorem 1: For all \( \nu > -\frac{1}{2} \) we have
\[
I_{\nu+\frac{1}{2}}(b)I_{\nu-\frac{1}{2}}(b) = \sum_{k \in \mathbb{Z}} I_{\nu+k+\frac{1}{2}}(b)I_{\nu-k-\frac{1}{2}}(b) \sin(t - k - \frac{1}{2}),
\]
uniformly on every compact \( t \)-subset of \( \mathbb{R} \).
Moreover, it holds
\[
I_{\nu+\frac{1}{2}}^2(b) = \frac{4}{\pi} \sum_{k \geq 0} \frac{(-1)^k}{2k+1} I_{\nu+(k+\frac{1}{2})}(b)I_{\nu-(k+\frac{1}{2})}(b),
\]
and
\[
\sinh^2 b = 2b \sum_{k \geq 0} \frac{(-1)^k}{2k+1} I_{k+1}(b)I_{-k}(b).
\]

Proof: Firstly, consider the integral [19, p. 716, Eq. 6.681.3]
\[
\int_0^\pi \cos(tx)I_{2\nu} \left( 2b \cos \frac{x}{2} \right) \, dx = \pi I_{\nu-\frac{1}{2}}(b)I_{\nu+\frac{1}{2}}(b),
\]
which holds for \( \Re\{\nu\} > -\frac{1}{2} \). Taking \( g(x) = I_{2\nu} \left( 2b \cos \frac{x}{2} \right) \),
which is in \( L^2(0, \pi) \) for \( \nu > -\frac{1}{2} \) and \( f(t) = \pi I_{\nu-\frac{1}{2}}(b)I_{\nu+\frac{1}{2}}(b) \),
from (2) it follows the first statement.
If we set here \( t = 0 \), bearing in mind that here there holds
\( \sum_{k \in \mathbb{Z}} = 2 \sum_{k \geq 0} \) immediately turns out the assertion (3).
Finally, setting \( \nu = \frac{1}{2} \) in (3), we obtain the Bessel function representation of the hyperbolic sine’s square.

Theorem 2: For \( |\nu| < \frac{1}{2} \) we have
\[
K_{\nu}(b) = \frac{2}{\pi} \sum_{k \in \mathbb{Z}} K_{2k+1}(b) \cos \left( \frac{\pi}{2} - \nu \right) \left( \frac{2}{2k+1} - \nu \right).
\]
Moreover
\[
e^{-b} = \frac{16\sqrt{b}}{\sinh \pi b} \sum_{k \geq 0} \left( \frac{-1}{2k+1} \right) K_{2k+1}(b),
\]
where (5) holds for large \( b \to \infty \).

Proof: By virtue of the formula [19, p. 716, Eq. 6.681.4]
\[
\int_0^\pi \cos(\nu x)K_{\nu} \left( 2b \cos x \right) \, dx = \frac{\pi}{2} I_{\nu}(b)K_{\nu}(b), \quad \Re\{\nu\} < 1,
\]
choosing \( g(x) = K_{\nu} \left( 2b \cos x \right) \) and \( f(\nu) = \frac{\pi}{2} I_{\nu}(b)K_{\nu}(b) \),
Theorem B ensures the formula (4).
Moreover, for \( \nu = \frac{1}{2} \) by (4) we can see that
\[
e^{-b} = \frac{4\sqrt{b}}{\sin \pi b} \sum_{k \geq 0} \left( \frac{-1}{2k+1} K_{2k+1}(b) \right),
\]
Because of the parity with respect to the order of \( K_{\nu}(t) \), i.e. \( K_{-\nu}(t) = K_{\nu}(t) \), this implies (5).

Finally, it remains the discussion of (5) in the case when \( b \) is growing to infinity. Denote
\[
R(b) := \frac{16\sqrt{b}}{\sinh \pi b} \sum_{k \geq 0} \left( \frac{-1}{2k+1} \right) K_{2k+1}(b).
\]
It is well known [21, p. 202] that
\[
K_{\nu}(z) = e^{-z} \sqrt{\frac{\pi}{2z}} \left( 1 + O(z^{-1}) \right), \quad |z| \to \infty.
\]

Letting \( b \to \infty \) above, we have
\[
R(b) = \frac{16}{\pi \sqrt{2}} \left( 1 + O(b^{-1}) \right) \sum_{k \geq 0} \left( \frac{-1}{2k+1} \right) \frac{1}{4k+1} (4k+3)
\]
\[
= \frac{1}{\pi \sqrt{2}} \left( 1 + O(b^{-1}) \right) \sum_{k \geq 0} \left( \frac{-1}{2k+1} \right) \left( \frac{1}{k+\frac{1}{2}} \right)
\]
\[
= \frac{1}{\pi \sqrt{2}} \left( 1 + O(b^{-1}) \right) \left[ \Phi \left( -1, \frac{1}{2} \right) + \Phi \left( -1, \frac{3}{2} \right) \right],
\]
where
\[
\Phi(z, s, a) := \sum_{k \geq 0} \frac{z^k}{(a+k)^s},
\]
stands for the familiar Lerch transcendent (or Hurwitz–Lerch Zeta-function), defined for all \( |z| < 1 \), or \( |z| = 1, \Re\{s\} > 1; \ a \notin \mathbb{Z}_0^* = \{0, -1, -2, \cdots \} \), see e.g. [22, p. 27, Eq. 1.11 (1)].
Being
\[
\Phi \left( -1, 1, \frac{1}{2} \right) + \Phi \left( -1, \frac{1}{2} \right) = \frac{\pi}{2} \left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) = \pi \sqrt{2},
\]
the proof of Theorem is complete.

Below, we derive another summation formula for modified Bessel function of the second kind \( I_{\nu} \), which we believe to be new.
Theorem 3: The modified Bessel function of the first kind \( I_\nu \) admits the sampling expansion
\[
I_\nu(t) = 2(2t)^\nu \cosh t \sum_{k \in \mathbb{Z}} \frac{(-1)^k (\pi + 2k\pi)^{1-\nu}}{(\pi + 2k\pi)^2 + 4t^2} \times J_{\nu} \left( \pi \left( k + \frac{1}{2} \right) \right), \quad t \in \mathbb{R}; \ |\Re\{\nu\}| > 0. \quad (6)
\]

Moreover
\[
\tanh(\pi t) = \frac{8t}{\pi} \sum_{k \geq 1} \frac{1}{(2k-1)^2 + 4t^2}, \quad t \in \mathbb{R}.
\]

Proof: For \( \cos(tx) \in L^2(0, 1) \), taking
\[
E = \{ t_k = \pi(k + \frac{1}{2}), k \in \mathbb{Z} \},
\]
the orthogonal family of functions \( \{ K(x, t_k) \}_{k \in \mathbb{Z}} \) is complete in \( L^2(0, 1) \). Set \( K(x, t) = \cos(tx) \equiv \cosh(tx) \).

Now, rewriting the formula [21, p. 79, Eq. (9)]
\[
I_{\nu}(t) = \frac{2^{1-\nu} \nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} \cosh(tx) \, dx,
\]
where \( \Re\{\nu\} > -\frac{1}{2} \) into
\[
\frac{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})}{2^{1-\nu} \nu} I_{\nu}(t) = \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} \cosh(tx) \, dx,
\]
and taking \( g(x) = (1-x^2)^{\nu-\frac{1}{2}} \), which belongs to \( L^2(0, 1) \) when \( \Re\{\nu\} > 0 \), by Theorem A it follows that
\[
t^{-\nu}I_{\nu}(t) = \cosh t \sum_{k \in \mathbb{Z}} \frac{(-1)^k \nu^{1-\nu}}{t_k - t^2} I_{\nu}(t_k). \quad (7)
\]
Using the well-known identities \( I_{\nu}(t) = i^{-\nu}J_{\nu}(it) \) and \( J_{\nu}(-t) = (-1)^\nu J_{\nu}(it) \), after some simplification, from (7) it follows (6).

For \( \nu = \frac{1}{2} \), expressing \( J_{\frac{1}{2}} \) and \( I_{\frac{1}{2}} \) via sine and hyperbolic sine, respectively, the expansion (6) becomes
\[
\sinh t \cosh t = \sum_{k \in \mathbb{Z}} \frac{4t}{(\pi + 2k\pi)^2 + 4t^2} = \sum_{k \geq 0} \frac{8t}{(\pi + 2k\pi)^2 + 4t^2},
\]
i.e. there holds the stated sampling expansion formula for hyperbolic tangent.

Notice that one can derive a new summation formula for modified Bessel function of the second kind \( K_\nu \), by using the previous result and connection between \( I_{\nu} \) and \( K_\nu \):
\[
K_{\nu}(t) = \frac{\pi}{2 \sin(\nu \pi)} (I_{-\nu}(t) - I_{\nu}(t)),
\]
which, for \( \nu = \frac{1}{2} \), becomes the well-known identity
\[
e^{-t} = \cosh t - \sinh t.
\]

Modified Bessel function of the second kind \( K_\nu \) can also be represented through \( I_{\nu} \) and Bessel function of the second kind \( Y_\nu \) as for noninteger \( \nu \notin \mathbb{Z} \):
\[
K_{\nu}(t) = \frac{\pi}{2} \left( \frac{(it)^{\nu} \cos(\nu \pi)}{t^{2\nu}} - 1 \right) I_{\nu}(t) \frac{\sin(\nu \pi)}{i^{\nu}} - \frac{\pi(1)^{\nu}}{2^{\nu}} Y_{\nu}(it),
\]
so, it is easy to deduce a new summation formula for \( K_\nu \), using a summation formulae for \( I_{\nu} \) and \( Y_{\nu} \). For that purpose, we derive a new summation formula for \( Y_\nu \).

Theorem 4: Let for some \( g \in L^2(0, a), a > 0 \) the function \( f \) possesses an integral representation
\[
f(t) = \int_0^a g(x) \sqrt{x} Y_\nu(tx) \, dx.
\]
Then, for all \( t \in \mathbb{R} \) and \( \nu \in [0, 1) \), \( f \) admits the sampling expansion
\[
f(t) = 2Y_{\nu}(at) \sum_{k \geq 1} \frac{y_{\nu,k}}{(y_{\nu,k}^2 - a^2 t^2)} Y_{\nu+1}(y_{\nu,k}), \quad (8)
\]
where \( y_{\nu,k}, k \in \mathbb{N} \) are the positive real zeros of the Bessel function \( Y_{\nu}(t) \).

Proof: For \( K(x, t) = \sqrt{2} Y_{\nu}(tx) \) and \( t_k = a^{-1} y_{\nu,k}, k \) positive integer, we have complete orthogonal family of functions \( \{ K(x, t_k) \}_{k \in \mathbb{N}} \) in \( L^2(0, a) \). Further, using the integrals [19, p. 624, Eqs. 5.54.1, 5.54.2]
\[
\int x Y_{p}(ax) Y_{p}(-\beta x) \, dx = \frac{x^2}{2} \left\{ Y_p(ax)^2 - Y_{p-1}(ax)Y_{p+1}(ax) \right\},
\]
by Kramer’s Theorem A it follows that
\[
S_{\nu}(t) = \frac{2t_k Y_{\nu}(at)}{a(t_k^2 - t^2) Y_{\nu+1}(at_k)}, \quad |\Re\{\nu\}| < 1.
\]
Now, bearing in mind the previous considerations, again by Theorem A we deduce the formula (8), where we set \( \nu \) to be a non–negative real number, because of the positivity of the zeros \( y_{\nu,k} \).

Corollary 4.1: For \( \Re\{\nu\} > 0, t \in \mathbb{R} \) we have
\[
Y_{\nu}(t) J_{\nu}(t) = 2 \sum_{k \geq 1} \frac{(2t)^{\nu-1} Y_{\nu}(2t) J_{\nu}(y_{\nu,k}) Y_{\nu}(y_{\nu,k})}{y_{\nu,k}^2 - 4t^2} \cdot Y_{\nu+1}(y_{\nu,k}). \quad (9)
\]

Moreover, for \( t \neq k - \frac{1}{2}, k \in \mathbb{N} \)
\[
\tan(\pi t) = \frac{8t}{\pi} \sum_{k \geq 1} \frac{1}{(2k-1)^2 + 4t^2}.
\]

Proof: Considering the formula [19, p. 672, Eq. 6.676.10], we conclude
\[
J_{\nu}(\frac{t}{2}) Y_{\nu}(\frac{t}{2}) = \frac{2^{1-\nu} \nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 x^{\nu-1} (1-x^2)^{\nu-\frac{1}{2}} Y_{\nu}(tx) \, dx,
\]
where \( t > 0 \), \( \Re\{\nu\} > -\frac{1}{2} \), and choose \( g(x) = (x-x^3)^{\nu-\frac{1}{2}} \in L^2(0, 1) \). From the previous theorem, setting \( a = 2 \), we obtain the asserted result.

For \( \nu = \frac{1}{2} \), having in mind that zeros of \( Y_{\frac{1}{2}} \) are of the form \( y_{\frac{1}{2},k} = \frac{\pi}{2} + \pi(k - 1), k \in \mathbb{N} \), the expansion formula (9) one reduces to well–known partial fraction series expansion of the tangent function.
III. Truncation Error Upper Bounds in $Y$–Bessel Sampling Expansions

In this section we would derive uniform upper bound for the truncation error for the Bessel–sampling expansion (8) setting $a = 1$ for the sake of simplicity.

The truncated sampling reconstruction sum of the size $N \in \mathbb{N}$, for Bessel–sampling formula (8) we define as

$$\mathcal{S}_N^Y(f; t) = 2Y(t) \sum_{k \geq 1} f(y_{\nu,k}) \frac{y_{\nu,k}}{(y_{\nu,k}^2 - t^2)} Y_{\nu+1}(y_{\nu,k}),$$

where $t \in \mathbb{R}$, $\nu \in [0, 1)$ and the function $f$ has a band–region contained in $(0, 1)$. Also, the truncation error of the order $N$ is the quantity $\mathcal{E}_N^Y(f; t) = |f(t) - \mathcal{S}_N^Y(f; t)|$, that is

$$\mathcal{E}_N^Y(f; t) = \left| \sum_{k \geq N+1} f(y_{\nu,k}) \frac{2y_{\nu,k}Y(t)}{(y_{\nu,k}^2 - t^2)Y_{\nu+1}(y_{\nu,k})} \right|.$$

Our main goal is to specify an upper bound for the truncation error $\mathcal{E}_N^Y(f; t)$ valid for the widest possible $t$–range assuming that the input function possesses a polynomially decaying upper bound like

$$|f(t)| \leq A |t|^{-(r+1)}, \quad A > 0, \quad r > \frac{1}{2}, \quad t \neq 0.$$  \hspace{1cm} (10)

Thus, for all $\nu \in [0, 1)$, it is

$$\mathcal{E}_N^Y(f; t) \leq 2A \sum_{k \geq N+1} \frac{|Y(t)|}{y_{\nu,k}^2 |y_{\nu,k}^2 - t^2| |Y_{\nu+1}(y_{\nu,k})|}; \quad (11)$$

indeed, being $\nu \geq 0$ all zeros $y_{\nu,k}$ are positive [20, p. 370].

Having in mind the well known interlacing inequalities for the positive zeros $j_{\nu,j}, j'_{\nu,j}$, $y_{\nu,k}$ and $y'_{\nu,k}$ of Bessel functions $J_{\nu}(t), J'_{\nu}(t), Y_{\nu}(t)$ and $Y'_{\nu}(t)$, respectively [20, p. 370]

$$\nu \leq j_{\nu,1} < y_{\nu,1} < j'_{\nu,1} < j_{\nu,2} > y'_{\nu,2} < y_{\nu,2} < \cdots,$$

from (11) it follows for a $t$ fixed

$$\mathcal{E}_N^Y(f; t) \leq 2A \frac{\sup_{\nu < \nu_k} |Y(t)|}{\min_{k \geq N+1} y_{\nu,k}^2 |Y_{\nu+1}(y_{\nu,k})|} \sum_{k \geq N+1} \frac{1}{y_{\nu,k}^2 - t^2}.$$

Consider now the integral expression [21, p. 178, Eq. (1)]

$$Y_{\nu}(z) = \frac{1}{\pi} \int_0^\infty \sin(z \sin x - \nu x) \, dx$$

$$- \frac{1}{\pi} \int_0^\infty e^{-z \sin x} (e^{-\nu x} \cos \nu x + e^{\nu x}) \, dx,$$

where $|\arg(z)| < \frac{\pi}{2}$. Thus, for real $z = t > 0$ we get

$$|Y_{\nu}(t)| \leq 1 + \frac{2}{\pi} \int_0^\infty e^{-t \sin x} \cos \nu x \, dx$$

$$\leq 1 + \frac{2}{\pi} \int_0^\infty e^{-t \sin x} \cosh \nu x \, dx$$

$$= 1 + \frac{2}{\pi \nu t},$$

in turn by increasing $t \mapsto \sinh t$ for $t > 0$. Hence

$$\sup_{\nu < \nu_k} |Y_{\nu}(t)| \leq 1 + \frac{2}{\pi \nu t} =: H(t).$$

Now, using the inequality [23, p. 68, Eq. (1.6)]

$$j_{\nu,k} > \nu + k\pi - \frac{3}{2} + \frac{1}{2}, \quad \nu > -\frac{1}{2}, \quad k \in \mathbb{N}$$

we conclude that

$$\mathcal{E}_N^Y(f; t) \leq 2Q_N(t) \sup_{\nu < \nu_k} \frac{2Y(t)}{y_{\nu,k}^2 (y_{\nu,k}^2 - t^2) Y_{\nu+1}(y_{\nu,k})},$$

$$< \sum_{k \geq N} \frac{2Q_N(t)}{(k + (\nu + 1 - \frac{1}{2} + \frac{1}{2\pi}))^2 - \pi - 2y_{\nu,k}^2} = \frac{Q_N(t)}{\pi y_{\nu,2}} \left\{ \psi \left( N + \frac{1 - \pi + 2(\nu + y_{\nu,2})}{2\pi} \right) \right\}, \quad (12)$$

where

$$Q_N(t) := \min_{k \geq N+1} \frac{AH(t)}{y_{\nu,k}^2 |Y_{\nu+1}(y_{\nu,k})|}$$

and $\psi$ stands for the familiar digamma function.

Indeed, let us recall that the partial fraction sum expansion of the digamma function reads

$$\psi(x) = \sum_{k \geq 1} \left( \frac{1}{k} - \frac{1}{k + x - 1} \right) - \gamma,$$

where $\gamma$ stands for the Euler–Mascheroni constant. Therefore

$$\sum_{k \geq N} \frac{1}{(k + x - N)^2 - \beta^2} = \frac{1}{2\beta} \left( \psi(x + \beta) - \psi(x - \beta) \right).$$

Putting

$$x = N + \frac{1}{2} - \frac{\pi}{2}, \quad \beta = \pi^{-1} y_{\nu,2},$$

we arrive at the bound (12). Next, by means of the Lagrange’s theorem we conclude that for every $x \pm \beta \notin \mathbb{Z}_0^-$ there holds

$$\psi(x + \beta) - \psi(x - \beta) < 2\beta \psi'(x + \beta (2\theta - 1)),$$

for certain $\theta \in [0, 1]$.

Using the inequality [24, Lemma 2]

$$\psi'(t) < e^{1/t - 1}, \quad t > 0,$$

for every $x, \beta > 0$ it follows

$$\psi(x + \beta) - \psi(x - \beta) < 2\beta \left( e^{(x + \beta (2\theta - 1))} - 1 \right)$$

$$< 2\beta \left( e^{(x - \beta)} - 1 \right).$$

Taking

$$x = N + \frac{1 - \pi + 2\nu}{2\pi}; \quad \beta = \pi^{-1} y_{\nu,2},$$

we have that

$$\psi(x + \beta) - \psi(x - \beta) < 2\frac{y_{\nu,2}}{\pi} \left\{ e^{\left( \frac{N + 1 - \pi + 2(\nu + y_{\nu,2})}{\pi} \right)} - 1 \right\}$$

$$= 2\frac{y_{\nu,2}}{\pi} M_N(\nu),$$
i.e. \[
\mathcal{P}_N^Y(f; t) < \frac{2AH(t) M_N(\nu)}{\pi^2 \min_{k \geq N+1} y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})|}.
\] (13)

It remains to minimize the denominator of (13). Knowing that \[20, \text{p. 364}\]
\[Y_\nu(t) \sim \sqrt{\frac{2\pi}{t}} \sin\left(t - \frac{\pi}{4}(2\nu + 1)\right), \quad t \to \infty,
\]
(14) obviously it is \(\sqrt{t} |Y_\nu(t)| = \mathcal{O}(1)\). Therefore
\[
\min_{k \geq N+1} y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})| \geq (y_{\nu,N+1})^{r-\frac{1}{2}} \min_{k \geq N+1} \sqrt{y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})|} > 0.
\]
Finally, we deduce
\[
\mathcal{P}_N^Y(f; t) < \frac{2AH(t) M_N(\nu)}{\pi^2 (y_{\nu,N+1})^{r-\frac{1}{2}} \min_{k \geq N+1} \sqrt{y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})|}}.
\]
This finishes the proof of the following result.

**Theorem 5:** Let \(f \) be satisfy condition (10) and \(\nu \in [0,1]\). Then for all \(t \in (0,y_{\nu,N+1})\), \(A > 0, r > \frac{1}{2}\) and all \(N \geq 2\) there holds the truncation error upper bound
\[
\mathcal{P}_N^Y(f; t) < U_N^Y(t) := \frac{2AH(t) M_N(\nu) (y_{\nu,N+1})^{\frac{1}{2}-r}}{\pi^2 \min_{k \geq N+1} \sqrt{y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})|}},
\]
in the \(Y\text{–}Bessel\) sampling expansion formula
\[
f(t) = 2Y_\nu(t) \sum_{k \geq 1} \frac{y_{\nu,k} j(Y_{\nu,k})}{(y_{\nu,k} - t^2) Y_{\nu+1}(y_{\nu,k})}. \tag{15}
\]

Here
\[
H(t) = 1 + \frac{2}{\pi \nu t}, \quad M_N(\nu) = \exp\left\{\left(N + 1 - \frac{\pi}{2} + 2(\nu - y_{\nu,2})\right)^{-1}\right\} - 1.
\]

Moreover, for any fixed \(t \in (0,y_{\nu,2})\) and growing \(N\) the truncation error has asymptotic behaviour
\[
\mathcal{P}_N^Y(f; t) = \mathcal{O}\left( \left(N^{-r-\frac{1}{2}}\right) \right). \tag{16}
\]

**Proof:** As the truncation error upper bound \(U_N^Y(t)\) has already remained, it follows to show the asymptotics of the truncation error \(\mathcal{P}_N^Y(f; t)\). In this goal consider the upper bound function \(U_N^Y(t)\) for some \(t\) fixed, for \(N\) enough large. We have
\[
\mathcal{P}_N^Y(f; t) = \mathcal{O}\left(U_N^Y(t)\right) = \mathcal{O}\left(\frac{M_N(\nu)}{(y_{\nu,N+1})^{r-\frac{1}{2}}}\right) = \mathcal{O}\left(\frac{\nu^{N-1} - 1}{(y_{\nu,N+1})^{r-\frac{1}{2}}}\right) = \mathcal{O}\left(\left(N^{-r-\frac{1}{2}}\right)\right). \tag{17}
\]

Indeed, since \(Y_{\nu+1}(y_{\nu,k})\) cannot vanish, the asymptotic formula (14) ensures that \(\min_{k \geq N+1} \sqrt{y_{\nu,k} |Y_{\nu+1}(y_{\nu,k})|}\) is a positive constant. So is (16).

However, the last step in (17) follows by the MacMahon expansion [21, p. 506] (see also Schlüff’s footnote [25, p. 137])
\[
y_{\nu,N} = \left(N + \frac{\nu}{\pi} - \frac{1}{2}\right) \pi + \mathcal{O}(N^{-1}), \quad N \to \infty,
\]
associated with the large zeros of the cylinder functions. The proof of (15) is complete.

**IV. CONCLUSION. FURTHER REMARKS**

The main aim of this short note is twofold. Firstly, to establish a set of summation formulæ by virtue of the Kramer’s sampling theorem and the underlying various properties of Bessel functions \(J_\nu, Y_\nu\) and their modified variants \(J_\nu, K_\nu\) taking for the set \(E\) of sampling nodes the zeros of these cylinder functions (up to a constant). The results concerns sampling of modified Bessel function \(I_\nu\) (Theorem 3); sampling restoration of the product of two modified Bessel functions of the first kind \(I_\nu\) (Theorem 1); sampling of modified Bessel function of the second kind \(K_\nu\) (Theorem 2), while Theorem 4 and Corollary 4.1 are devoted to the \(Y\text{–}Bessel\) sampling of the Hankel–transform with the Bessel function of the second kind \(Y_\nu\) in the kernel. All these results are novelties in the field.

To demonstrate the Bessel–sampling approximation behaviour we decide to present Corollary 4.1. Let us signify
\[
h(t) = \frac{Y_\nu(t) J_\nu(t)}{\nu Y_\nu(2t)} \quad \mathcal{P}_N^Y(h; t) = \sum_{k \geq 1} \frac{2^{\nu+1} J_\nu(y_{\nu,k}) Y_\nu(y_{\nu,k})}{y_{\nu,k}^2 - 4t^2 Y_\nu(y_{\nu,k})}.
\]

We present below on Fig. 1. the input function \(h(t)\) and the truncated sampling \(Y\text{–}Bessel\) sampling approximation sums \(\mathcal{P}_N^Y(h; t)\) for \(N = 1,10,90\) respectively on the \(t\)-domain \([\frac{1}{2}y_{0,1}, \frac{1}{2}y_{0,2}] \approx [0.446788, 1.97884]\), in the case \(\nu = 0\).

Secondly, to derive sharp truncation error upper bound appearing in the sampling series approximation in a currently obtained summation formulæ, when the input signal vanishes by polynomial rate for a growing values of its argument. Theorem 5 consists from the related results.

As we have mentioned above, Helms and Thomas proved a sampling expansion for the generalized Hankel–transform (employing \(J_\nu\)) [8], [14, p. 324, Eq. (7)], which counterpart is actually our Theorem 4, with a slightly different input band–limited function (employing Bessel’s \(Y_\nu\)). Helms and Thomas derive a truncation error upper bound of magnitude \(\mathcal{O}(N^{-1})\) using the guard–band technique [8], [14, p. 324, Eq. (7)], when the input signal is bounded. Yao [9] estimated a truncation error of the generalized sampling expansion by using reproducing kernel Hilbert space approach, getting a bound which \(J\text{–}Bessel\) type sampling variant produces a computationally requiriable form, see also [14, p. 325, Eq. (15)]. Further, Jerri and Joslin [14, p. 326, Eq. (16)] reported on a computationally simpler upper bound.
then Yao’s:

$$\mathcal{T}_n^d(f;t) \leq \frac{2K_f}{\sqrt{2\pi}} \left\{ \log \frac{1}{1-2r} \right\} \frac{J_\nu(t)}{J_\nu(j_{\nu,N})} \times \left\{ \frac{1}{\lfloor j_{\nu,N} + t \rfloor} + \frac{1}{\lfloor j_{\nu,N} - t \rfloor} \right\},$$

where the guard–band $r \in (0, \frac{1}{2})$ and the absolute constant $K_f$ depends on the nature of the input signal function $f$. This bound has a magnitude $O\left( N^{-1} |J_\nu(j_{\nu,N})|^{-1} \right)$ for band-limited input signals. Jerri and Joslin used complex integration approach under guard–band assumption.

Our current results are superior to earlier reported ones in convergence rate; however, the faster convergence we pay with the $t$–domain reduction – our bounds posses finite domain $t \in (0, y_{\nu,2}), \nu \in [0, 1)$ (see Theorem 5), while the above mentioned bounds’ domain is $\mathbb{R}$.

REFERENCES


1We point out that the guard–band’s range $(0, 1)$ in [14, p. 326, Eq. (16)] is faulty, and it has to be corrected to $(0, \frac{1}{2})$. 


\[ J_\beta_1(x), J_\beta_2(x), \ldots, \]

wo $\beta_1, \beta_2, \beta_3, \ldots$ die positiven Wurzeln der Gleichung $\beta(x) = 0$ vorstehlen”, Math. Ann., vol. 10, pp. 137–142, 1876.