Dynamical behaviors of Cohen–Grossberg neural networks with discontinuous activation functions

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Abstract

In this paper, we discuss dynamics of Cohen–Grossberg neural networks with discontinuous activations functions. We provide a relax set of sufficient conditions based on the concept of Lyapunov diagonally stability (LDS) for Cohen–Grossberg networks to be absolutely stable. Moreover, under certain conditions we prove that the system is exponentially stable globally or convergent globally in finite time. Convergence rate for global exponential convergence and convergence time for global convergence in finite time are also provided.

Keywords: Cohen–Grossberg neural networks; Differential inclusions, Set-valued map; Filippov solution, Lyapunov diagonally stable; Global stability

1. Introduction

Research of recurrently connected neural networks (RCNN) is an important topic in neural network theory. Among them, Cohen–Grossberg neural networks were firstly proposed in pioneering works of Cohen and Grossberg (1983). They can be modelled as follows

\[
\frac{dx_i}{dt} = a_i(x_i) \left[ -b_i(x_i) + \sum_{j=1}^{n} I_{ij} g_j(x_j) + J_i \right], \quad i = 1, 2, \ldots, n
\]

or

\[
\frac{dx}{dt} = A(x)[-d(x) + Tg(x) + J]
\]

where \( x=(x_1, x_2, \ldots, x_n)^T \) is the state vector of the neural network, \( A(x)=\text{diag}\{a_1(x_1), \ldots, a_n(x_n)\} \) is composed of the gain functions \( a_i(x_i) \), \( d(x)=(d_1(x_1), d_2(x_2), \ldots, d_n(x_n))^T \) with \( d_i(\cdot) \) modelling self-inhibition of \( i \)th neuron. \( T=(t_{ij}) \in \mathbb{R}^{n,n} \) is the connection matrix, and \( J=(J_1, J_2, \ldots, J_n)^T \in \mathbb{R}^n \) is input vector, \( g(x)=(g_1(x_1), g_2(x_2), \ldots, g_n(x_n))^T \) with \( g_i(\cdot) \) modelling the non-linear input–output activation of \( i \)th neuron. In this paper, \( g_i(\cdot) \) is not necessarily continuous.

It must be pointed out that Cohen–Grossberg neural networks include Hopfield neural networks (Hopfield, 1984; Hopfield & Tank, 1986) as special cases. The latter can be described as:

\[
\frac{dx_i}{dt} = -d_i(x_i) + \sum_{j=1}^{n} t_{ij} g_j(x_j) + I_i, \quad i = 1, 2, \ldots, n
\]

Research on the dynamical behavior of the RCNN networks can be dated back to the early days of neural networks science. For instance, multi-stable and oscillatory behaviors were studied by Amari (1971, 1972) and Wilson and Cowan (1972); Chaotic behaviors were investigated by Sompolinsky and Crisanti (1988); Hopfield (1984) and Hopfield and Tank (1986) looked into the dynamics stability of symmetrically connected networks and showed their practical applicability to optimization problems. It should be noted that Cohen and Grossberg (1983) present more rigorous and analytical results on the globally stability of the RCNN networks.

Michel, Farrel, and Porod (1989), Michel and Gray (1990), as well as Yang and Dillon (1994), obtained the sufficient conditions for the local stability and of the equilibrium point. However, they did not address


All these papers are based on the assumption that the activation functions are continuous and even Lipshizean. A brief review on some common neural network models reveals that neural networks with discontinuous activation functions are of importance and do frequently arise in practice. For example, consider the classical Hopfield neural networks with graded response neurons (see Hopfield, 1984). The standard assumption is that the activations used are in high-gain limit, where they closely approach discontinuous and comparator functions. As shown by Hopfield (1984, 1986), the high-gain hypothesis is crucial to make negligible the connection to the neural network energy function of the term depending on neuron self-inhibitions, and to favor binary output formation. For example: the activation function \( g_2(s) = \text{sign}(s) \).

A conceptually analogous model based on hard comparators is also used to describe the Discrete-Time Neural Networks by Harrer, Nossek, and Stelzl (1992). Another important example is the neural networks introduced in Kennedy and Chua (1988) to solve linear and non-linear programming problems. Those networks exploit constrained neurons with a diode-like input–output activations. Again, in order to satisfy the constraints, the diodes are required to possess a very high slope in the conducting region, i.e. they should approximate the discontinuous characteristic of an ideal diode (see Chua, Desoer, & Kuh, 1987). When dealing with dynamical systems possessing high-slope non-linear elements, it is often of advantage to model them with a system of differential equations with discontinuous right-hand side, rather than studying the case where time slope is high but of finite value (Utkin, 1978).

In this paper, without assumption of boundedness and continuity of the activation functions, we will present some sufficient conditions for the global stability and exponential stability of a class of the Cohen–Grossberg neural networks by using the LDS, and provide an estimate of the convergence rate.

2. Preliminaries

2.1. Set-value maps, LDS, and hypotheses about the model

In this section, we present some definitions and basic concepts concerning set-value maps and matrices, which will be used throughout the paper.

**Definition 1.** Suppose \( E \subseteq \mathbb{R}^n \). Map \( x \mapsto F(x) \) is called a set-value map from \( E \subseteq \mathbb{R}^n \) if, to each point \( x \) of a set \( E \subseteq \mathbb{R}^n \), there corresponds to a non-empty set \( F(x) \subseteq \mathbb{R}^m \). A set-value map \( F \) with non-empty values is said to be upper semicontinuous at \( x_0 \in E \), if for any open set \( N \) containing \( F(x_0) \), there exists a neighborhood \( M \) of \( x_0 \) such that \( F(M) \subseteq N \). \( F(x) \) is said to have closed (convex, compact) image, if for each \( x \in E \), \( F(x) \) is closed (convex, compact).

More details about set-value maps can be found in Aubin and Frankowska (1990).

**Definition 2.** A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be Lyapunov diagonally stable (LDS), if there exists a positive diagonal matrix \( P \) such that the symmetric part of \( PA \) is positive definite, i.e.

\[
(PLA)^T = \frac{1}{2}(PA + A^TP) > 0
\]

**Definition 3.** Class \( \bar{A} \) of functions: Let \( A(x) = \text{diag}[a_1(x_1), a_2(x_2), \ldots, a_n(x_n)] \). If for all \( i = 1, 2, \ldots, n \), \( a_i(s) > 0 \) is continuous and

\[
\int_0^{+\infty} \frac{s ds}{a_i(s)} = \int_0^{-\infty} \frac{s ds}{a_i(s)} = +\infty
\]

Then we say that \( A(x) \in \bar{A} \).
It is easy to see that if $A(x) \in \bar{A}$, then for any $s^* \in R$, we have
\[
\int_0^\infty \frac{s}{a(s + s^*)} \, ds = \int_0^\infty \frac{s}{a(s + s^*)} \, ds = +\infty
\] (2)

**Definition 4.** Class $\bar{A}_i$ of functions: let $A(x) = \text{diag}\{a_1(x_1), a_2(x_2), \ldots, a_p(x_p)\}$. It is said that $A(x) \in \bar{A}_1$, if $A(x) \in \bar{A}$ and there exist $\tilde{q} > 0$ such that $a_i(s) \geq \tilde{q}$ for all $s \in R$ and $i = 1, 2, \ldots, n$.

**Definition 5.** Class $\bar{D}$ of functions: let $D = \text{diag}\{D_1, D_2, \ldots, D_p\}$, where $D_i > 0$, $i = 1, 2, \ldots, n$. Then it is said that $d(x) = (d_1(x_1), d_2(x_2), \ldots, d_p(x_p))^T \in \bar{D}$, if the functions $d_i(\cdot)$ is continuous and satisfy
\[
\frac{d_i(\xi) - d_i(\zeta)}{\xi - \zeta} \geq D_i
\]
for $i = 1, 2, \ldots, n$, and any $\xi \neq \zeta$.

**Definition 6.** Class $\bar{G}$ of functions: let $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_p(x_p))^T$. Then it is said that $g(x) \in \bar{G}$ if for all $i = 1, 2, \ldots, n$, $g_i(\cdot)$ is non-decreasing and in every compact set of $R$, every $g_i(\cdot)$ has only finite discontinuous points. Therefore, in any compact set in $R$, except a finite points $\{p_k\}$, where there exist finite right and left limit $g_i(\rho^+)$ and $g_i(\rho^-)$ with $g_i(\rho^+) > g_i(\rho^-)$. $g_i(\cdot)$ is continuous.

Let $x_0 = x_{01}, x_{02}, \ldots, x_{0n} \in R^n$, and $y = x - x_0$. We can rewrite Eq. (1) as follows
\[
\frac{dy}{dt} = \tilde{a}_i(y_i) \left[ -d_i(y_i) + \sum_{j=1}^{n} t_j \tilde{g}_j(y_j) + \tilde{J}_i \right] \quad i = 1, 2, \ldots, n
\]
where
\[
\tilde{a}_i(s) = a_i(s + x_0) \quad \tilde{d}_i(s) = d_i(s + x_0) - d_i(x_0) \quad \tilde{g}_i(s) = g_i(s + x_0) - g_i(x_0) \\
\tilde{J}_i = -d(x_0) + \sum_{j=1}^{n} t_j \tilde{g}_j(x_0) + J_i \quad i = 1, 2, \ldots, n
\]

Therefore, by a suitable translation, in the sequel, for all $i = 1, 2, \ldots, n$ we assume
1. $d_i(0) = 0$;
2. $0 \in K[\tilde{g}_i(0)]$;
3. If $g_i(\cdot)$ is non-trivial, i.e. $g_i(s)$ is not a constant for all $s$, then for each $s_1 > 0$ and $s_2 < 0$
$$g_i(s_1) > 0 \quad \text{and} \quad g_i(s_2) < 0$$

2.2. **Filippov solutions of the model**

Note that the function $g_i(\cdot)$ is undefined at the points, where $g_i(\cdot)$ is discontinuous. Such discontinuous functions include a number of neuron activations of interest for applications. For example, the standard hard comparator function $\text{sign}(\cdot)$:
$$\text{sign}(s) = \begin{cases} 1 & s > 0 \\ -1 & s < 0 \end{cases}$$

If $g(\cdot) \in \bar{G}$, the right-hand side of (1) is discontinuous, it is needed to explain what is meant by a solution of Cauchy problem associated with (1). A possible definition, which we will adopt in this paper, is that of Filippov (1988).

Consider the following system:
\[
\frac{dx}{dt} = f(x)
\] (3)

**Definition 7.** A set-value map defined as
\[
\phi(x) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} K[f(B(x, \delta) - N)]
\] (4)
where $K(E)$ is the closure of the convex hull of set $E$, $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(N)$ is Lebesgue measure of set $N$. A solution of the Cauchy problem of (3) with initial conditions $x(0) = x_0$ is an absolutely continuous function $x(t)$, $t \in [0, T]$, which satisfies: $x(0) = x_0$, and differential inclusion:
\[
\frac{dx}{dt} \in \phi(x) \quad \text{a.e.} \ t \in [0, T]
\] (5)

In view of engineering applications, the concept of the solutions in the sense of Filippov is useful. Because it is a good approximation to solutions of actual systems with very high-gain non-linearities, see Aubin and Cellina (1984), Paden and Sastry (1987), and Utkin (1978).

Now, we use this definition to discuss dynamical behavior of the model (1).

Let
\[
K[g(x)] = (K[g_1(x_1)], K[g_2(x_2)], \ldots, K[g_p(x_p))]^T
\]
where
\[
K[g_i(x_i)] = [g_i(x_i^-), g_i(x_i^+)]
\]
and
\[
\phi(x) = A(x)[-d(x) + TK[g(x)] + J]
\]

We define the Filippov solution of (1) as follows:

**Definition 8.** A solution (in the sense of Filippov) of Cauchy problem with initial condition $x(0) = x_0$ of the model (1) is an absolutely continuous function $x(t)$ on $t \in [0, T]$ such that $x(0) = x_0$, and
\[
\frac{dx}{dt} \in A(x)[-d(x) + TK[g(x)] + J] \quad \text{a.e.} \ t \in [0, T]
\] (6)

**Definition 9 (Equilibrium)** $x^*$ is said to be an equilibrium of a set-value map $F(x)$ if
\[
0 \in F(x^*)
\] (7)
perticularly, $x^*$ is said to be an equilibrium of system (6) if there exists $\gamma^* \in K[g(x^*)]$ such that
\[
0 = A(x^*)[-d(x^*) + T\gamma^* + J]
\]
Definition 10. If $x^*$ is an equilibrium of system (3), $x^*$ is said to be globally asymptotically stable, if for any solution $x(t)$ of (3), whose existence interval is $[0, + \infty)$, we have
\[
\lim_{t \to \infty} x(t) = x^*
\]
Moreover, $x(t)$ is said to be globally exponentially asymptotically stable if there exists $\epsilon > 0$ and $M > 0$, such that
\[
\|x(t) - x^*\| \leq M e^{-\epsilon t}
\]
Remark 1. Concerning solution of ordinary differential equation with discontinuous right-hand, there are various definitions, such as generalized sampling solution, Euler solution, etc. Among them, Carathéodory solution is widely used, which is an absolutely continuous function $x(t)$ satisfying:
\[
\frac{dx}{dt} = f(x) \quad \text{for all almost} \; t
\]
\[x(0) = x_0\] (8)
It is easy to see that if $f(x) \in \phi(x)$ for all $x$, then any Carathéodory solution in the sense of (8) is also a Filippov solution in the sense of (5). Instead, for the following system
\[
\dot{x} = f(x) = \begin{cases} 
1 & x \neq 0 \\
0 & x = 0 
\end{cases} \quad x(0) = 0 \] (9)
The Carathéodory solutions of (9) are $x(t) = t$ and $x(t) = 0$. But the Filippov solution of (9) is only $x(t) = t$.
Furthermore, Spraker and Biles (1996) discussed the relation between the Filippov solution set and Carathéodory solution set. They obtained that for one-dimensional case, the two solution sets are equivalent if and only if the two sets $\{x: f(x) = 0\}$ and $\{x: 0 \in \phi(x)\}$ are equivalent. For instance
\[
\dot{x} = \text{sign}(x) = \begin{cases} 
1 & x > 0 \\
0 & x = 0 \\
-1 & x < 0 
\end{cases} \quad x(0) = 0 \] (10)
where the Filippov solutions satisfy
\[
\dot{x} \in \phi(x) = \begin{cases} 
1 & x > 0 \\
I & x = 0 \\
-1 & x < 0 
\end{cases} \quad x(0) = 0
\]
where $I = [-1, 1]$.
The Carathéodory solutions of (10) are $x(t) = 0$, $x(t) = |t|$, $x(t) = -|t|$, which are same with the Filippov solutions, because of $\{x: \text{sign}(x) = 0\} = \{x: 0 \in \phi(x)\}$. However, if the two sets $\{x: f(x) = 0\}$ and $\{x: 0 \in \phi(x)\}$ are not equivalent, Carathéodory solutions and Filippov solutions are not equivalent. Consider the following system
\[
\dot{x} = \psi(x) = \begin{cases} 
1 & x > 0 \\
0 & x = 0 \\
-1 & x < 0 
\end{cases} \quad x(0) = 0
\]
(11)
In this case, Carathéodory solutions are $x(t) = \pm |t|$. But Filippov solutions are $x(t) = 0$, $x(t) = |t|$, $x(t) = -|t|$, which are same with those of (10). They are not equivalent.
In the study of dynamical system (6), the following problems should be answered:
1. Is there a solution of (6) for any Cauchy problem on $t \in [0, + \infty)$, which is said to be viable in the terms of differential inclusion?
2. Does there exist an equilibrium point for the system (6)? Firstly, what is the meaning of equilibrium for the differential inclusion?
3. If there exists an equilibrium, look for conditions for the equilibrium being stable? Firstly, what is the meaning of stability of the system?
4. Uniqueness of the equilibrium.

We will address these issues in this paper. The paper is organized as follows. In Section 3, we will discuss the existence of the equilibrium point of the system (6). We will investigate the viability of the model (6) in Section 4. The stability and exponential stability of the Cohen–Grossberg networks are studied in Sections 5 and 6. Some numerical examples are presented in Section 7. We conclude this paper in Section 8.

3. Existence of equilibrium

In this section, we always make following assumption
\[
A(x) \in \tilde{A}, \quad d(x) \in \tilde{D}, \quad g(x) \in \tilde{G} \tag{12}
\]
We will prove existence of equilibrium for the system (1) by Equilibrium Theorem. Firstly, we give some necessary definitions concerning equilibrium and equilibrium theorem to set-value maps. All these materials can be found in Aubin and Frankowska (1990).

Definition 11. $K$ is a convex subset of $R^n$. The tangent cone $T_k(x)$ to $K$ at $x \in K$ is defined as
\[
T_k(x) = \bigcup_{h > 0 \atop k > 0} \frac{k - x}{h} \tag{13}
\]
where $\bigcup$ is the closure of the union set.

Proposition 1. The necessary and sufficient condition for $v \in T_k(x)$ is that there exist $h_n \to 0^+$, and $v_n \to v$, as $n \to + \infty$, such that
\[
x + h_nv_n \in K
\]
hold for all $n$. Moreover, if $x \in \text{Int}(K)$, where $\text{Int}(K)$ is the set of the inner points of $K$, then $T_k(x) = R^n$.
Definition 12 (Viability Domain) Let $F$: $X \rightarrow X$ be a non-trivial set-value map. We shall say that a subset $K \subset \text{Dom}(F)$ is a viability domain of $F$, if for all $x \in K$, we have $F(x) \cap T_K(x) \neq \emptyset$ (14)

where $\text{Dom}(F)$ is the domain of $F$.

The following theorem is used below.

Theorem 1 (Equilibrium Theorem) (see Aubin & Frankowska, 1990, p. 84) Assume that $X$ is a Banach Space and $F: X \rightarrow X$ is an upper semicontinuous set-value map with closed convex image. If $K \subset X$ is a convex compact viability domain of $F(x)$, then $K$ contains an equilibrium $x^*$ of $F(x)$, i.e.

$$0 \in F(x^*)$$

Now, we use Equilibrium Theorem to prove existence of the equilibrium for the system (1).

Lemma 1. Suppose that the assumption (12) is satisfied, and each $g_i(\cdot)$ is non-trivial, $P_i > 0$, $i = 1, 2, \ldots, n$. Define

$$\bar{V}(x) = \sum_{i=1}^{n} P_i \int_0^1 g_i(\rho) d\rho$$ (15)

For any $M > 0$, denote $Q_M = \{x : \bar{V}(x) \leq M\}$, and

$$\partial Q_M = \{x : \bar{V}(x) = M\},$$

and

$$K_1 = \left\{ v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n : \sum_{i=1}^{n} v_i P_i \gamma_i \leq 0, \text{ for all } \gamma_i \in K[g(x_i)] \right\}$$

Then $K_1 \subset T_{Q_M}(x)$ whenever $x \in \partial Q_M$.

Proof. For each $x \in \partial Q_M$, i.e. $\bar{V}(x) = M$, and $v \in \text{int}(K_1)$ satisfying

$$\sum_{i=1}^{n} v_i P_i \gamma_i < 0 \text{ for all } \gamma_i \in K[g(x_i)]$$

Denote

$$y_n = x + h_n v$$

where $0 < h_n \rightarrow 0$, as $n \rightarrow +\infty$. We will prove that $\bar{V}(y_n) \leq M$, namely, $y_n \in Q_M$.

Denote

$$\gamma_i^c = \begin{cases} g_i(x_i^+) & \text{if } v_i > 0 \\ g_i(x_i^-) & \text{if } v_i < 0 \\ \text{any value} & \text{if } v_i = 0 \end{cases}$$

Then we have

$$\sum_{i=1}^{n} v_i P_i \gamma_i^c \leq \sum_{i=1}^{n} v_i P_i \gamma_i^c \text{ for all } \gamma_i \in K[g(x_i)]$$

Thus, pick

$$\epsilon = -\sum_{i=1}^{n} v_i P_i \gamma_i^c > 0$$

we have

$$\bar{V}(y_n) - \bar{V}(x) = \sum_{i=1}^{n} P_i \int_{y_i}^{y_i + h_n v_i} g_i(\rho) d\rho = \sum_{i=1}^{n} P_i \int_{y_i}^{y_i + h_n v_i} g_i(\rho) d\rho = \left( \sum_{i=1}^{n} v_i P_i \gamma_i^c \right) h_n + o(h_n) = -\epsilon h_n + o(h_n)$$

If $n$ is large enough, we have $\bar{V}(y_n) < \bar{V}(x) = M$, which implies $v \in T_{Q_M}(x)$, i.e. $\text{int}(K_1) \subset T_{Q_M}(x)$. Since $T_{Q_M}(x)$ is closed, $K_1 \subset T_{Q_M}(x)$. Lemma 1 is proved. \qed

Lemma 2 (Ky Fan Inequality) (see Aubin & Frankowska, 1990) Let $K$ be a compact convex subset in a Banach Space $X$ and $\phi$: $X \times X \rightarrow \mathbb{R}$ be a function satisfying:

1. For all $y \in K$, $x \mapsto \phi(x, y)$ is lower semicontinuous;
2. For all $x \in K$, $y \mapsto \phi(x, y)$ is concave, i.e. for all $\lambda_i > 0$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$, and $y_i \in K$

$$\phi \left( x, \sum_{i=1}^{n} \lambda_i y_i \right) \geq \sum_{i=1}^{n} \lambda_i \phi(x, y_i)$$

3. For all $y \in K$, $\phi(y, y) \leq 0$.

Then there exists $\bar{x} \in K$, such that, for all $y \in K$, $\phi(\bar{x}, y) \leq 0$.

Theorem 2. Suppose the assumption given in (12) is satisfied, and $-T \in \text{LDS}$. Then there exists an equilibrium $x^*$ of system (1), i.e.

$$0 \in F(x^*)$$

where

$$F(x^*) = [-d(x^*) + TK[g(x^*)] + J].$$

Proof Because $-T \in \text{LDS}$, there exists a diagonal matrix $P = \text{diag}(P_1, P_2, \ldots, P_n)$, with $P_i > 0$, $i = 1, 2, \ldots, n$, such that $\lambda(PT + T^TP) < 0$. Let

$$\bar{V}(x) = \sum_{i=1}^{n} P_i \int_0^1 g_i(\rho) d\rho$$

There are two possible cases:

Case 1. All $g_i(\cdot)$, $i = 1, 2, \ldots, n$ are non-trivial.
Case 2. There exist some indices $i$ such that $g_i(x) = 0$, for $x \in R$.

We prove theorem for two cases separately.

Case 1. It is easy to see that $Q_M$, defined in Lemma 1, is a convex compact subset of $\mathbb{R}^n$. 
Let\( \alpha = \min i(\{ PT\}) > 0, \ I = \sum_{j=1}^{n}(1/\alpha) P_{j}^{2}/f_{j}^{2}, \ l = \min \{ D_{j}\} \) and \( M_{0}=(ll)\). We claim that if \( M>M_{0}\), then \( \Omega_{M}\) is a viability domain of \( F(x)\).

In fact, if \( x \in \text{Int}(\Omega_{M})\), then \( T_{\Omega_{M}}(x) = R\) and \( F(x) \cap T_{\Omega_{M}}(x) = \emptyset\).

Now, we will prove that if \( x \in \partial \Omega_{M}\), then \( F(x) \cap T_{\Omega_{M}}(x) \neq \emptyset\). For this purpose, define \( \phi(g_{1}, g_{2}) : K[g(x)] \times K[g(x)] \rightarrow R\) as follows

\[
\phi(g_{1}, g_{2}) = \sum_{i=1}^{n} g_{1,i} f_{i} \left[ -d_{i}(x_{i}) + \sum_{j=1}^{n} t_{ij} g_{2,j} + J_{i} \right]
\]

where \( g_{1} = (g_{1,1}, g_{1,2}, \ldots, g_{1,n})^{T}\), and \( g_{2} = (g_{2,1}, g_{2,2}, \ldots, g_{2,n})^{T}\).

If we can find \( g_{2} \in K[g(x)]\), such that \( \phi(g_{1}, g_{2}) \leq 0 \)
holds for all \( g_{1} \in K[g(x)]\). Then by Lemma 1, we have

\( F(x) \cap T_{\Omega_{M}}(x) \neq \emptyset\)

It can be shown that for each \( g_{1} \in K[g(x)]\), \( g_{2} \mapsto \phi(g_{1}, g_{2}) \) is continuous; for each \( g_{2} \in K[g(x)]\), \( g_{1} \mapsto \phi(g_{1}, g_{2}) \) is concave; Moreover, let \( \lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})^{T}\), where \( \lambda_{i} \in K[g(x)]\). Then it is easy to see that

\[
f_{i} g_{i} \geq \int_{0}^{T} g_{i}(p) dp
\]

which implies

\[
\phi(f, \lambda) = -\sum_{i=1}^{n} f_{i} P_{i} d_{i}(x_{i}) x_{i} + f^{T} P T f + f^{T} P J
\]

\[
\leq -l f^{T} P x - \alpha f^{T} f + f^{T} P J
\]

\[
= -l f^{T} P x - \alpha f^{T} f + \sqrt{f^{T} f (P J)^{T} P J}
\]

\[
\leq -l f^{T} P x - \alpha f^{T} f + \alpha f^{T} f/2 + (P J)^{T} P J/2\alpha
\]

\[
\leq -l f^{T} P x - \frac{\alpha}{2} f^{T} f + l \leq -l M + l \leq 0
\]

By Lemma 2 (Ky Fan Inequality), we can find \( \bar{g} \in K[g(x)]\) such that

\( \phi(\bar{g}, \bar{g}) \leq 0 \)

holds for all \( g \in K[g(x)]\). Therefore, for each \( x \in \Omega_{M}\), we have

\( F(x) \cap T_{\Omega_{M}}(x) \neq \emptyset\)

By Theorem 1, \( \Omega_{M}\) contains an equilibrium of \( F(x)\).

Case 2. Without loss of generalization, we assume that \( g_{s}(x) = 0\), for all \( s \in R \) and \( g_{1}, g_{2}, \ldots, g_{n-1} \) are non-trivial. Considering \( \bar{x} = (x_{1}, x_{2}, \ldots, x_{n-1})^{T}\), the discussion of the case 1, there exists an equilibrium \( \bar{x} = (x_{1}^{*}, x_{2}^{*}, \ldots, x_{n-1}^{*})^{T}\), such that

\[
0 = -d_{i}(x_{i}^{*}) + \sum_{j=1}^{n} t_{ij} K[g_{j}(x_{j})] + J_{i} \quad i = 1, 2, \ldots, n-1
\]

i.e. there exist \( \gamma_{i} \in K[g_{j}(x_{j})] \), for \( i = 1, 2, \ldots, n-1 \), such that

\[
0 = -d_{i}(x_{i}^{*}) + \sum_{j=1}^{n} t_{ij} \gamma_{j} + J_{i} \quad i = 1, 2, \ldots, n-1
\]

It can also be seen that there exists \( x_{n}^{*} \) such that

\[
-d_{n}(x_{n}^{*}) + \sum_{j=1}^{n} t_{nj} \gamma_{j} + J_{n} = 0
\]

Therefore, \( x^{*} = (x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*})^{T}\) is an equilibrium of \( F(x)\).

Theorem 2 is proved.

Let \( x^{*} = (x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*})^{T}\) be an equilibrium point of the system (1), i.e. there exist \( \gamma = (\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n})^{T} \in K[g(x)] \) such that

\[
-d_{i}(x_{i}^{*}) + \sum_{j=1}^{n} t_{ij} \gamma_{j} + J_{i} = 0 \quad \text{for all } i = 1, 2, \ldots, n
\]

and \( u(t) = x(t) - x^{*} \) be a translation of \( x(t)\). Then \( u(t) = (u_{1}(t), u_{2}(t), \ldots, u_{n}(t))^{T} \) satisfies

\[
\frac{du_{i}(t)}{dt} \in a_{i}(u_{i}(t) + x_{i}^{*}) \left[ -d_{i}(u_{i}(t)) + \sum_{j=1}^{n} t_{ij} K[g_{j}(u_{j}(t))] \right]
\]

\( i = 1, 2, \ldots, n \)

where \( d_{i}^{*}(s) = d_{i}(s + x_{i}^{*}) - d_{i}(x_{i}^{*}), \quad g_{j}^{*}(s) = g_{j}(s + x_{i}^{*}) - \gamma_{j}, \quad i = 1, 2, \ldots, n \). To simplify, instead of \( g_{j}^{*}(s), d_{i}^{*}(s) \), we still use \( g_{j}(s), d_{i}(s) \). Therefore, the above equations can be rewritten as:

\[
\frac{du_{i}(t)}{dt} \in a_{i}(u_{i}(t) + x_{i}^{*}) \left[ -d_{i}(u_{i}(t)) + \sum_{j=1}^{n} t_{ij} K[g_{j}(u_{j}(t))] \right]
\]

\( i = 1, 2, \ldots, n \) \quad (16)

4. Boundedness and viability

In this section, we investigate the viability of the system (16) and its dynamical behavior. Firstly, we need following Viability Theorem.

Theorem 3. (Viability Theorem) (Aubin, 1991, p. 91) Consider a non-trivial upper semicontinuous set-value map \( F(x) \) with compact, convex image from Banach Space \( X \) to \( X \), and a closed subset \( K \subset \text{Dom}(F) \). If \( K \) is a viability domain, then for any initial state \( x_{0} \in K \), there exist a positive \( L \) (can be \( +\infty \)) and an absolutely continuous solution \( x(t) \), \( t \in [0, L) \), satisfying \( x(0) = x_{0}, \dot{x}(t) \in F(x) \), for a.e. \( t \in [0, L) \), viable in \( K \), and

\[
\begin{cases}
\text{either } & L = +\infty \\
& \text{or } L < +\infty \quad \lim \sup_{t \to L-} ||x(t)|| = \infty
\end{cases}
\]

By Theorem 3, if one can prove the solution \( x(t) \) is bounded on its existence interval, then its existence interval
is \([0, +\infty)\). Therefore, in the following, we will prove the solution of the system (16) is bounded. Before, that we need following chain rule.

**Definition 13** (Clarke Regular, see Clarke, 1983) V(x): \(R^n \rightarrow R\) is said to be C-regular, if for each \(x \in R^n\), and \(v \in R^n\)

1. There exists the usual right directional derivative:

\[
D^+ V(x, v) = \lim_{h \to 0^+} \frac{V(x + hv) - V(x)}{h}
\]

2. Define

\[
\tilde{D}_c V(x, v) = \limsup_{h \to 0^+, x \to v} \frac{V(y + hv) - V(y)}{h}
\]

Then

\[
\tilde{D}_c V(x, v) = D^+ V(x, v)
\]

**Definition 14** (Clarke generalized gradient. See Clarke, 1983) Clarke generalized gradient \(\partial_c V(x)\) is defined as follows

\[
\partial_c V(x) = \{p \in R^N : D^+_c V(x, v) \leq p^T v \leq \tilde{D}_c V(x, v), \text{ for all } v \in R^N\}
\]

where \(\tilde{D}_c V(x, v)\) is defined as in Definition 13 and

\[
D^+_c V(x, v) = \liminf_{h \to 0^+, x \to v} \frac{V(y + hv) - V(y)}{h}
\]

**Theorem 4** (Chain Rule, see Bacciotti, Conti, & Marcellini, 2000, chap. 2.3) If V(x): \(R^n \rightarrow R\) is C-regular, and \(\phi(t)\) is absolutely continuous. Then for any measurable function \(\gamma(t) \in \partial_c V(\phi(t))\), we have

\[
\frac{d}{dt} V(\phi(t)) = \gamma^T(t) \phi(t) \quad \text{for a.e. } t
\]

**Theorem 5** (Boundedness Theorem) Suppose the assumption given in (12) is satisfied, \(-T \in LDS\). Then for any \(u_0 \in R^n\), system (16) has a bounded absolutely continuous solution \(u(t)\) for \(t \in [0, +\infty)\), and satisfying \(u(0) = u_0\).

**Proof.** By Theorem 3 (Viability Theorem), there exists an absolutely continuous solution \(u(t)\) of system (16) on \([0, L]\), and if we can prove that \(u(t)\) is bounded, then \(L = +\infty\).

Because \(-T \in LDS\), there exists a positive definite diagonal matrix \(P = \text{diag}(P_1, P_2, \ldots, P_n)\) with \(P_i > 0\), \(i = 1, 2, \ldots, n\), such that \(\lambda(PT + T^TP) < 0\).

Define a Lyapunov function as

\[
V_k(u) = \sum_{i=1}^{n} \int_{0}^{u_i} \frac{\rho \, d\rho}{a_i(\rho + x_i^*)} + k \sum_{i=1}^{n} \int_{0}^{u_i} \frac{g_i(\rho) \, d\rho}{a_i(\rho + x_i^*)} \quad (17)
\]

It can be seen that \(V(u, k)\) is C-regular. Let

\[
\gamma(t) = T^{-1} [d(u(t)) + A(u + x^*)^{-1} \dot{u}(t)]
\]

Then \(\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))^T \in K[g(u(t))]\) is measurable and

\[
\dot{u}(t) = A(u + x^*)[-d(u) + T\gamma(t)]
\]

holds almost everywhere.

By Theorem 4 (Chain Rule), we have

\[
\frac{d}{dt} V_k(u(t)) = \sum_{i=1}^{n} u_i \left[-d_i(u_i) + \sum_{j=1}^{n} P_{ij} \gamma_j(t) \right]
\]

\[
+ k \sum_{i=1}^{n} P_{ii} \gamma_i(t) \left[-d_i(u_i) + \sum_{j=1}^{n} P_{ij} \gamma_j(t) \right]
\]

\[
\leq -lu^T u + u^T T\gamma(t) - k\alpha \gamma(t)^T \gamma(t)
\]

\[
- k\gamma(t)^T P d(u) - lu^T u + u^T T\gamma(t)
\]

\[
- \frac{1}{4l} \gamma(t)^T T^T T\gamma(t) + \frac{1}{4l} \gamma(t)^T T^T T\gamma(t)
\]

\[
- k\alpha \gamma(t)^T \gamma(t) \leq -\left( \sqrt{lu + \frac{1}{2l}} T\gamma(t) \right)^T
\]

\[
\times \left( \sqrt{lu + \frac{1}{2l}} T\gamma(t) \right) + \frac{1}{4l} \gamma(t)^T T^T T\gamma(t)
\]

\[
- k\alpha \gamma(t)^T \gamma(t)
\]

where

\[
l = \min_i D_i, \quad \alpha = \min \lambda(|TP|^T)
\]

If \(k \geq (|T|)^2/4\alpha\), then

\[
\frac{d}{dt} V_k(u(t)) \leq 0
\]

holds for almost all \(t \in [0, T]\). Therefore,

\[
\sum_{i=1}^{n} \int_{0}^{u_i} \frac{\rho \, d\rho}{a_i(\rho + x_i^*)} \leq V_k(x(0)) < +\infty
\]

Since \(A(\cdot) \in \hat{A}\), by the inequalities (2), \(u(t)\) is bounded. Theorem 5 is a consequence of the Viability Theorem. □

**Remark 2.** We have investigated the viability of the system (16). It is natural to raise the following question: Is the solution unique? Because Filippov solution includes set-valued function, it is hard to assure the uniqueness and there are few papers concerning this issue. Here we present a simple discussion of the uniqueness of Filippov solution. According to Viability Theorem (Theorem 3), the key is whether \(K\) is viable, i.e. \(F(x) \cap T_K(x) \neq \emptyset\), for all \(x \in K\). We explain it as follows: the map \(F(x)\) defines the expected direction of the trajectory \(x(t)\) and \(T_K(x)\) is composed of all directions which assures that the trajectory stays in \(K\). Of course, \(F(x) \cap T_K(x) \neq \emptyset\) assures the trajectory \(x(t)\) stays in \(K\) for at least an interval of time. It can be seen that if the direction of \(F(x) \cap T_K(x)\) is composed of a single direction, then the Filippov solution of Cauchy problem of (5) has a unique solution.

Consider the following one-dimensional system

\[
\dot{x} = 3 + \text{sign}(x), \quad x(0) = 0
\]

In this case, \(K = R\), the direction
of corresponding set-valued function $F(x)$ is a singlet, i.e. \{\nu: \nu > 0\}. Therefore, the system has a unique solution $x(t) = 4t$, for $t \geq 0$.

Instead, for the system, $\dot{x} = \text{sign}(x)$, $x(0) = 0$. When $x > 0$, $F(0) \cap T_K(0)$ has two directions. Hence, the system has solutions $x(t) = \pm |t|$ and $x(t) = 0$.

Moreover, though the direction set of $F(x) \cap T_K(x)$ is not a singlet, but at the equilibrium of $F(x)$, the trajectory has unique direction to go. Namely the direction set \{v : \lim_{h \to +} F(x + hv) = v, v \in F(x) \cap T_K(x)\} is a singlet direction. Thus, the solution is also unique. For example, consider $\dot{x} = -\text{sign}(x)$, $x(0) = 0$. Letting $K = R$, at $x = 0$, the direction set of $F(0) \cap T_K(0)$ is not a singlet. But from the results given in Spraker and Biles (1996), one can see that the solution $x(t)$ must be monotone, i.e. either non-decreasing or non-increasing. If $x(t)$ is strictly increasing then $x(t) > 0$ for $t > 0$. It implies that $\dot{x} = -1$ and $x(t)$ is strictly decreasing, a contradiction. Instead, if $x(t)$ is strictly decreasing, then $x(t) < 0$ for $t > 0$. It implies that $\dot{x} = 1$ and $x(t)$ is strictly increasing. Therefore, the equation has a unique solution: $x(t) = 0$.

In this paper, we will not discuss the uniqueness of the solution in detail. Even though the solution is not unique, we still prove the global stability in spite of multi-trajectories with same initial condition. In other words, no matter what initial data are and which solution is, all trajectories converge to the unique equilibrium.

5. Global asymptotic stability

In this section, we address global asymptotical stability. Before doing so, let us review the following basic lemma, whose details can be found in Miller and Michel (1982).

Consider the autonomous systems of ODEs:

$$\frac{dx}{dt} = f(x)$$ (18)

**Definition 15.** $w(x): R^n \to R$ is said to be radially unbounded if the following conditions are satisfied:

1. $w(0) = 0$
2. $w(x) > 0$ for $x \in R^n - \{0\}$
3. $w(x) \to \infty$ as $|x| \to \infty$

where $0 = (0, 0, \ldots, 0)^T$

**Lemma 3 (Theorem 11.14 of Miller & Michel, 1982)** Assume that there exists a continuous, differentiable positive definite and radially unbounded function $L: R^n \to R$, such that:

1. $(dL(x)/dt) \leq 0$, for all $x \in R^n$
2. The origin is the only invariant subset of the set $E = \{x \in R^n : (dL(x)/dt) = 0\}$

Then the equilibrium $x = 0$ of Eq. (18) is asymptotically stable for $R^n$.

**Theorem 6 (Global Asymptotic Stability)** Suppose that the assumption given in (12) is satisfied, $-T \in L^D$. Then the system (1) is globally asymptotically stable, i.e. there exists a unique equilibrium $x^* \in R^n$, such that for any solution $x(t)$ on $[0, + \infty)$, we have

$$\lim_{t \to \infty} x(t) = x^*$$

**Proof.** We need only to prove that any solution $u(t)$ of system (16) satisfies

$$\lim_{t \to \infty} u(t) = 0$$

Let $V_2(u)$ be defined by (17). It is easy to see that $V_2(u)$ is continuous, positive definite, and radially unbounded. On the other hand, if $0 < \epsilon < 1$ and $k \geq \|T\|^2/4a(1 - \epsilon)$, we have

$$\frac{d}{dt} V_2(u(t)) = \sum_{i=1}^{n} u_i \left[ -d_i(u_i) + \sum_{j=1}^{n} t_{ij} \gamma_j(t) \right] + k \sum_{i=1}^{n} P_i \gamma_i(t) \left[ -d_i(u_i) + \sum_{j=1}^{n} t_{ij} \gamma_j(t) \right]$$

$$\leq -\epsilon u^T \gamma(t) - (1 - \epsilon) u^T \gamma(t) + u^T T \gamma(t)$$

$$\leq -\epsilon u^T \gamma(t) + (1 - \epsilon) u^T \gamma(t) - k \gamma^T \gamma(t)$$

$$\leq -\epsilon u^T \gamma(t) + \left( 1 - \epsilon \right) u^T \gamma(t)$$

Therefore, $(d/dt)V_2(u(t))$ is negative definite. By Lemma 3, one can see that

$$\lim_{t \to \infty} u(t) = 0$$

Uniqueness is a direct consequence of the proof. Theorem is proved. □

**Remark 3.** The criterion for the stability given here is independent of the bound of $A(u)$. This is in contrast with the result given by Wang and Zou (2002), where the upper bound of $A(u)$ is a prerequisite condition. Here, we only assume that $A(x) \in A$.

**Remark 4.** In the paper Lu and Chen (2003), the authors discussed a class of Cohen–Grossberg neural networks. Under assumptions that $A(\cdot) \in A$, $d(\cdot) \in D$, and $g_k(\cdot)$,
$i = 1, 2, \ldots, n$, satisfy

$$0 \leq \frac{g_i(\xi) - g_i(\zeta)}{\xi - \zeta} \leq G_i \quad G_i > 0 \quad i = 1, 2, \ldots, n \quad \xi \neq \zeta$$

If $DG^{-1} - T \in L^2$, then the Cohen–Grossberg neural network is globally asymptotically stable. It can be seen that if $G_i \to \infty$, which implies $g_i(\cdot)$ are high-slope activation functions. Then $DG^{-1} - T \in L^2$ becomes $-T \in L^2$. The relation between the results here and results given in Lu and Chen (2003) indicates that the case of non-continuous activation functions is a limit case of high-slope activation functions.

By Gershgorin’s theorem (Horn & Johnson, 1985), we have

**Corollary 1.** Suppose that $A(\cdot) \in \tilde{A}$, $g(\cdot) \in \tilde{G}$, and $d(\cdot) \in \tilde{D}$. If there exist constant $\xi_i < 0$, $i = 1, 2, \ldots, n$, such that

$$-t_i \xi_i - \sum_{j \neq i} \left| \frac{\xi_i j_i + \xi_j j_i}{2} \right| > 0 \quad i = 1, 2, \ldots, n$$

Then there exists a unique equilibrium point $x^* \in R^n$ such that

$$\lim_{t \to \infty} \|x(t) - x^*\| = 0$$

### 6. Further result of exponential stability analysis

In this section, firstly, we investigate the exponent stability of system (1), as $A(x) \in \tilde{A}_1$. Then we establish a condition ensuring that the neural network (1) is globally convergent in finite time.

**Theorem 7 (Exponential Stability)** Suppose that the assumption given in (12) is satisfied, $-T \in L^2$, and $A(x) \in \tilde{A}_1$. Then the system (1) is exponentially asymptotically stable with rate $lq/2$, where $l = \min D_i$, i.e. there exists $M > 0$ such that

$$\|x(t) - x^*\| \leq M e^{-lq/2}$$

where $x^*$ is the unique equilibrium.

**Proof.** We need to prove that for any solution $u(t)$ of the system (16), there exists $M > 0$ such that

$$\|u(t)\| \leq M e^{-lq/2}$$

Define another Lyapunov Function

$$V_{1,k}(u) = u^T u + k \sum_{i=1}^n \int_0^{u_i(t)} \frac{g_i(\rho) d\rho}{a_i(\rho + x_i^*)}$$

where $P = \text{diag}\{P_1, P_2, \ldots, P_n\}$, such that $\{-PT\}$ is positive definite, $\alpha = \lambda\min\{\{-PT\}\}$, is the smallest eigenvalue of $\{-PT\}$.

From Theorem 6, one can see that $\lim_{t \to \infty} u(t) = 0$. So there must exist a positive $\bar{a} > 0$ and $t_1 > 0$, such that

$$\alpha \leq a_i(u(t) + x^*) \leq \bar{a} \quad \text{for} \; t > t_1$$

Differentiating $V_{1,k}(u)$: for a.e. $t \in [t_1, \infty)$ we have

$$\frac{d}{dt} V_{1,k}(u(t)) = -2u^T A(u + x^*) du(t) + 2u^T A(u + x^*) T^T \gamma(t)$$

$$-k\gamma(T^T P d(u(t)) - k\alpha \gamma(T^T \gamma(t))$$

$$\leq -a_i u^T u - k\gamma(T^T P d(u(t))$$

$$+ 2u^T A(u + x^*) T^T \gamma(t) - \frac{1}{a_i} \gamma(T^T T^T A(u + x^*)^2 T \gamma(t)$$

$$\leq -a_i u^T u - k\gamma(T^T P d(u(t))$$

Since

$$\gamma(t) P_i d_i(u_i(t)) \geq l\gamma(t) P_i u_i(t) \geq l q P_i \int_0^{u_i(t)} \frac{g_i(\rho) d\rho}{a_i(\rho + x_i^*)}$$

Therefore, if $k \geq (\alpha^2 \|T\|_{\text{max}}^2) / 2$, we have

$$\frac{d}{dt} V_{1,k}(u(t)) \leq -a_i u^T u - a_i k \sum_{i=1}^n P_i \int_0^{u_i(t)} \frac{g_i(\rho) d\rho}{a_i(\rho + x_i^*)}$$

$$\leq -a_i V_{1,k}(u(t)) \quad \text{for a.e.} \; t \in [0, \infty)$$

Thus

$$V_{1,k}(u) = O(e^{-a_i t})$$

and $\|u(t)\| = O(e^{-a_i t})$.

Theorem is proved. \Box

By Gershgorin’s theorem (Horn & Johnson, 1985), we have

**Corollary 2.** Suppose $A(\cdot) \in \tilde{A}_1$, $g(\cdot) \in \tilde{G}$, and $d(\cdot) \in \tilde{D}$. If there exist constant $\xi_i > 0$, $i = 1, 2, \ldots, n$, such that

$$-t_i \xi_i - \sum_{j \neq i} \left| \frac{\xi_i j_i + \xi_j j_i}{2} \right| > 0 \quad i = 1, 2, \ldots, n$$
Then there exist an unique equilibrium point \( x^* \in \mathbb{R}^n \), such that
\[
\|x(t) - x^*\| = O(e^{-\eta t^2})
\]

**Theorem 8.** Suppose (12) is satisfied, \(-T \in \text{LDS}\). Furthermore, suppose that every \( g_i(x_i), i=1, 2, \ldots, n \), is discontinuous at equilibrium point \( x^* = (x_1^*, \ldots, x_n^*) \) of (1). If \( \gamma^* = T^{-1} \) \( \{D(x^*) - J\} \) is an internal point of \( K\{g(x^*)\} \), i.e., \( g_i(x_i^*) < \gamma_i^* < g_i(x_i^+) \). Then any solution of (1) converges to \( x^* \) in finite time. To be more precise, if
\[
t \geq t^* = \frac{1}{N_0 \delta^2} \sum_{i=1}^{n} \int_{0}^{|x_i^*-x_i^+|} \frac{g_i(p)}{a_i(p + x_i^*)} \, dp
\]
where \( x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T \) is the initial value, then
\[
x(t) = x^*
\]
where
\[
\alpha = \min \lambda \{ -PT \}^s
\]
and
\[
\delta = \min_{i} \left| \left| \gamma_i^* - g_i(x_i^*) \right| \right|, \left| \left| \gamma_i^* - g_i(x_i^+) \right| \right|
\]

**Proof.** Let
\[
0 = (0, 0, \ldots, 0)^T
\]
From (19) in the proof of Theorem 6, it can be seen that
\[
\frac{d}{dt} V_k(u(t)) \leq \frac{1}{4(l - \epsilon)} \gamma^T(t) T \gamma(t) - k \alpha \gamma^T(t) \gamma(t)
\]
Thus, if \( k \geq (\|T\|_2^2)/4(l - \epsilon) \), \( V(u(t), k) \) is decreasing. Moreover, \( g_i(x_i^*) < \gamma_i^* < g_i(x_i^+) \) implies \( \delta > 0 \), and if \( u(t) \neq 0 \), we have
\[
\gamma^T(t) \gamma(t) \geq n \delta^2
\]
Therefore
\[
\frac{d}{dt} V_k(u(t)) \leq n \left[ -k \alpha + \|T\|_2^2 \right] \delta^2 \text{ for almost}
\]
\( t : u(t) \neq 0 \)
In summary, if \( k \geq (\|T\|_2^2)/4 \alpha (l - \epsilon) \) and
\[
t \geq t^*(k) = \frac{V_k(u(0))}{n \left[ k \alpha - \frac{\|T\|_2^2}{4(l - \epsilon)} \right] \delta^2}
\]
we have \( V_k(u(t)) = 0 \), which means \( x(t) = x^* \). Because of \( t^* = \inf \left\{ t^*(k), k \geq (\|T\|_2^2)/4 \alpha (l - \epsilon) \right\} \), the theorem is proved.

**Remark 5.** Theorems 7 and 8 address the convergence rate of the neural networks (1): Theorem 7 gives the exponential convergence rate of the system; Theorem 8 shows that if the equilibrium point of the system satisfies certain condition, then the system can be asymptotically stable in finite time.

**7. Numerical examples**

In this section, we give several numerical examples to verify the theorems given in this paper and show how the trajectories converge to the equilibrium points.

**7.1. Example 1**

Let us consider the following system
\[
\begin{align*}
\dot{x}_1 &= \frac{1}{|x_1| + 1} \left[ -x_1 - \frac{1}{4} (\text{sign}(x_1) + x_1) + 2 (\text{sign}(x_2) + x_2) \right] \\
\dot{x}_2 &= \frac{1}{|x_2| + 1} \left[ -x_2 - 10 (\text{sign}(x_1) + x_1) - \frac{1}{4} (\text{sign}(x_2) + x_2) \right]
\end{align*}
\]
(19)
where
\[
a_i(s) = \frac{1}{|s| + 1}, \quad d_i(s) = -s
\]
\[
g_i(s) = \text{sign}(s) + s, \quad T = \begin{bmatrix} -\frac{1}{4} & 2 \\ -10 & -\frac{1}{4} \end{bmatrix}
\]
It is clear that the activation functions are unbounded. Now, pick \( P = \text{diag} \{ 5, 1 \} \), then \( -PT^T \) is positive definite with \( \alpha = (1/4) \), which implies that the equilibrium \( (0, 0) \) of the system (19) is globally asymptotically stable. Fig. 1 shows the trajectory of the system (19) with initial condition \( x_1 = 10, x_2 = -4 \).

**7.2. Example 2**

In this simulation, we illustrate neural network with discontinuous activation functions is a proper
approximation of neural networks with high-slope activation functions. Together with the system (19), we consider the following two systems

$$\begin{align*}
    x_1 &= -x_1 - \frac{1}{4}(\tanh(5x_1) + x_1) + 2(\tanh(5x_2) + x_2), \\
    x_2 &= -x_2 - 10(\tanh(5x_1) + x_1) - \frac{1}{4}(\tanh(5x_2) + x_2).
\end{align*}$$

(20)

$$\begin{align*}
    x_1 &= -x_1 - \frac{1}{4}(\tanh(10x_1) + x_1) + 2(\tanh(10x_2) + x_2), \\
    x_2 &= -x_2 - 10(\tanh(10x_1) + x_1) - \frac{1}{4}(\tanh(10x_2) + x_2).
\end{align*}$$

(21)

with the same initial condition.

**Fig. 2** indicates three trajectories of the systems (19)–(21), where dark line represents the trajectory of system (19), dark dot represents that of system (20), broken line represents that of system (21). It can be seen that they are very close.

### 7.3. Example 3

Consider the following system:

$$\begin{align*}
    \dot{x}_1 &= -x_1 - \frac{1}{4}(\text{sign}(x_1) + x_1) + 2(\text{sign}(x_2) + x_2), \\
    \dot{x}_2 &= -x_2 - 10(\text{sign}(x_1) + x_1) - \frac{1}{4}(\text{sign}(x_2) + x_2).
\end{align*}$$

(22)

The equilibrium point \( x^* \) of the system (22) is \((0, 0)^T\) and \( \gamma^* = (0, 0)^T \). It can be seen that \( x^* \in D, \gamma^* \in \text{Int}(K[g(x^*)]) \) and \( \delta = 1 \). From Theorem 8, the state of the system (22) converges in finite time. Start from initial state \( x_0 = (10, -4)^T \), we have

$$t^* = \frac{1}{2\alpha \beta^2} \sum_{i=1}^{n} p_i \int_{t_0}^{t_{n-1}} g(\rho) \frac{g(\rho)}{a_i(\rho + x_i^*)} d\rho$$

$$= 2 \times [5 \times (10 + 50) + 1 \times (4 + 8)] = 664$$

**Fig. 3** indicates the trajectory of the system (22) converges to zero in finite time. It is easily seen that when \( t \geq t^* \), \( x(t) = 0 \).

### 8. Conclusions

In this paper, we discuss dynamics of the Cohen–Grossberg neural networks with discontinuous activation functions. We provide a relax set of sufficient conditions based on Lyapunov diagonally stable (LDS) property for the Cohen–Grossberg neural networks to be absolutely stable. Moreover, under certain conditions we prove that the system is exponentially stable globally or globally converges in finite time.

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**References**


