Delayed Neural Networks with Multistable Almost Periodic Solutions

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Abstract—In this paper, we are concerned with the multistability of almost periodic solutions of a class of delayed neural networks. We derive conditions guaranteeing $2^n$ asymptotically stable almost periodic trajectories for neural networks with $n$-neurons. Furthermore, we investigate the attraction basin of each almost periodic solution. Compared with the existing literature, we obtain a more general criteria for the multistability of delayed neural networks and depict the attraction basins more precisely.

I. INTRODUCTION

RECENTLY, the neural networks have been extensively studied both in theory and applications. There have been a lot of papers concerned with the dynamical behaviors of delayed neural networks, such as stability, periodic oscillations, almost periodic oscillations and so on. Most of them focus on the uniqueness and global stability of the attractor, which is called as monostability, see [1] - [4]. However, it is worth noting that there may exist more than one attractors, which is called as multistability and it has many applications [5] - [7]. For example, for pattern recognition, multiple attractors correspond to the possible patterns and the converging to certain attractor means that the system can recognize the given pattern.

In the pioneering paper [8], the authors have found out that the one neuron model \( \frac{dx(t)}{dt} = -u(t) + (1+\epsilon)g(u(t)), \) where \( \epsilon \) is a small positive number and \( g(u) = \tanh(u) \), has three equilibrium points and two of them are locally stable, one is unstable. Recent studies have generalized it to the $n$-neuron neural networks and derived many results in literature. In [9], it has been shown that the $n$-neuron neural networks can have $2^n$ locally stable equilibrium points, and they are attractive in a class of subsets with positive invariance. In [10], by decomposition of state space into $3^n$ subsets, the authors gave some conditions on the multiperiodicity of delayed cellular neural networks with saturated activation function \( f(x) = \frac{|x+1|-|x-1|}{2} \), to show that a $n$-neural network can have $2^n$ periodic orbits located in $2^n$ subsets of $R^n$ and are attractive in the corresponding subsets, respectively. In [11], by using the \( \infty \)-norm, the multistability of almost periodic solutions of delayed neural networks was also discussed and a set of sufficient conditions guaranteeing the existence and local stability of $2^n$ almost periodic solutions were presented. For more references, see [12] - [17] and references therein.

It should be noted that most of existing results are focused on the coexistence of multiple attractors in $2^n$ subsets and their locally stability in the subsets where they are located, not to mention the dynamics in other $3^n - 2^n$ subsets of $R^n$. However, it is of great interest in both theory and applications to deal with such questions: Would solutions with initial states in other $3^n - 2^n$ subsets stay in the subsets or go out? Can they be attracted by $2^n$ attractors? In other words, can the attraction basins of attractors be extended to other $3^n - 2^n$ subsets?

In this paper, we are going to study these questions, to investigate the multistability of almost periodic solutions and to see whether their attraction basins can be extended. Consider the neural networks described by

\[
\frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} \int_{0}^{\infty} g_j(x_j(t - \tau_{ij}(t) - \lambda))d_sK_{ij}(t, \lambda) + I_i(t),
\]

where $x(t) = [x_1(t), \cdots, x_n(t)]$ is the state vector at time $t$; $d_i(t) > 0$ denotes the rate with which the $i$-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time $t$; $a_{ij}(t)$ is the connection weight at time $t$; $\tau_{ij}(t) \geq 0$ denotes the transmission delay; $d_sK_{ij}(t, \lambda)$, for any fixed $t \geq 0$, is a Lebesgue-Stieltjes measure with respect to $s$, and satisfies that $\int_{0}^{\infty}|d_sK_{ij}(t, \lambda)| < \infty$; $g_j(\cdot)$ is the activation function, and $I_i(t)$ stands for the external input at time $t$.

In the following, we are to develop analysis on the multistability of almost periodic solutions of system (1) and their respective attraction basins. The rest of this paper is organized as follows. In section II, some definitions and assumptions are presented. In section III, some sufficient conditions are obtained that guarantee the coexistence of $2^n$ asymptotically stable almost periodic solutions, and new attraction basins are depicted. In section IV, two illustrative examples are provided to clarify the effectiveness of our results. In section V, conclusions are derived.

II. PRELIMINARIES

First, we list some definitions and assumptions used in the following paper.

Definition 1 (Almost Periodic Function) Let $x(t): R \rightarrow R^n$ be a continuous function, we call it an almost periodic
function, if for any $\epsilon > 0$, there exist $l = l(\epsilon) > 0$ and $\omega = \omega(\epsilon)$ in any interval with the length of $l(\epsilon)$, such that for all $t \in R$, it holds $|x_i(t+\omega) - x_i(t)| < \epsilon$, for $i = 1, \ldots, n$.

**Definition 2 (\(\xi\)-norm)** (See [4]): For an $n$-dimensional vector $x = [x_1, \ldots, x_n]$, \[ ||x||_\xi = \max_{1 \leq i \leq n} \xi_i^{-1} |x_i|, \]
where $\xi_1, \ldots, \xi_n$ are positive constants.

**Assumption 1** Suppose that $d_i(t), a_{ij}(t), \tau_{ij}(t), I_i(t)$ and $\int_0^\infty d_\lambda K_{ij}(t, \lambda)$ for $t$ are continuous, and all possess the property such that for any $\epsilon > 0$, there exist $l = l(\epsilon)$ and $\omega = \omega(\epsilon)$ in any interval with the length of $l(\epsilon)$ such that

\[ |d_i(t+\omega) - d_i(t)| < \epsilon, \quad |a_{ij}(t+\omega) - a_{ij}(t)| < \epsilon, \]
\[ |\tau_{ij}(t+\omega) - \tau_{ij}(t)| < \epsilon, \quad |I_i(t+\omega) - I_i(t)| < \epsilon, \]
\[ \int_0^\infty |d_\lambda K_{ij}(t+\omega, \lambda) - d_\lambda K_{ij}(t, \lambda)| < \epsilon, \]
hold for all $i, j = 1, 2, \ldots, n$ and $t \in R$.

Then, since $d_i(t), a_{ij}(t), \tau_{ij}(t), I_i(t)$ and $\int_0^\infty d_\lambda K_{ij}(t, \lambda)$ are almost periodic functions, they are all bounded. Denote \[ d_i = \inf_t d_i(t), \quad a_{ij} = \sup_t |a_{ij}(t)|, \quad \tau_{ij} = \sup_t \tau_{ij}(t), \]
\[ K_{ij} = \sup_t \int_0^\infty |d_\lambda K_{ij}(t, \lambda)|, \quad I_i = \sup_t |I_i(t)|. \]

**Definition 3 (Function Class \(A\))** A continuous nondecreasing function $g_i(x) \in A$ if there exist constants $p_i < q_i, m_i < M_i$ such that
\[ g_i(x) = \begin{cases} m_i & -\infty < x < p_i, \\ M_i - m_i(x - p_i) + m_i & p_i < x < q_i, \\ M_i & q_i \leq x < +\infty. \end{cases} \]

**Definition 4 (Function Class \(B\))** A continuous nondecreasing function $g_i(x) \in B$ if it is sigmoid-like, that is, the left-hand and right-hand derivative of $g_i$ at point $p_i, q_i$ exist, denoted as $g_i^-(p_i), g_i^+(p_i), g_i^-(q_i), g_i^+(q_i)$, respectively, and it is differentiable at other points, such that
\[ \lim_{x \to -\infty} g_i(x) = m_i, \quad g_i^-(p_i) \leq g_i^-(q_i), \quad g_i^+(p_i) \leq g_i^+(q_i), \]
\[ \lim_{x \to +\infty} g_i(x) = M_i, \quad g_i^+(x) \leq g_i^+(q_i), \]
where $p_i, q_i, M_i, m_i, \sigma_i$ are constants with $p_i < q_i, m_i < M_i, \sigma_i \in (p_i, q_i)$.

It is easy to see that, for $g_i \in A$ as well as $g_i \in B$, if we denote
\[ (-\infty, p_i] = (-\infty, p_i] \times (p_k, q_k) \times [q_k, +\infty)^0, \]
\[ (p_k, q_k) = (-\infty, p_i] \times (p_k, q_k) \times [q_k, +\infty)^0, \]
\[ [q_k, +\infty) = (-\infty, p_i] \times (p_k, q_k) \times [q_k, +\infty)^1, \]
then $R^n$ can be divided into 3 subsets:
\[ \prod_{k=1}^n (\infty, p_i]^a_k \times (p_k, q_k)^b_k \times [q_k, +\infty)^c_k : \]
\[ (a_k^0, b_k^0, c_k^0) = (1, 0, 0), (0, 1, 0), (0, 0, 1), \]
\[ k = 1, \ldots, n. \]

Denote
\[ \Phi_1 = \bigcup_{a_k^0 \neq 1} \left( \prod_{k=1}^n (-\infty, p_k)^{a_k^0} \times (p_k, q_k)^{b_k^0} \times [q_k, +\infty)^{c_k^0} \right) \times [q_k, +\infty)^1; \]
\[ \Phi_2 = \prod_{k=1}^n (p_k, q_k), \quad \Phi_3 = R^n - \Phi_1 - \Phi_2. \]

Then, consider system (1) with initial states
\[ x(\theta) = \phi(\theta) \quad \text{for} \theta \in [-\bar{\tau}, 0], \]
where $\bar{\tau} = \max \bar{\tau}_{ij}, \phi(\theta) = [\phi_1(\theta), \ldots, \phi_n(\theta)]$, and $\phi_i \in C([-\bar{\tau}, 0]), i = 1, \ldots, n$. In the following, we are to investigate the coexistence of $2^n$ almost periodic solutions and its stability in subset $\Phi_1$, and to see whether its attraction basin can be extended to other subsets.

**III. MAIN RESULTS**

In this section, we address our main results. Corresponding to the two kinds of activation $g_i \in A$ and $g_i \in B$, we have

**Theorem 1:** Under Assumption 1, suppose $p_i < q_i$ and there exists a small constant $\delta > 0$ such that there hold
\[ -d_i(t)p_i + a_{ii}(t)p_i + \sum_{j \neq i} \max(a_{ij}(t)m_j, a_{ij}(t)M_j) \]
\[ + \sum_{j \neq i} \max(m_j, M_j) \int_0^\infty |d_\lambda K_{ij}(t, \lambda)| \]
\[ + I_i(t) \leq -\delta < 0, \quad \text{for} \quad t \geq 0, \]
\[ -d_i(t)q_i + a_{ii}(t)q_i + \sum_{j \neq i} \min(a_{ij}(t)m_j, a_{ij}(t)M_j) \]
\[ - \sum_{j \neq i} \max(m_j, M_j) \int_0^\infty |d_\lambda K_{ij}(t, \lambda)| \]
\[ + I_i(t) \geq \delta > 0, \quad \text{for} \quad t \geq 0, \]
for all $t \geq 0, i, j = 1, \ldots, n$. Then, for $g_i \in A$, there exist $2^n$ exponentially stable almost periodic solutions of system (1), and their attraction basins component can be extended to $(-\infty, \Psi_i)$ or $(\Gamma_i, +\infty)$, from $(-\infty, p_i]$ or $[q_i, +\infty)$, respectively, where
\[ \Gamma_i \triangleq \sup_{t \geq 0} \{ - \int_0^t e^{-\int_0^s (-d_i(u) + a_{ii}(u)I_i(u))du} \]
\[ - \sum_{j \neq i} \max(m_j, M_j) \int_0^\infty |d_\lambda K_{ij}(s, \lambda)| \]
\[ + a_{ii}(s)c_i + I_i(s) \} \}
\[ ds \]}.
be extended to for all \( t \)

and \( l_i = \frac{M_i - m_i}{q_i - p_i}, c_i = m_i - \frac{p_i(M_i - m_i)}{q_i - p_i}. \)

And

\[ \Xi_i \triangleq \inf_{t \geq 0} \left\{ - \int_0^t e^{-\int_0^s (-d_i(u) + a_{ii}(u))du} \left( \sum_{j \neq i} \max(a_{ij}(s)m_j, a_{ij}(s)M_j) \\
+ \sum_{j=1}^n \max(|m_j|, |M_j|) \int_0^\infty |d_{ij}K_{ij}(s, \lambda)| \\
+ a_{ii}(s)c_i + I_i(s) \right) ds \right\}. \]

Then, we study the attraction basin for each region. Obviously, for each almost periodic solution, its attraction basin contains the corresponding region of \( \Phi_1 \), where it is located. Furthermore, for system (1) with \( \tau_i \in \mathcal{A} \), it can be proved that the component solution with initial state \( \Gamma_i < x_i(0) < \Psi_i \) would be attracted to \( [q_i, +\infty) \) when \( t \) is sufficiently large; while the component solution with initial state \( \psi_i < x_i(0) < \Psi_i \) would be attracted to \( (-\infty, p_i) \) when \( t \) is sufficiently large. Therefore, the attraction basin of each attractor can be extended in such a way that the region components are extended to \( (-\infty, \Psi_i) \) or \( (\Gamma_i, +\infty) \), from \( (-\infty, p_i) \) or \( [q_i, +\infty) \), respectively.

For example, suppose \( x^*(t) \) is an almost periodic solution of system (1) located in region

\[ \Omega = \bigcap_{k \in N_1} (-\infty, \Psi_k) \times \bigcap_{k \in N_2} [q_k, +\infty), \]

where \( N_1, N_2 \) are subsets of \( \{1, 2, \cdots, n\} \), and \( N_1 \cup N_2 = \{1, 2, \cdots, n\} \), \( N_1 \cap N_2 = \emptyset \). Then, the attraction basin of \( x^*(t) \) can be extended to

\[ \bigcap_{k \in N_1} (-\infty, \Psi_k) \times \bigcap_{k \in N_2} (\Gamma_k, +\infty) \]

from \( \Omega \).

For system (1) with \( \tau_i \in \mathcal{B} \), similar methods can be applied and similar conclusions can be derived. However, due to the limit of space, the detailed proof cannot be presented here.

In addition, as special cases of system (1), we can derive several models adopted in literature. Let \( d_{ij}K_{ij}(t, \lambda) = b_{ij}(t)\delta(\lambda) \), where \( \delta(\lambda) \) is the Dirac function. Then, (1) reduces to the following neural networks with time-varying delays

\[ \frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)g_j(x_j(t)) \\
+ \sum_{j=1}^n b_{ij}(t)g_j(x_j(t - \tau_ij(t))) + I_i(t), \]

\[ i = 1, \cdots, n. \] (10)

As direct consequences of Theorem 1, 2, corresponding results can be obtained as follows.

\[ \text{Corollary 1: Under Assumption 1, suppose } p_i < 0 < q_i \text{ and there exists a small constant } \delta > 0 \text{ such that there hold} \]

\[ -d_i(t)p_i + a_{ii}(t)g_i(p_i) + \sum_{j \neq i} \max(a_{ij}(t)m_j, a_{ij}(t)M_j) \\
+ \sum_{j=1}^n \max(|m_j|, |M_j|)|b_{ij}(t)| + I_i(t) \leq -\delta < 0, \] (11)
respectively, where 

\[ -d_i(t)q_i + a_{ii}(t)g_i(q_i) + \sum_{j \neq i} \min(a_{ij}(t)m_j, a_{ij}(t)M_j) \\
- \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(t)| + I_i(t) \geq \delta > 0, \quad (12) \]

for all \( t \geq 0, i, j = 1, \ldots, n \). Then, for \( q_i \in \mathcal{A} \), there exist \( 2^n \) exponentially stable almost periodic solutions of system (10), and their attraction basins component can be extended to \((-\infty, \tilde{\Psi}_i)\) or \((\tilde{\Gamma}_i, +\infty)\), from \((-\infty, p_i)\) or \([q_i, +\infty)\), respectively, where

\[ \tilde{\Gamma}_i \triangleq \sup_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \min(a_{ij}(u)m_j, a_{ij}(u)M_j) \\
+ a_{ii}(u)g_i(u) + I_i(u) \right) du \right\}, \]

\[ \tilde{\Psi}_i \triangleq \inf_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \min(a_{ij}(u)m_j, a_{ij}(u)M_j) + a_{ii}(u)g_i(u) + I_i(u) \right) du \right\}, \]

and \( l_i = M_i - m_i, c_i = m_i - p_i(M_i - m_i). \Box \)

**Corollary 2:** Under Assumption 1, suppose \( p_i < 0 < q_i \) and there exist positive constants \( \delta, \xi_1, \ldots, \xi_n, \eta, \gamma \) such that there hold (11), (12) and

\[ -\xi_i(d_i(t) - \eta) + \sum_{j=1}^{n} \xi_j |a_{ij}(t)| \\
+ |b_{ij}(t)|e^{\eta \tau} \max(g_j(q_{j}^-), g_j(q_{j}^+)) \leq -\gamma, \quad (13) \]

for all \( t \geq 0, i, j = 1, \ldots, n \). Denote \( p_i = \min(g_i^-(p_i), g_i^+(q_i)) \). If there also hold (8) and (9), then, for \( q_i \in \mathcal{B} \), there are \( 2^n \) almost periodic solutions of system (10), and their attraction basins component can be extended to \((-\infty, \tilde{\Psi}_i)\) or \((\tilde{\Gamma}_i, +\infty)\), from \((-\infty, p_i)\) or \([q_i, +\infty)\), respectively, where

\[ \tilde{\Gamma}_i \triangleq \sup_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \min(a_{ij}(u)m_j, a_{ij}(u)M_j) \\
- \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(u)| + I_i(u) \right) du \right\}, \]

\[ \tilde{\Psi}_i \triangleq \inf_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \max(a_{ij}(u)m_j, a_{ij}(u)M_j) + \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(u)| + I_i(u) \right) du \right\}, \]

and \( l_i = M_i - m_i, c_i = m_i - p_i(M_i - m_i). \Box \)

**Corollary 4:** Under Assumption 1, suppose \( p_i < 0 < q_i \) and there exist positive constants \( \delta, \xi_1, \ldots, \xi_n, \eta, \gamma \) such that

\[ \frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) \\
+ \sum_{j=1}^{n} b_{ij}(t) \int_{0}^{\infty} k_{ij}(\lambda)|x_j(t - \tau_{ij}(t) - \lambda)|d\lambda \\
+ I_i(t), \quad i = 1, \ldots, n. \quad (14) \]

As direct consequences of Theorem 1, 2, corresponding results can be obtained as follows.

**Corollary 3:** Under Assumption 1, suppose \( p_i < 0 < q_i \) and there exists a small constant \( \delta > 0 \) such that there hold

\[ -d_i(t)p_i + a_{ii}(t)p_i + \sum_{j \neq i} \max(a_{ij}(t)m_j, a_{ij}(t)M_j) \\
+ \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(t)| + I_i(t) \leq -\delta < 0, \quad (15) \]

\[ -d_i(t)q_i + a_{ii}(t)q_i + \sum_{j \neq i} \min(a_{ij}(t)m_j, a_{ij}(t)M_j) \\
- \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(t)| + I_i(t) \geq \delta > 0, \quad (16) \]

for all \( t \geq 0, i, j = 1, \ldots, n \). Then, for \( q_i \in \mathcal{A} \), there exist \( 2^n \) exponentially stable almost periodic solutions of system (14), and their attraction basins component can be extended to \((-\infty, \tilde{\Psi}_i)\) or \((\tilde{\Gamma}_i, +\infty)\), from \((-\infty, p_i)\) or \([q_i, +\infty)\), respectively, where

\[ \tilde{\Gamma}_i \triangleq \sup_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \min(a_{ij}(u)m_j, a_{ij}(u)M_j) \\
- \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(u)| + I_i(u) \right) du \right\}, \]

\[ \tilde{\Psi}_i \triangleq \inf_{t \geq 0} \left\{ \int_{0}^{t} e^{-\int_{0}^{u}(-d_i(u) + a_{ii}(u)\lambda)} dm_u \left( \sum_{j \neq i} \max(a_{ij}(u)m_j, a_{ij}(u)M_j) + \sum_{j=1}^{n} \max(|m_j|, |M_j|)|b_{ij}(u)| + I_i(u) \right) du \right\}, \]

and \( l_i = M_i - m_i, c_i = m_i - p_i(M_i - m_i). \Box \)
there hold (15), (16) and

\[-\xi_i(d_i(t) - \eta) + \sum_{j=1}^{n} \xi_j [(a_{ij}(t)) + e^{\eta \tau} |b_{ij}(t)| \int_{0}^{\infty} e^{\eta \lambda} \cdot |k_{ij}(\lambda)| d\lambda \cdot \max(g_{j-}(p_j), g_{j+}(q_j)) \leq -\gamma, \hspace{1cm} (17)\]

for all \( t \geq 0, i, j = 1, \cdots, n \). Denote \( \rho_i = \min(g_{j-}(p_j), g_{j+}(q_j)) \). If there also hold (8) and (9), then, for \( g_i \in \mathcal{B} \), there are \( 2^n \) almost periodic solutions of system (14), and their attraction basins component can be extended to \((-\infty, \Psi_i)\) or \((\tilde{\Gamma}, +\infty)\), from \((-\infty, \rho_i)\) or \([q_i, +\infty)\), respectively, where

\[\tilde{\Gamma} = \sup_{t \geq 0} \left\{ -\int_{0}^{t} e^{-\int_{0}^{\tau} (d_i(u) + a_{ii}(u)\rho_i) du} \right\}, \]

\[\Psi_i = \inf_{t \geq 0} \left\{ -\int_{0}^{t} e^{\int_{0}^{\tau} (d_i(u) + a_{ii}(u)\rho_i) du} \right\}.

\] IV. COMPARISON AND SIMULATIONS

In [9], the authors also discussed multistability of the following recurrent neural networks:

\[\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t - \tau_{ij})) + J_i, \hspace{1cm} (18)\]

\( i = 1, \cdots, n \), and gave some conditions guaranteeing the existence of \( 2^n \) stable equilibrium points. Basins of attraction for these stationary equilibria were also estimated, which are with positive invariance property and there hold that

\[-b_i + \sum_{j=1}^{n} |w_{ij}| g'(\eta_j) < 0, \hspace{1cm} (19)\]

for \( i = 1, \cdots, n \), and all \( \eta = [\eta_1, \cdots, \eta_n] \) in the basins. However, our results break through these restrictions: we do not require that the attraction basins are positive invariant, nor necessarily satisfying the above conditions (19). In fact, the attraction basins in the sense of [9] are subsets of ours.

Moreover, we provide the following examples to illustrate the effectiveness of our results.

**Example 1** Consider the following 2-neuron neural network model described by

\[\begin{aligned}
dx_1(t) &= -0.8x_1(t) + 2g(x_1(t)) + 0.2g(x_1(t - 0.1)) \\
dx_2(t) &= -0.5x_2(t) + 0.1g(x_1(t)) + 0.1g(x_1(t - 0.1)) \\
&\quad + 1.2g(x_2(t)) + 0.1g(x_2(t - 0.2)) + 0.3\sin(\pi t),
\end{aligned}\]

where the activation function \( g(\xi) = \frac{1}{1 + |\xi|}. \)

According to the previous results, in order to guarantee the positive invariance of attraction basins, we see that \(-1 \leq p_1, q_1, p_2, q_2 \leq 1\) must satisfy

\[-0.8p_1 + 2g(p_1) + 0.5 + 0.6\sin t < 0, \]

\[-0.8q_1 + 2g(q_1) - 0.5 + 0.6\sin t > 0, \]

\[-0.5p_2 + 1.2g(p_2) + 0.3 + 0.3\sin(\pi t) < 0, \]

\[-0.5q_2 + 1.2g(q_2) - 0.3 + 0.3\sin(\pi t) > 0, \]

for all \( t \geq 0 \), from the conditions (5) and (6). Hence, \(-1 \leq p_1 < \frac{11}{12}, q_1 < 1, -1 \leq p_2 < -\frac{6}{7}, q_2 < 1 \).

Compared with it, applying our new results, we have \( p_1 \leq \frac{172}{200}, q_1 > -0.665, p_2 < 0.339, q_2 > -0.523. \) According to the Theorem 1, we know that there are 4 almost periodic solutions of system (20), and they are stable in subsets

\[\begin{aligned}
\tilde{\Gamma}_1 &= (-\infty, -0.665) \times (0.339, +\infty), \\
\tilde{\Gamma}_2 &= (0.172, +\infty) \times (0.339, +\infty), \\
\tilde{\Gamma}_3 &= (-\infty, -0.665) \times (-\infty, -0.523), \\
\tilde{\Gamma}_4 &= (0.172, +\infty) \times (-\infty, -0.523),
\end{aligned}\]

In fact, by simulations, the dynamics of \( x_1(t) \) and \( x_2(t) \) are illustrated in Fig 1, 2, where evolutions of 4 initial states \([-0.8, 0.4], [0.2, 0.35], [-0.7, -0.55], [0.18, -0.6]\), plotted in green, black, blue, and red, respectively, have been tracked. There show that they are converging to 4 almost periodic solutions in \( \tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{\Gamma}_4 \), respectively, as confirmed by our theory.
4.32 Let \( \delta = 0.5, \xi_1, \cdots, \xi_n = 1 \), it can be validated that (5), (6), (7), (8) and (9) hold. Hence, according to the Theorem 2, we know that there are 4 almost periodic solutions of system (21), and they are stable in subsets \( S_1 = (-\infty, -0.41) \times (0.235, +\infty), S_2 = (0.228, +\infty) \times (0.235, +\infty), S_3 = (-\infty, -0.41) \times (-\infty, -0.424), S_4 = (0.228, +\infty) \times (-\infty, -0.424), \) respectively.

In fact, by simulations, the dynamics of \( X_1(t) \) and \( X_2(t) \) are illustrated in Fig 3, 4, where evolutions of 4 initial states \([ -0.5, 0.3], [0.3, 0.25], [ -0.42, -0.5], [0.25, -0.55]\), plotted in green, black, blue, and red, respectively, have been tracked. There show that they are converging to 4 almost periodic solutions in \( S_1, S_2, S_3, S_4 \), respectively, as confirmed by our theory.

Example 2 Consider the following 2-neuron neural network model described by

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -0.5x_1(t) + 2.2g(x_1(t)) + 0.05g(x_2(t)) + 0.05g(x_1(t-0.1)) + 0.05g(x_2(t-0.2)) + 0.3\sin(\pi t), \\
\frac{dx_2(t)}{dt} &= -0.3x_2(t) + 0.05g(x_1(t)) + 1.8g(x_2(t)) + 0.05g(x_1(t-0.1)) + 0.05g(x_2(t-0.2)) + 0.2\cos t,
\end{align*}
\]

where the activation function

\[
g(x) = \begin{cases} 
\tanh(0.15x) - \tanh(0.8) + \tanh(0.12), & -\infty < x \leq -0.8 \\
\tanh(x), & -0.8 < x < 0.8 \\
\tanh(0.15x) + \tanh(0.8) - \tanh(0.12), & 0.8 \leq x < +\infty.
\end{cases}
\]

According to the previous results, in order to guarantee (19), there must hold that

\[
\begin{align*}
-0.5 + 2.25g'(\eta_1) + 0.1g'(\eta_2) &< 0, \\
-0.3 + 1.85g'(\eta_2) + 0.1g'(\eta_1) &< 0,
\end{align*}
\]

which imply that \( g'(\eta_1) < 0.2222, g'(\eta_2) < 0.1622. \) Referred to system (21), since

\[
g'(x) \geq g'_+ (0.8) = g'_+ (-0.8) = 0.5591, \\
\text{for } x \in (-0.8, 0.8),
\]

\[
g'(x) \leq g'_+ (0.8) = g'_+ (-0.8) = 0.1479, \\
\text{for } x \in (-\infty, 0.8) \cup (0.8, +\infty),
\]

then, the component attraction basin is \((-\infty, -0.8), \) or \((0.8, +\infty). \)

Compared with it, applying our new results, we have \( \rho_i = 0.5591, \) and \( \bar{\Gamma}_1 \leq 0.228, \bar{\Psi}_1 > -0.41, \bar{\Gamma}_2 \leq 0.235, \bar{\psi}_2 > -0.424. \)
V. CONCLUSIONS

In this paper, we study the multistability of almost periodic solutions of delayed neural networks and their attraction basins. It shows that under some conditions, the \( n \)-neuron neural networks can have \( 2^n \) almost periodic solutions in \( 2^n \) subsets of \( \mathbb{R}^n \), respectively. Furthermore, the almost periodic solution is attractive not only in the subset where it is located, but expanded to other \( 3^n - 2^n \) subsets of \( \mathbb{R}^n \). A set of new criteria on their corresponding attraction basins are derived. The results obtained extend and improve the existing ones.

REFERENCES


