Synchronisation in complex networks of coupled systems with directed topologies

Wenlian Lu, Tianping Chen

* The School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China

To cite this Article Lu, Wenlian and Chen, Tianping 'Synchronisation in complex networks of coupled systems with directed topologies', International Journal of Systems Science, 40: 9, 909 — 921

To link to this Article: DOI: 10.1080/00207720802645238

URL: http://dx.doi.org/10.1080/00207720802645238
Synchronisation in complex networks of coupled systems with directed topologies

Wenlian Lu and Tianping Chen*

The School of Mathematical Sciences, Fudan University, Handan Road 220, Shanghai 200433, P.R. China

(Received 21 February 2008; final version received 8 October 2008)

The aim of this article is to provide a systematic review on the framework to analyse synchronisation in complex networks of coupled systems with a focus on the situation of directed graphs. Transverse stability of synchronisation subspace/manifold is used to describe synchronous motions, which presents the idea for synchronisation analysis. The stability of the variational systems in the transverse directions to the synchronisation manifold is studied to obtain local synchronisation. As for global synchronisation, certain structure matrices are defined to measure the distance from the collective states of the coupled systems to the synchronisation subspace, which serves as a candidate Lyapunov function. In this way, the synchronisability of a directed graph can be denoted by extending the algebraic connectivity to the underlying graph via Rayleigh–Ritz ratio. These methods and results depict how the interaction structure among individuals affects the global dynamics. Coupling delay and time-varying couplings are also considered. Furthermore, these ideas and methods can be used to investigate synchronisation of discrete-time networks of coupled maps and pinning control problem.

Keywords: stability; multi-agent systems; networked control; continuous-time systems; synchronisation; indirect graph; asymmetric coupling

1. Introduction

Complex networks have been widely used to model complex systems in science, engineering, biology, etc., since a mathematical term ‘graph’ is used to describe the interactions between individuals in complex systems. In such graphs, vertices represent individuals in the complex system and edges represent interactions among individuals. Typical examples of complex networks include Internet, WWW, cellular and metabolic networks, neural nets and so on (Albert and Barabási 2002). The complexity of such networks lies in two aspects: structure and dynamics. And these two aspects are linked to each other. Besides regular graph topologies, such as the k-nearest neighbourhood, complete graph and star-like wiring, complex structures, especially including randomness and evolution, have attracted much research attention since Erdős and Rényi (1959) for the first time proposed the random graph model. Watts and Strogatz (1998) introduced a random rewiring model (WS model). This network has similar clustering distribution as regular graph and similar short diameter as random graph, which is called ‘small-world’ property. Barabási and Albert (1999) proposed an evolving network model which has a power-law degree distribution named scale-free model (BA model). These works as well as many others afterwards enrich our understandings on the complexity of network structure.

Many complicated dynamical behaviours have been studied in the research of complex networks—for example, stability, bifurcation, chaos, etc. Among them, the synchronisation phenomenon has been a focal topic (Pikovsky, Rosenblum, Kurths, and Strogatz 2001; Wu 2007). The word ‘synchronisation’ comes from Greek, which means ‘share time’ and today it has come to be considered as ‘time coherence of different processes’ in science and technology. Since synchronisation of interacting systems was observed by Hegenii (1673), this phenomenon also appears in wide range of real-world systems. For example, fireflies have been known to flash in unison, and this phenomenon has been proved to occur in a group of integrate-fire cells (Mirollo and Strogatz 1990).

Besides coherence among two or several systems (Boccaletti, Kurths, Osipov, Valladares, and Zhou 2002), synchronisation in complex networks has attracted increasing attention in physics, mathematics, biology and engineering. In fact, synchronisation has diverse senses: for instance, phase synchronisation, lag synchronisation, partial synchronisation, and

*Corresponding author. Email: tchen@fudan.edu.cn
generalised synchronisation (Pikovsky et al. 2001 and Boccaletti, Latora, Moreno, Chavez, and Hwang 2006 for more details). In this article, we mainly focus on the most special one: complete synchronisation, i.e. \( \lim_{t \to \infty} |x(t) - x'(t)| = 0 \) holds for all \( i, j = 1, \ldots, m \), where \( | \cdot | \) denotes certain norm. Shortly, this is referred to as synchronisation in this article.

In mathematical terms, synchronisation can be regarded as transverse stability: the dynamical system possesses an invariant diagonal synchronisation manifold which is asymptotically stable. The means of transverse stability was introduced to study local synchronisation of networks of coupled differential systems by Pecora and Carroll (1998), Wang and Chen (2002a, b), and networks of coupled maps by Jost and Joy (2001) etc.

Relation between dynamics and structure is always the main focus (Boccaletti et al. 2006). The dynamical theoretical manners can help to investigate the influence of the network topological structure on synchronous motions. Graph Laplacians and their spectral properties play a crucial role in synchronisation analysis. Pecora and Carroll (1998), Wang and Chen (2002a, b), and Jost and Joy (2001) have found that the synchronisability of a graph topology can be described as a function of the spectral of the corresponding Laplacian, for example, the ratio between the smallest and largest nonzero eigenvalues. From then on, the spectral properties of the Laplacian of complex networks models have been discussed by physicists and mathematicians, theoretically and numerically (see Motter, Zhou, and Kurths 2005 and Zhou and Kurths 2006 for references).

Most of these works deal with networks of indirected topology, of which the Laplacian is symmetric or diagonally symmetrisable. This implies that the eigenvectors of the Laplacian construct a basis of the whole space. However, if the graph is directed and its Laplacian is asymmetric or/and reducible, symmetry or symmetrisability does not hold. In the past decade, several papers are concerned with synchronisation analysis of complex network with general topologies. For local synchronisation, the method developed by Pecora and Carroll (1998) can be directly employed with minor modification (Lu and Chen 2006a). Global synchronisation was studied by introducing a general class of potential function to describe the distance from the collective states of the coupled systems to the synchronisation manifold. The coupled system can be globally synchronised if such a potential function is a Lyapunov one. See Wu and Chua (1995) and others for reference.

In this article, we focus on the results described in the papers authored or co-authored by us: Lu and Chen (2004, 2006a, 2007), Lu, Chen, and Chen (2006b), Chen, Liu, and Lu (2007), Liu and Chen (2007), Lu, Atay, and Jost (2007), Wu and Chen (2008), and several other important papers: Wu and Chua (1995), Pecora and Carroll (1998), Wu (2003, 2005a, b). The main goal of this article is to present a cohesive overview of the key means for synchronisation analysis of complex networks with directed, asymmetric and reducible graphs, including continuous-time and discrete-time models, in a unified framework. We introduce the methods in dynamical theory for synchronisation analysis and synchronisability discussions that heavily rely on matrix analysis and algebraic graph theory. We also study the influence of coupling delay on synchronous dynamics. Moreover, the pinning control of complex network to a homogeneous trajectory of the uncoupled system can be investigated under the same framework.

This article is organised as follows. Basic concepts, theories and results on synchronisation of complex networks with directed topologies are presented in Section 2. Synchronisation in dynamical networks with time-varying topologies is studied in Section 3. Pinning control is investigated in the similar framework in Section 4. Finally, some conclusions are stated in Section 5.

Before starting, we present some necessary notations which will be used throughout this article. \( Z^T \) denotes the transpose of a matrix \( Z \), and the symmetry part of a square matrix \( Z \) is denoted by \( Z' = (1/2)(Z + Z^T) \). We say a square matrix \( Z > 0 \) if \( Z \) is positive (semi-positive) definite, and so it is with \( < 0 \) (\( \leq 0 \)). \( |Z| \) denotes certain matrix norm. In particular, \( |Z|_2 \) denotes its 2-norm. \( \lambda(Z) \) denotes the eigenvalue set of a square matrix \( Z \), among which \( \lambda_{\text{max}}(Z) \) and \( \lambda_{\text{min}}(Z) \) denote the maximum and minimum ones, respectively, if all eigenvalues of \( Z \) are real. \( I_p \) denotes the identity matrix with dimension \( p \). Symbol \( \otimes \) denotes the Kronecker product in the sense that for a matrix \( A = [a_{ij}]_{i,j=1}^m \in \mathbb{R}^{m \times m} \) and a matrix \( B = [b_{kl}]_{k,l=1}^n \in \mathbb{R}^{n \times n} \). Let \( C = [c_{ij}]_{i,j=1}^m \in \mathbb{R}^{m \times m} \) be a nonsingular matrix with \( c_{ij} \leq 0 \), \( i,j = 1,\ldots,m \), \( i \neq j \). \( C \) is said to be an \( M \)-matrix if all elements of \( C^{-1} \) are nonnegative. For more details about \( M \)-matrix, we refer readers to Berman (2003). \( Re(z) \) denotes the real part of a complex number \( z \).

### 2. Synchronisation in complex networks of coupled differential systems

A graph is defined by a triple set: \( G = [\mathcal{V}, \mathcal{E}, \mathcal{W}] \), where \( \mathcal{V} = \{1, \ldots, m\} \) denotes the vertex set, \( \mathcal{E} \) denotes the edge set by the way that \( e(i,j) \in \mathcal{E} \) if and only if there exists an edge from the vertex \( j \) to \( i \), and...
\( \mathcal{V} = \{ w_{ij} : e(i, j) \in \mathcal{E} \} \) denotes the weight set of edges. Here, we consider the case that the graph \( \mathcal{G} \) can be directed, which means \( e(i, j) \in \mathcal{E} \) does not imply \( e(j, i) \in \mathcal{E} \), and it can also be asymmetric, which means that \( w_{ij} \neq w_{ji} \) may hold even if two edges exist, and not strongly connected, which means that at least one vertex cannot access all other vertices.

The coupled system is set upon the graph \( \mathcal{G} \) and has the following form:

\[
\frac{d}{dt} x'(t) = f(x'(t), t) + c \sum_{j \in N(i)} w_{ij}[x'(t) - x'(t)],
\]

\[i = 1, \ldots, m.\] (1)

Here, \( t \in \mathbb{R}^+ \) denotes the continuous-time, \( x'(t) = [x'_1(t), \ldots, x'_m(t)]^T \in \mathbb{R}^m \) is the state variable vector of the vertex \( i \), its evolution depends on two factors: its own dynamical property described by a map \( f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n \) and the diffusion couplings from its neighbourhood \( \mathcal{N}(i) \), via the graph structure:

\[ \mathcal{N}(i) = \{ j \in \mathcal{V} : e(i, j) \in \mathcal{E} \}. \]

\( \Gamma = [\gamma_{kl}]_{k,j=1}^n \) presents the inner coupling structure, that is, \( \gamma_{kl} \neq 0 \) if and only if there is a coupling from the \( k \)th component of the vertex \( j \) to the \( k \)th component of the vertex \( i \) in the case of \( e(i, j) \in \mathcal{E} \); (see Pogromsky, Glad, and Nijmeijer 1999; Pogromsky and Nijmeijer 2001; Wu 2007 for references). Moreover, \( c \) is the coupling strength. We can alternatively describe the coupling coefficients by the graph Laplacian. Namely, \( L = [l_{ij}]_{i,j=1}^n \) satisfying: \( l_{ii} = -\sum_{j=1}^n l_{ij} \) if and only if \( e(i, j) \in \mathcal{E} \); otherwise, \( l_{ij} = 0 \); and all row sums are zero, i.e. \( l_{ii} = -\sum_{j=1}^n l_{ji} \). Then, the coupled system (1) can be rewritten as follows:

\[
\frac{d}{dt} x'(t) = f(x'(t), t) - c \sum_{j=1}^m l_{ij}[x'_i(t) - x'_j(t)],
\]

\[i = 1, \ldots, m.\] (2)

Denoting \( x(t) = [x'_1(t), \ldots, x'_m(t)]^T \in \mathbb{R}^m \), and \( F(x(t), t) = [f(x'_1(t), t), \ldots, f(x'_m(t), t)]^T \) a map \( \mathbb{R}^m \times \mathbb{R}^+ \rightarrow \mathbb{R}^m \), the coupled system (2) has the following compact form:

\[
\frac{d}{dt} x(t) = F(x(t), t) - c \left( L \otimes \Gamma \right) x(t),
\]

(3)

Synchronisation is defined by that all differences between trajectories of vertices disappear as time goes to infinity, namely, it holds

\[
\lim_{t \to \infty} |x'(t) - x'(t)| = 0, \quad i, j = 1, \ldots, m.
\]

Moreover, if this convergence is exponential, then it is called exponential synchronisation. It is clear that the diagonal subspace in \( \mathbb{R}^m \):

\[
S = \{ x = [x'_1, \ldots, x'_m]^T \in \mathbb{R}^m : x'_i = x'_i \text{ holds for all } i, j = 1, \ldots, m \},
\]

is invariant through the evolution of the dynamical system (3). Therefore, synchronisation can be equivalently regarded as the transverse stability of the invariant subspace \( S \).

### 2.1. Local synchronisation

Lu and Chen (2006a) studied the local synchronisation via a reference trajectory which is a weighted average \( \bar{x}(t) \) of the collective trajectories of the network (3). Then, synchronisation can be transformed into:

\[
\lim_{t \to \infty} |x'(t) - \bar{x}(t)| = 0, \quad i = 1, \ldots, m.
\]

The authors introduced a weighted average \( \bar{x}(t) = \sum_{i=1}^m \xi_i x_i(t) \) based on the left eigenvector of the Laplacian \( L \) with respect to the eigenvalue zero satisfying \( \xi_i \geq 0, \ i = 1, \ldots, m \), and \( \sum_{i=1}^m \xi_i = 1 \), which can be guaranteed by the famous Perron-Probenius theorem (Horn and Johnson 1985).

Let \( \delta x'(t) = x'(t) - \bar{x}(t) \). Linearising Equation (2) gives

\[
\frac{d\delta x_i(t)}{dt} = Df(\bar{x}(t), t)\delta x_i(t) - c \sum_{j=1}^m l_{ij}\Gamma \delta x_j(t).
\]

(4)

Denote \( \delta X(t) = [\delta x'_1(t), \delta x'_2(t), \ldots, \delta x'_m(t)]^T \in \mathbb{R}^{m,m} \) we have a matrix form of variational equation:

\[
\frac{d\delta X(t)}{dt} = Df(\bar{x}(t), t)\delta X(t) - c\Gamma \delta X(t)L^T,
\]

(5)

where \( Df(\bar{x}(t), t) \) is the Jacobian matrix with respect to the reference trajectory \( \bar{x}(t) \). Let \( L^T = SJS^{-1} \) be the Jordan decomposition. \( \lambda_k, \ k = 1, \ldots, l \), are distinguished eigenvalues of \( L \). Thus, letting \( \delta Y(t) = \delta X(t)S \), we have a variational equation in terms of \( \delta Y(t) \):

\[
\frac{d\delta Y(t)}{dt} = Df(\bar{x}(t), t)\delta Y(t) - c\Gamma \delta Y(t)J,
\]

(6)

where \( \delta Y(t) = [\delta y'_1(t), \ldots, \delta y'_l(t)]^T \), \( \delta y_k(t) = [\delta y_{k1}(t), \ldots, \delta y_{km}(t)]^T \), where \( \delta y_k(t) \equiv 0 \). And for all transverse directions to the synchronisation subspace \( S \), i.e. \( 2 \leq k \leq l \), we have

\[
\frac{d\delta y_{ki}(t)}{dt} = [Df(\bar{x}(t), t) - c\lambda_k \Gamma]\delta y_{ki}(t).
\]

(7)
\[
\frac{d\delta y_{k+1}(t)}{dt} = [Df(\bar{x}(t), t) - \omega_{\Delta} \Gamma] \delta y_{k+1}(t) + \Gamma \delta y_k(t)
\]
\[1 \leq p \leq m_k - 1 \tag{8}\]

where \(\lambda_k\) is an eigenvalue of the Laplacian \(L\), which may be a complex number.

Thus, the local synchronisation is equivalent to the local stability of the variational Equations (7) and (8). This fact is summarized by the following theorem:

**Theorem 1** (Lu and Chen 2006a): Let \(\lambda_2, \lambda_3, \ldots, \lambda_l\) be the nonzero eigenvalues of the coupling matrix \(L\). If all the variational equations

\[
\frac{d\mu(t)}{dt} = [Df(\bar{x}(t), t) - \omega_{\Delta} \Gamma] \mu(t) \quad k = 2, 3, \ldots, l \tag{9}\]

are exponentially stable, then the synchronisation manifold \(S\) is locally stable exponentially.

In the case of \(\Gamma = L_n\), then the stability of the variational equations can be described by their Lyapunov exponents:

\[
\mu - c \text{Re}(\lambda_k) < 0, \quad k = 2, \ldots, l,
\]

where \(\mu\) is the maximum Lyapunov exponent along the reference trajectory \(\bar{x}(t)\).

In the case that the graph is directed and symmetric, its Laplacian is symmetric. Then, the eigenvalues are all real. Assume that if there exists a threshold \(\rho^*\) such that for any \(\rho > \rho^*\), the following linear system:

\[
\frac{d\mu(t)}{dt} = [Df(\bar{x}(t), t) - \rho \Gamma] \mu(t) \tag{10}\]

is exponentially stable, then the region of the coupling strength by which the coupled system can be locally synchronised as

\[
c > \frac{\rho^*}{\lambda_2}. \tag{11}\]

This implies that the second largest nonzero eigenvalue of the Laplacian, which is also named the algebraic connectivity of the graph \(G\), can be used to measure the synchronisability of the complex network in the model (3), because a larger \(\lambda_2\) implies a larger region of the coupling strength which can synchronise the coupled system (3).

The method mentioned above is different but closely related to the well-known master stability function method proposed by Pecora and Carroll (1998).

In the literature, local synchronisation is studied via the means of local transverse stability analysis proposed by Ashwin, Buescu, and Stewart (1996) and the references therein.

Suppose that the uncoupled system

\[
\frac{d}{dt} z(t) = f(z(t)) \tag{12}\]

is an autonomous system and has an attractor \(A\), which can be an equilibrium, limit cycle, or chaotic/ strange one. Then, define

\[
S = S \cap A^m
\]

as the synchronisation manifold, where \(A^m\) denotes the Cartesian product. One can see that \(S\) is an invariant submanifold. Thus, local synchronisation can equivalently be defined by the fact that \(S\) is an attractor of the whole space \(\mathbb{R}^m\). Pecora and Carroll (1998) used this idea to study local synchronisation of a model of coupled oscillators which is more general than Equation (3):

\[
\frac{d}{dt} x^i(t) = f(x^i(t)) - c \sum_{j=1}^{m} l_{ij} H(x^j(t)), \quad i = 1, \ldots, m,
\]

where \(H(\cdot) : \mathbb{R}^m \to \mathbb{R}^m\) is an output map, which can be nonlinear.

Considering the local synchronisation of the coupled system (3), variational equations (9) can be replaced by along a trajectory \(z(t)\) of the uncoupled system (12):

\[
\frac{d\mu(t)}{dt} = [Df(z(t)) - \omega_{\Delta} \Gamma] \mu(t) \quad k = 2, 3, \ldots, l \tag{13}\]

Thus, we have the following proposition as a direct consequence from the results by Ashwin et al. (1996).

**Proposition 1:** Suppose \(\lambda_2, \lambda_3, \ldots, \lambda_l\) are the different eigenvalues of the coupling matrix \(L\), excluding the eigenvalue 0.

Let \(\mu_{\text{max}}(\mathcal{B})\) be the maximum Lyapunov exponent of all variational equations (13) corresponding to the ergodic measure \(\mathcal{B}\) over the synchronisation manifold \(S\).

1. If \(\sup_{\mathcal{B} \in \text{Erg}(S)} \mu_{\text{max}}(\mathcal{B}) < 0\), where \(\text{Erg}(S)\) denotes the ergodic measure space over \(S\), then the coupled system is asymptotically synchronised, i.e., there exists neighbourhood of \(S\) which is contained in the attracting basin of \(S\);
2. If \(\mathcal{B}\) is an SBR measure and \(\mu_{\text{max}}(\mathcal{B}) < 0\), then the coupled system is synchronised in the Milnor sense, i.e., there exists a measured set with positive measure which is contained in its attracting basin;
3. If \(\mathcal{B}\) is absolutely continuous with respect to the Lebesgue measure and \(\mu_{\text{max}}(\mathcal{B}) < 0\), then the coupled system is synchronised in the essential sense, i.e.,

\[
\lim_{\delta \to 0} \frac{l(BA(S) \cap B_\delta(S))}{l(B_\delta(S))} = 1
\]
where $BA(S)$ denotes the attracting basin of $S$, $B_δ(S)$ denotes the $δ$-neighbourhood of $S$ and $k(·)$ denotes the Lebesgue measure.

It can be seen that the definitions of local synchronisation depend on the definitions of attractors. For more details, readers can refer to Ashwin et al. (1996) and references therein.

In this proposition, the key assumption is that the trajectory $z(t)$ of the uncoupled system should contain an attractor. More precisely, if $z(t)$ is an equilibrium, then it should be a stable equilibrium; if $z(t)$ is a periodic trajectory, then it should be a limit cycle. In both cases, the synchronised coupled system (3) is actually asymptotically stable at the equilibrium or periodic trajectory. That is, all variational equations

$$\frac{du_k(t)}{dt} = [Df(z(t)) - cλ_k Γ]u(t), \quad k = 1, 2, \ldots, l \quad (14)$$

should be stable (note that the equation for $λ_1 = 0$ is included).

Also, one should note that in the case that the uncoupled system is chaotic, the synchronised coupled system (3) never implies

$$\lim_{t \to \infty} |x^i(t) - z(t)| = 0, \quad i = 1, \ldots, m$$

despite

$$\lim_{t \to \infty} |x^i(t) - x^j(t)| = 0, \quad i, j = 1, \ldots, m$$

hold. All transverse directions to the synchronisation manifold $S$ is stable, but the trajectory on $S$ has chaotic dynamics which is sensitive with respect to the initial conditions. Recall the reference trajectory $\bar{x}$ as defined above, which satisfies $\lim_{t \to \infty} |x^i(t) - \bar{x}(t)| = 0$ for all $i = 1, \ldots, m$, if the system is synchronised. One can see that $\bar{x}(t)$ satisfies

$$\dot{\bar{x}} = f(\bar{x}(t)) + u(t).$$

Here, $u(t)$ is an exponentially decaying term which must be nonzero in the case that the initial data $[x^1(0), \ldots, x^m(0)]^T \notin S$.

Even if $z(t)$ has the same initial value as $\bar{x}(t)$, the nonzero exponentially decaying term will lead the difference between $\bar{x}(t)$ and $z(t)$ in the finite time. Even if we can omit the exponential term after a sufficiently large while, noting that at this moment, $z(t)$ and $\bar{x}(t)$ are different even slightly, the chaotic dynamics (sensitive with respect to the initial conditions) of uncoupled system (12) causes that

$$\lim_{t \to \infty} |\bar{x}(t) - z(t)| > 0$$

holds.

In summary, investigating local synchronisation by discussing the local stability of the following systems

$$\frac{d(x^i(t) - z(t))}{dt} = [Df(z(t)) - cλ_k Γ](x^i(t) - z(t)), \quad (15)$$

where $k = 2, \ldots, l$, $i = 1, \ldots, m$, as done in Wang and Chen (2002a, b) and other papers, is inappropriate.

**Remark 1:** The method used by Lu and Chen (2006a) does not demand knowing the attractor structure of the uncoupled system. Without any information about the uncoupled system, the master stability function method might fail but the method in Theorem 1 still works. Moreover, in the case that the coupled system is a nonautonomous system, i.e. $f(x(t), t)$ as the right-hand side, the master stability function should be developed with certain uniform condition, the method proposed in Lu and Chen (2006a) can work without any modifications.

### 2.2. Global synchronisation

Different from the local synchronisation, the global synchronisation describes the image that starting with any initial data, the differences between the collective state trajectories converge to zero. Wu and Chua (1995) studied global synchronisation of the coupled system (3) by introducing a class of matrices with the following special structure.

**Definition 1:** Set $M^n_1$: $M^n_1$ is composed of matrices with $m$ columns. Each row (for instance, the $i$th row) of $M$ has exactly one entry $α_i$ and the other entry $-α_i$, where $α_i \neq 0$. All other entries are zeros.

**Definition 2:** Set $M^n_2$: $M^n_2$ is a subset of $M^n_1$. It consists of those matrices $M \in M^n_1$ with following property: for any pair of indices $i$ and $j$, we can find an integer $l$, indices $j_1, \ldots, j_l$ and $p_1, p_2, \ldots, p_l$, such that $j_1 = i$ and $j_l = j$, such that $M_{p_l j_1} \neq 0$ and $M_{p_l j_1+1} \neq 0$ for all $l \leq q < l$.

A direct lemma on these matrices is:

**Lemma 1** (Wu and Chua 1995): Let $L = M^T M$, $L \in \mathbb{R}^{m \times m}$ be a Laplacian of certain indirected graph $G$. And, $G$ is connected if and only if $M \in M^n_2$.

Moreover, for $x \in \mathbb{R}^m$, $M \in M^n_2$, and a positive definite matrix $P \in \mathbb{R}^n$, $x^T (M \otimes P)^T (M \otimes P) x = 0$ holds if and only if $x \in S$. Thus, the matrix belonging to $M^n_2$ can be used to construct a candidate Lyapunov function to measure the distance from the collective trajectories of the coupled system to the synchronisation manifold

$$V(x) = x^T (M^T M \otimes P)x.$$
For some positive definite matrix $P \in \mathbb{R}^{n \times n}$, a function $f: \mathbb{R}^{n} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{n}$ is said to be $P$-decreasing for some $\alpha \in \mathbb{R}$ if there exists a positive number $\delta > 0$ such that

$$(\xi - \zeta)^{T} P[f(\xi, t) - f(\zeta, t) - \alpha \Gamma(\xi - \zeta)]$$

$$< - \delta (\xi - \zeta)^{T} (\xi - \zeta)$$

(16)

holds for all $\xi, \zeta \in \mathbb{R}^{n}$. Then, we have

**Theorem 2** (Wu and Chua 1995): Suppose that $f$ is $P$-decreasing for some positive definite matrix $P$ satisfying $P \Gamma$ is semi-positive definite and a real number $\alpha$. If there exists $M \in \mathbb{R}^{m}_{+}$ such that

$$M^{T} (L - \alpha I_{m}) \geq 0$$

(17)

holds, then the coupled system (3) is globally exponentially synchronised.

Even though this condition is rather general, how to find $M$ such that inequality (17) holds is a problem. What more is, it is difficult to obtain the information from this result how the graph topology (indicated by its Laplacian) influences the synchronous dynamics. Thus, Lu and Chen (2006a) proposed a definitive description of the distance from the collective trajectories of the coupled system to the synchronisation subspace $S$. For the case that $L$ is irreducible, namely, the graph $G$ is strongly connected, this description is based on the left eigenvector of the Laplacian associated with the eigenvalue 0. Let $\xi = [\xi_{1}, \ldots, \xi_{m}]^{T}$ be the left eigenvector of $L$ associated with eigenvalue 0. Since $L$ is irreducible, $\xi$ has all components with the same sign. Therefore, we can safely suppose $\sum_{i=1}^{m} \xi_{i} = 1$ and $\xi_{i} > 0$ for all $i = 1, \ldots, m$. This can be summarised as the following theorem.

**Theorem 3** (Lu and Chen 2006a): Suppose that the graph $G$ is strongly connected. Let $\Xi = \text{diag}(\xi_{1}, \ldots, \xi_{m})$, where $[\xi_{1}, \ldots, \xi_{m}]^{T}$ is the left eigenvector corresponding to eigenvalue 0 with $\xi_{i} > 0$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \xi_{i} = 1$. If there exists a positive definite matrix $P \in \mathbb{R}^{m \times m}$ such that the $P$-decreasing condition (16) holds and $P \Gamma$ is semi-positive definite, and

$$\tilde{C}^{T} [\Xi (cL - \alpha I_{m})] \tilde{C} \geq 0,$$

(18)

where

$$\tilde{C} = \begin{bmatrix}
\frac{I_{m-1}}{\xi_{1}} & \frac{I_{m-1}}{\xi_{2}} & \cdots & \frac{I_{m-1}}{\xi_{m}}
\end{bmatrix},$$

then the coupled system (3) is globally exponentially synchronised.

In the situation that the graph is not strongly connected. Without loss of generality, we can rewrite its Laplacian as follows:

$$L = \begin{bmatrix}
L_{qq} & L_{qp} \\
0 & L_{pp}
\end{bmatrix},$$

(19)

where $L_{pp} \in \mathbb{R}^{m \times m}$ is irreducible.

First, we consider the subsystem

$$\frac{dx_{i}}{dt} = f(x(t), t) - c \sum_{j \in S_{i}} I_{q_{j}} x_{i}^{(j)}$$

$$i = m_{1} + \ldots + m_{p-1} + 1, \ldots, m.$$  

(20)

Suppose that the condition in Theorem 3 holds, i.e.

$$|x_{i}(t) - x_{j}(t)| = O(e^{-\beta t})$$

(21)

holds for all $i, j = m_{p} + 1, \ldots, m$ and some positive $\beta$. Define $y_{i}(t) = x_{i}(t) - x_{m_{p}}(t)$. Then, for $i = 1, \ldots, m - m_{p}$, we have

$$\dot{y}_{i} = f(x_{i}(t), t) - f(x_{m_{p}}(t), t) - c \sum_{j=1}^{m_{p}} I_{q_{j}} y_{j}(t) + O(e^{-\beta t}).$$

(22)

Furthermore, denote $Y(t) = [y_{1}(t), \ldots, y_{m_{p}}(t)]^{T}$, $\tilde{F}(Y(t), t) = \frac{1}{2} \sum_{i=1}^{m_{p}} (f(y_{i}(t), t) - f(x_{m_{p}}(t), t) - f(x_{i}(t), t))^{2}$, and $Y(t) = Y(t) + O(e^{-\beta t})$. Then, Equation (22) has the following compact form:

$$\dot{Y}(t) = \tilde{F}(Y(t), t) - c (L_{qq} \otimes \Gamma) Y(t) + O(e^{-\beta t}).$$

(23)

For certain positive definite diagonal matrix $\Xi$ with $m - m_{p}$ dimension and positive definite matrix $P$ with $n$ dimension, define a candidate Lyapunov function as follows:

$$V_{1}(Y) = \frac{1}{2} Y^{T} (\Xi \otimes P) Y^{T}.$$
there exists certain positive definite diagonal matrix \( \tilde{\Xi} \) such that
\[
[\tilde{\Xi}(cL_{qq} - \alpha I_{m-m_p})] \succeq 0, \tag{24}
\]
then the coupled systems (3) is globally exponentially synchronised.

Theorems 3 and 4 are closely related to Theorem 2. For instance, for the case that \( L \) is irreducible, by letting \( M^T M = \Xi - \xi \xi^T \), we can get Theorem 3 from Theorem 2. However, introducing \( \tilde{x}(t) \) as a reference state for the coupled system (1) in the synchronisation manifold and investigating synchronisation by \( x'(t) - \tilde{x}(t) \) gives more insight to the topic. Furthermore, more information about influence of the graph on synchronisation can be obtained. One example is to find the doorsill for chaos synchronisation. Suppose that the condition (16) holds for some \( \alpha > 0 \), which implies that the uncoupled system (12) is unstable, even has a chaotic attractor and \( P \Gamma \) is nonnegative definite. Irreducibility of \( L \) implies that \( \Xi L + L^T \Xi \) is positive definite, from which it can be concluded that a sufficient large \( c \) can guarantee that \([\Xi(cL - \alpha I_p)] \succeq 0\) for \( \alpha > 0 \). As for the situation that \( L \) is reducible and has Perron-Frobenius form
\[
L = \begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1p} \\
0 & L_{22} & \cdots & L_{2p} \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & L_{pp}
\end{bmatrix}, \tag{25}
\]
where \( L_{kk} \in \mathbb{R}^{m_k \times m_k} \), \( k = 1, \ldots, p \), are all irreducible matrices. So, if \( L_{kk} \), \( k = 1, \ldots, p - 1 \), are all \( M \)-matrix, then \( L_{qq} \) in the form (19) is an \( M \)-matrix, which implies that the condition (24) can hold for sufficiently large \( c \). This condition is equivalent to that at least one \( L_{kl} \neq 0 \), for all \( l = k + 1, \ldots, p \). That is, the vertex corresponding to the submatrix \( L_{pp} \) can access any other vertex. This statement can be summarised as follows:

**Theorem 5 (Wu 2005b):** Suppose that all assumptions in Theorem 2 holds for some \( \alpha > 0 \). The coupled system (3) can be globally exponentially synchronised for some coupling strength \( c \) if and only if the underlying graph \( G \) has spanning trees.

For the case that \( L \) is symmetric and irreducible, since the left eigenvector of \( L \) associated with the eigenvalue 0 is \( \xi = \frac{1}{m}[1, \ldots, 1]^T \) same with the right one, the condition (17) can be rewritten as:
\[
c > \frac{\alpha}{\lambda_2^G}, \tag{26}
\]
which also implies that the second largest nonzero eigenvalue of \( L \) (algebraic connectivity) can measure the synchronisability of the graph \( G \).

**Remark 2:** In the case of \( f = 0 \) and \( \Gamma = I_m \), the coupled system (3) becomes
\[
\frac{d}{dt}x'(t) = m \sum_{j=1}^{m} w_j [x'(t) - x'(t)], \quad i = 1, \ldots, m, \tag{27}
\]
which is just a mathematical generalisation of consensus algorithms in the static network. See Olfati-Saber, Fax, and Murray (2007) and references therein. As a direct consequence, it can reach a consensus if and only if the underlying graph has a spanning tree.

### 2.3. Synchronisability and algebraic connectivity of directed graph

From the discussions above, the synchronisability of the network can be described as a function of the graph Laplacian. For the indirected graph, it can be measured by the second largest nonzero eigenvalue due to inequalities (11) and (26). However, for a directed, asymmetric, and strongly connected graph, inequality (18) is equivalent to
\[
c \geq \alpha \frac{u^T \Xi u}{u^T (\Xi L) u} \tag{28}
\]
holds for all \( u \perp e \). Wu (2005a) introduced a new inner product
\[
\langle u, v \rangle_\Xi = u^T \Xi v,
\]
to define a generalised Rayleigh–Ritz ratio:
\[
\lambda_2^\Xi = \min_{u \perp e} \frac{\langle Lu, u \rangle_\Xi}{\langle u, u \rangle_\Xi}, \tag{29}
\]
which is named the algebraic connectivity of the strongly connected directed graph. From this inequality (28) can be rewritten as
\[
c \geq \frac{\alpha}{\lambda_2^\Xi}.
\]
This implies that \( \lambda_2^\Xi \) can serve to estimate the synchronisability of the directed graph.

As for a directed graph which is not strongly connected, the Perron-Frobenius theorem tells that the condition (24) in Theorem 4 can be divided into several parts according to the diagonal blocks of the Laplacian in the form (25). Since the block \( L_{pp} \) is irreducible, the algebraic connectivity of the subgraph corresponding to \( L_{pp} \) can be defined similar to the above and denoted by \( \lambda_2^\Xi \). For the block \( L_{ll} \), \( l = 1, \ldots, p - 1 \), which is also irreducible and a \( M \)-matrix, define another matrix \( \tilde{L}_{ll} \).
which has the same off-diagonal elements with $L_d$ and all row sums zero by letting each diagonal element equal to the minus of the sum of all off-diagonal elements in the same row. Let $\xi^i = [\xi^i_1, \ldots, \xi^i_n]^T$ be the left eigenvector of $L_{ij}$ associated with the eigenvalue 0 with $\sum_{j=1}^n \xi^i_j = 1$ and $\xi^i_j > 0$, $j = 1, \ldots, m$. Let $\Xi_i = \text{diag}(\xi^i_1, \ldots, \xi^i_n)$. One can conclude that $(\Xi L_d)^l$ is positive definite. Define the following Rayleigh–Ritz ratio:

$$\lambda^l_1 = \min_{u \in \mathbb{R}^n} \frac{(L_d u, u)}{(u, u)} \Xi_i,$$

where $(\cdot, \cdot)_{\Xi_i}$ is defined via the positive definite diagonal matrix $\Xi_i$. Thus, as a consequence from the condition (24), we have

$$c > \frac{\alpha}{\lambda^l_1}, \quad l = 1, \ldots, p - 1.$$  

(30)

Therefore, the synchronisability can be estimated as follows:

$$\min\{\lambda^1_1, \ldots, \lambda^{p-1}_1, \lambda^p_2\}.$$  

(31)

A larger value implies a large region of the coupling strength $c$ which can synchronise the coupled system (3) due to inequality (30), namely, a better synchronisability of a directed graph.

### 2.4. Synchronisation with a coupling time-delay

Due to the finiteness of signal transmission and switching speeds, coupling delay in a real-world network is inevitable, for example, see Nakamura, Tominaga, and Munakata (1994) and Rosenblum and Pikovsky (2004). In this article, as formulated by Buric and Todorovic (2003), we consider that a coupling delay occurs when the signals are transmitted to their interconnected nodes. In such situation, the coupled system (1) with a coupling delay is described as follows:

$$\frac{dx_i(t)}{dt} = f(x_i(t), t) - c \sum_{j=1, j \neq i}^m l_{ij} [x_j(t - \tau) - x_i(t)],$$  

(32)

where $\tau$ is the coupling delay. We define the following set:

$$S_c = \{x = [x^1(\theta), \ldots, x^m(\theta)]^T : x^i(\theta) \in C_T, x^i(\theta) = x^i(\theta),$$  

$$i, j = 1, \ldots, m \text{ and } \theta \in [-\tau, 0]\}$$

as a subspace of a Banach space $C_T = C([-\tau, 0], \mathbb{R}^n)$. In the case that the coupled system (32) can synchronise, i.e. the subspace $S_c$ is invariant through the evolution, with $x^i(t) = x^2(t) = \cdots = x^m(t) = z(t)$, we have the following synchronised state equation:

$$\frac{dz(t)}{dt} = f(z(t), t) - ca_i \Gamma [z(t - \tau) - z(t)], \quad i = 1, \ldots, m,$$

(33)

where $a_i = \sum_{j=1, j \neq i}^m l_{ij}$. Obviously, the synchronised trajectory $z(t)$ is uniform, i.e. the synchronisation subspace $S$ is invariant for the coupled system (32) for general $f(\cdot, \cdot)$, if and only if the following assumption is imposed:

$$a_1 = a_2 = \cdots = a_m = a$$  

(34)

for a uniform $a$. So, without loss of generality, we can assume $a = -1$ and put the common row sum to the coupling strength $c$.

Let $\tilde{x}(t)$ be the weighted average of $x^i(t)$, $i = 1, \ldots, m$ via the left eigenvector associated with the eigenvalue 0 by the way mentioned above. By linearization, the variational equation with $\delta x^i = x^i(t) - \tilde{x}(t)$ is written as follows:

$$\frac{d\delta x^i(t)}{dt} = Df(\tilde{x}(t), t)\delta x^i(t) - c \sum_{j=1}^m l_{ij} \Gamma \delta x^j(t - \tau)$$  

$$+ c\Gamma [\delta x^i(t) - \delta x^i(t)],$$

(35)

where $Df(\tilde{x}(t), t)$ is the Jacobian matrix of $f(\cdot, \cdot)$ at $\tilde{x}(t)$ and $i = 1, \ldots, m$. Let $L^T = VJ^T$ be the Jordan form of $L$. Similar to the means used in Section 2.1, let $\delta X(t) = [\delta x^1(t), \ldots, \delta x^m(t)] \in \mathbb{R}^{n,m}$ and $\delta Y(t) = \delta X(t)V = [\delta y^1(t), \ldots, \delta y^m(t)]$. Then, the variational equation of $\delta Y(t)$ is

$$\frac{d\delta Y(t)}{dt} = [Df(\tilde{x}(t), t) - c\Gamma] \delta Y(t)$$  

$$+ c\Gamma \delta Y(t)(-J + I_m).$$

(36)

Since the first column of $V$ is the left eigenvector of $L$, $\delta y^1(t)$ can be regarded as variance in the synchronisation subspace $S_c$. Extending from the concept of the master stability function introduced by Pecora and Carroll (1998), the stability of the following variational systems in the Banach space $C_T$ transverse to $S_c$:

$$\frac{d\varphi(t)}{dt} = [Df(\tilde{x}(t), t) - c\Gamma] \varphi(t) + c(1 - \lambda_c) \Gamma \varphi(t - \tau),$$

$$k = 2, \ldots, m$$  

(37)

is utilised to analyse the local stability of the synchronisation manifold. This leads to the following theorems.

**Theorem 6** (Lu et al. 2006b): Suppose that system (33) has an asymptotically stable attractor $A$ and $z(t)$ is a
solution of the coupled system (33) included in A. Let $Df(\tilde{x}(t), t)$ be the Jacobian of $f(\tilde{x}(t), t)$, $\mu = \lim_{t \to \infty} |Df(\tilde{x}(t), t)|$, denote the time average, $k_1 = \lim_{t \to \infty} |Df(\tilde{x}(t), t) - c I|$, and $k_2 = c \max_{k \geq 2} |1 - \lambda_k|$. If the following inequality holds:

$$c > 1 - \max_{k \geq 2} |1 - \lambda_k|$$

(38)
or the following two inequalities

$$\beta = c \min_{k \geq 2} Re(\lambda_k) - (|Df(\tilde{x}(t), t)|) > 0$$

$$e^{2\beta \tau} < \frac{\beta}{c(k_1 + k_2) \max_{k \geq 2} |1 - \lambda_k|}$$

(39)

hold, then the coupled system (32) is locally exponentially synchronised.

One can see that if the graph $G$ has spanning trees, then we have $|1 - \lambda_k| \leq 1$ for all $k \geq 2$ due to the Gershgorin disc theorem. If $f(\cdot, \cdot)$ is a uniformly Lipschitz function, which implies $\mu$ can be estimated independently of both $c$ and $\tau$, then a sufficiently large coupling strength $c$ can guarantee that the coupled system synchronises with an arbitrary coupling delay $\tau$. And, from the inequality (38), one can use the quantity

$$cap := 1 - \max_{k \geq 2} |1 - \lambda_k|$$

as an index to estimate the synchronisability of a coupling configuration with an arbitrary delay. The larger the $cap$ is, the smaller the coupling strength $c$ is needed to synchronise the coupled system (32) no matter how large the coupling delay $\tau$ is. Because $|1 - \lambda_k| \leq 1 < 1 + Re(\lambda_k)$ holds for all $k \geq 2$, the first inequality of (39) is easier to be satisfied than (38). Furthermore, if $\tau = 0$, one can see that inequalities (39) can hold if $c$ is large enough because of the existence of a synchronised compact attractor. According to the continuous dependence, although we might not find $c$ to guarantee the synchronisation for any delays, we can find $c$ to synchronise the coupled system with a small delay.

Remark 3: In fact, synchronisation of the coupled system (32) is rather complex. For example, as reported by Dhamala, Jirsa, and Ding (2004), a large coupling delay might enhance synchronisation in coupled Hindmarsh–Rose (HR) neurons with coupling delay. This phenomenon is because the synchronised state of system (33) itself depends on parameters $c$ and $\tau$. Different delays indicate different possible dynamics of the synchronisation manifold.

Global synchronisation can be studied by similar means in Section 2.2 and more details can be referred to in Lu et al. (2006b).

2.5. Synchronisation in discrete-time networks

The alternative form of complex networks is named networks of coupled maps:

$$\dot{x}(t) + 1 = g(x(t)) - \varepsilon \sum_{j=1}^{n} l_j [g(x(t)) - g(x(j))],$$

(40)

$$i = 1, \ldots, m,$$

where $t \in \mathbb{Z}^+$ denotes the discrete-time, $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous map, and $\varepsilon$ is the coupling strength. Let $x(t) = [x^1(t), \ldots, x^m(t)]^T$ and $G(x(t)) = [g(x^1(t))^T, \ldots, g(x^m(t))^T]^T$. Equation (40) can be rewritten as the following compact form:

$$x(t + 1) = [(I_m - \varepsilon L) \otimes I_d] G(x(t)).$$

(41)

The similar methods can be employed to study its local and global synchronisation.

For local synchronisation, linearising the equation near the weighted average $\tilde{x}(t)$ of all trajectories via the left eigenvector of $L$ associated with the eigenvalue 0 with certain linear transformation transverse to the synchronisation subspace $S$ gives

$$u(t + 1) = (1 - \varepsilon \lambda_k) Dg(\tilde{x}) u(t), \quad k = 2, \ldots, m,$$

(42)

where $0 = \lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $L$. Noting that $\lambda_1 = 0$ corresponds to the synchronisation subspace, the Lyapunov exponents transverse to $S$ are $\mu + \log |1 - \varepsilon \lambda_k|$, $k = 2, \ldots, m$, where $\mu$ is the maximum Lyapunov exponent along $\tilde{x}(t)$. Therefore, the local synchronisation criterion is

$$\mu + \log |1 - \varepsilon \lambda_k| < 0, \quad k = 2, \ldots, m.$$  (43)

To discuss global synchronisation, we define candidate Lyapunov function similar to Section 2.2:

$$V(x) = x^T (M^T M \otimes P)x,$$

(44)

where $M \in \mathcal{M}_n^m$ and $P \in \mathbb{R}^{n \times n}$ is positive definite. This leads to the following theorem:

Theorem 7 (Lu and Chen 2004): Suppose that $g(\cdot)$ satisfies

$$[g(\xi) - g(\zeta)]^T P [g(\xi) - g(\zeta)] \leq \kappa^2 (\xi - \zeta)^T P (\xi - \zeta).$$

(45)

for some $\kappa > 0$, some positive definite matrix $P$ and all $\xi, \zeta \in \mathbb{R}^n$. If there exists some $M \in \mathcal{M}_n^m$ such that

$$(I_m - \varepsilon L)^T M^T M (I_m - \varepsilon L) \leq \frac{1}{\kappa^2} M^T M,$$

(46)

then the coupled system (40) is globally exponentially synchronised.

A realization of Theorem 7 is made by using the similar idea as in theorem 4 via the left eigenvalue of $L$.
associated with the eigenvalue 0: \( \xi = [\xi_1, \ldots, \xi_m] \) with \( \xi_i \geq 0, i = 1, \ldots, m \), and \( \sum_{i=1}^{m} \xi_i = 1 \).

**Theorem 8** (Lu and Chen 2007): Let \( \Xi = e \otimes \xi \), where \( e = [1, 1, \ldots, 1]^T \). Suppose condition (45) is satisfied. If \( \kappa (I_m - \Xi - \epsilon L) \) is diagonally stable, i.e. there exists a positive definite diagonal matrix \( T \) such that

\[
\kappa^2 (I_m - \Xi - \epsilon L)^T (I_m - \Xi - \epsilon L) < T
\]

holds, then the coupled system (40) is globally exponentially stable.

Denote by \( PD \) the set of all positive definite diagonal matrices. Define the following Rayleigh–Ritz ratio:

\[
\gamma (e) = \max_{P \in PD} \min_{x \in \mathbb{R}^n} \frac{\langle x, x \rangle_P}{(y(x), y(x))_P},
\]

where \( y(x) = (I_m - \Xi + \epsilon L)x \). If \( \gamma (e) > \kappa \), then by Theorem 8, the coupled system (40) can be globally synchronised. Thus, we can define \( \kappa (L) = \sup_{e > 0} \gamma (e) \) as an estimation of the synchronisability for the coupled system with the coupling matrix \( L \). If \( g(\cdot) \) is Lipschitz continuous with Lipschitz constant \( \kappa \) and \( \eta (L) > \kappa \), then the coupled system (40) can be globally synchronised for some properly chosen coupling strength \( \epsilon \). A basic problem arises as to which kind of directed graph topologies can be a door in as a chaotic synchroniser, i.e. \( \eta (L) > 1 \). The following tells that the result is same with the continuous-time sutation:

**Theorem 9** (Lu and Chen 2007): A graph \( G \) contains a spanning directed tree if and only if \( \eta (G) > 1 \).

Synchronisation under a coupling delay was studied by Atay, Jost, and Wende (2004).

### 3. Time-varying topologies

In many scenarios, complexity of networks lies on dynamical topologies which is time-varying due to vertex/edge failure or creation, movement of agents, asynchronous updating and topology evolution and so on. In the case that the time scale of topology variation and that of dynamics evolution are same or comparable, one should consider complex network with time-varying topologies.

A time-varying graph is a triple set series \( G(t) = \{V, E(t), W(t)\} \), where \( V = \{1, \ldots, m\} \) is a fixed vertex set, \( E(t) = \{e(i, j)\} \) denotes the edge set at time \( t \) and \( W(t) \) denotes the corresponding weight set at time \( t \). The Laplacian of the time-varying graphs \( G(t) \) is a matrix function upon time:\( L(t) = \{l_{ij}(t)\}_{i,j=1}^{m} \) denoted by the same way with the static case: for \( i \neq j \), \( l_{ij}(t) = -w_{ij}(t) \) if \( e(i, j) \in E(t) \) and \( l_{ij}(t) = 0 \) otherwise; the diagonal elements are row balanced, i.e.

\[
l_{ii}(t) = -\sum_{j=1,j \neq i}^{m} l_{ij}(t). \]

Thus, the coupled system with time-varying topologies can be written as:

\[
\frac{dx(t)}{dt} = f(x(t), t) - c \sum_{j=1}^{m} l_{ij}(t) \Gamma x(t), \quad i = 1, \ldots, m.
\]

Wu (2003) directly employed the means of the static topology to the case of time-varying topologies, which can be summarised as the following theorem:

**Theorem 10** (Wu 2003): Suppose the assumptions in Theorem 2 satisfied for some \( \alpha \in \mathbb{R} \). If there exists \( M \in \mathcal{M}_2^m \) such that

\[
M^T M \{\alpha L(t) - a I_m\} \geq 0,
\]

then the coupled system (48) is globally exponentially synchronised.

Wu and Chen (2008) gave a criterion for global synchronisation of coupled neural networks with time-varying couplings, which is easy to be verified. In many cases, time-varying topologies are not simply a function of time but have certain internal regularity, for example, induced by certain stochastic signal. Belykh, Belykh, and Hasler (2004) proposed a blinking small-world model based on the famous small-world model by rewiring vertices every fixed time period, following an independent identical Bernulli distribution. This model actually induces a time-varying topology process. They studied its global synchronisation by defining a Lyapunov function to estimate the distance from the collective states to the synchronisation subspace. Stilwell, Bolitt, and Roberson (2006) studied the local synchronisation of coupled systems with time-varying topologies with assumption that the time-average of the Laplacians is fixed for time intervals of a given length \( T \). They both proved that if the time-average topology can synchronise the coupled system, then a sufficient fast switching can guarantee synchronous dynamics through the time-varying topologies.

More generally, time variation of topologies can be defined by a metric dynamical system \{\( \Omega, \mathcal{F}, \mathbb{P}, \theta^{(t)}\)\}, where \( \Omega \) is the state space, \( \mathcal{F} \) is the Borel-\sigma algebra, \( \mathbb{P} \) is the probability measure and \( \theta^{(t)} \) is a shift map upon \( \Omega \) satisfying the semi-group property: \( \theta^{(t + s)} = \theta^{(t)} \circ \theta^{(s)} \) (id denotes the identity map). The time-varying graph is denoted by \( G(\theta^{(t)} \omega) \) and its Laplacian is \( L(\theta^{(t)} \omega) \) accordingly. Thus, the coupled system with time-varying topologies can modelled as:

\[
\frac{dx(t)}{dt} = f(x(t), t) - c \sum_{j=1}^{m} l_{ij}(\theta^{(t)} \omega) \Gamma x(t), \quad i = 1, \ldots, m.
\]
This equation is in fact a random dynamical system (RDS), where \( \theta(t) \) can include both deterministic (\( L(\cdot) \) is a fixed function of time) and stochastic (\( L(\cdot) \) is induced by a stochastic process) cases. Lu et al. (2007) studied the discrete-time version of this model—networks of coupled maps with time-varying topologies:

\[
x'(t+1) = g(x'(t)) - \epsilon \sum_{j=1}^{m} l_{ij}(\theta(t))g(x'(t)), \quad i = 1, \ldots, m,
\]

and obtained local synchronisation criterion by calculating the transverse Lyapunov exponents:

\[
\mu + \lambda_2 < 0,
\]

where \( \mu \) is the maximum Lyapunov exponent of the uncoupled system and \( \lambda_2 \) is the largest Lyapunov exponent of the linear system:

\[
\dot{u}'(t+1) = u'(t) - \epsilon \sum_{j=1}^{m} l_{ij}(\theta(t))u'(t), \quad i = 1, \ldots, m,
\]

except the Lyapunov exponent along the synchronisation subspace \( S \), which equals 0. The synchronisation analysis of continuous-time networks with time-varying topologies (50) is still ongoing.

4. Pinning control

Synchronisation, or equivalently, the stability of the synchronisation manifold, is a phenomenon that all trajectories over vertices converge to the synchronisation manifold. However, the final synchronised trajectory depends on the initial data of the coupled system and is unknown at the initial time if the uncoupled system is sensitive dependent on whether the initial condition, for example, possesses a chaotic attractor.

Different from synchronisation, pinning control is a technique to make a complex network converge to a given homogeneous trajectory of the uncoupled system. Due to the large sizes of complex networks, global control of networked system is rather expensive. Li, Wang, and Chen (2004) conceived a local alternative means by setting a small fraction of controllers:

\[
\frac{d}{dt}x'(t) = f(x'(t), t) - \epsilon \sum_{j=1}^{m} l_{ij}\Gamma x'(t) + c\delta_D(i)[z(t) - x'(t)], \quad i = 1, \ldots, m,
\]

where \( \epsilon \) is the pinning feedback strength, \( c \) is the pinning feedback gain, \( D \) is a subset of vertex set which is pinned, \( \delta_D(i) \) is Dirac-\( \delta \) function:

\[
\delta_D(i) = \begin{cases} 1 & i \in D \\ 0 & i \notin D \end{cases}
\]

and \( z(t) \) is a given trajectory of the uncoupled system (12). This aim of control is to pin all trajectories \( x'(i), i = 1, \ldots, m, \) to the given homogeneous one \( z(t) \), i.e. \( \lim_{t \to \infty} |x'(i) - z(t)| = 0 \) holds for all \( i = 1, \ldots, m \).

We can use the idea of synchronisation analysis of directed networks to interpret the pinning control problem. Consider the initial graph \( G = \{ V, E, W \} \). The consequence of setting controllers on vertices belong to \( D \) is obtaining another directed graph \( G' = \{ V', E', W' \} \). Here, the vertex set \( V' = V \cup \{ v_0 \} \), where \( v_0 \) represents one vertex over which is the uncoupled system (12), \( E' = E \cup \{ e(i,v_0) : i \in D \} \) is added by edges linking from \( v_0 \) to the vertices which is set by controllers, \( W' = W \cup \{ w_{i,v_0} = \epsilon : i \in D \} \), where the weights of additional edges are all set by \( \epsilon \). See Figure 1. Thus, we can define the corresponding Laplacian by the same way as above: \( L' = [\ell_{ij}' \mid i,j=1,\ldots,m+1] \). Therefore, the coupled system (52) can be written as follows:

\[
\frac{d}{dt}x'(t) = f(x'(t), t) - c \sum_{j=1}^{m+1} \ell_{ij}'\Gamma x'(t), \quad i = 1, \ldots, m,
\]

where \( v_0 \) is labelled by \( m+1 \). By this transformation, pinning the complex network to a homogeneous trajectory of the uncoupled system is equivalent to synchronisation of the new master-slave networks of coupled system (53) (see Chen et al. 2007) and the analysis means that synchronisation of directed complex networks can be directly employed.

Based on this idea, Chen et al. (2007) proposed a minimum-cost method of pinning control by setting only one controller to the root vertex of a network possessing spanning trees. More generally, Lu, Li, and Rong (2009) proved the minimum number of

Figure 1. The initial graph is in the dotted frame and the vertex outside is set to control vertices linked from it.
controllers to realise pinning control to arbitrary directed graphs. Furthermore, Li et al. (2004), Sorrentino (2007), Sorrentino, di Bernardo, Garofalo, and Chen (2007), and Lu et al. (2009) discussed the controllability of the pinning strategies, which is equivalent to the synchronizability of the new network $G$ and found out that a selective pinning strategy is much better than a random one.

5. Conclusions

It is well known that the core point of theoretical analysis of synchronisation in complex networks is to reveal the relation between graph topology and synchronisability. In spite of much literature concerned with the indirected graph with the symmetry weights, with which the synchronisability of a graph can be represented by the spectral properties of the corresponding Laplacians, synchronisation analysis of complex networks with directed topologies and asymmetry Laplacians is the focal one in this article. Transverse stability theory is used for synchronisation analysis. Algebraic graph theory bridges the graph theory and matrix analysis, which helps to understand the structure of directed graphs. By these two techniques, some concepts for indirected graph, for example, algebraic connectivity, are extended to the directed case and used to describe the synchronisability of a directed graph. It has been reported that the directed topology and asymmetry weights can either enhance or reduce synchronisability. In fact, many real-world complex networks should be modelled by directed and asymmetry graphs, for instance, the protein function networks, food web and neuronal networks. Therefore, it is significant to reveal the direction and the actual influence of the directed topology and asymmetry weights on the synchronisability, which is still in mist. Moreover, control of complex networks is another important and interesting topic. As we have pointed out, pinning control can be regarded as synchronisation of certain directed network, even if the original is indirected. With a given pinning gain strength and graph topology, the minimum vertex number of pinning strategy is another significant topic. As one can see, this problem is still related to the influence of topology of directed graph on synchronisability.

References


