GLOBAL CONVERGENCE OF DELAYED NEURAL NETWORK SYSTEMS

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In this paper, without assuming the boundedness, strict monotonicity and differentiability of the activation functions, we utilize a new Lyapunov function to analyze the global convergence of a class of neural networks models with time delays. A new sufficient condition guaranteeing the existence, uniqueness and global exponential stability of the equilibrium point is derived. This stability criterion imposes constraints on the feedback matrices independently of the delay parameters. The result is compared with some previous works. Furthermore, the condition may be less restrictive in the case that the activation functions are hyperbolic tangent.

Keywords: Neural networks; time delays; exponential stability; asymptotic stability; global convergence; Lyapunov function.

1. Introduction

It is well-known that cellular neural networks (CNNs), proposed by L. O. Chua and L. Yang in 1988,\textsuperscript{10,11} have been extensively studied both in theory and applications. They have been successfully applied in signal processing, pattern recognition and associative memories, especially in static image treatment.\textsuperscript{12} Such applications heavily rely on the dynamic behaviors of the neural networks. Therefore, the analysis of these dynamic behaviors is a necessary step for the practical design of neural networks.

In hardware implementation, time delays inevitably occur due to the finite switching speed of the amplifiers and communication time. What’s more, to process moving images, one must introduce time delays in the signals transmitted among the cells.\textsuperscript{9} Neural networks with time delays have much more complicated dynamics due to the incorporation of delays.\textsuperscript{9} Nevertheless, some useful results on the stability analysis of delayed cellular neural networks (DCNNs) have already been obtained in Refs. 3–5, 8, 13, 17, 18, 23, 31. In Refs. 3, 13 and 23, some criteria related to the global asymptotic stability independent of delays were obtained by use of the Lyapunov function method. In Refs. 4, 5 and 8, some sufficient conditions ensuring the global exponential stability of DCNNs were given. In Refs. 17 and 18, the asymptotic stability and absolute stability of the delayed system were considered. In Ref. 31, a model of CNNs which contains variable, unbounded delays was investigated.

In this paper, we present some sufficient conditions for the uniqueness and global exponential stability of the equilibrium point for a class of delayed neural networks models. The conditions in our results are less restrictive, so some previous works of other researchers are extended.

This paper is organized in the following way. In Sec. 2, we give the model description and establish some lemmas related to the existence and uniqueness
of the system. In Sec. 3, we apply the lemmas obtained in Sec. 2 to give detailed stability analysis of the delayed model. We also compare and contrast our results with others from the literature in the remarks following the main theorem. A less restrictive criterion for a special case of the dynamical system is obtained in Sec. 4. In Sec. 5, we investigate the existence and exponential stability of unique equilibria of bidirectional associative memory (BAM) neural network. And conclusions follow in Sec. 6.

2. Model Description and Uniqueness of the Equilibrium Point

In this paper, we deal with delayed neural networks systems described by the following differential equations with time delays:

\[ \frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \tau_j)) + I_i \quad i = 1, 2, \ldots, n \]

where \( n \) corresponds to the number of units in a neural network, \( u_i(t) \) corresponds to the state of the \( i \)th unit at time \( t \), \( g_j(u_j(t)) \) denotes the output of the \( j \)th unit at time \( t \). \( a_{ij}, b_{ij}, d_i \) are constants, \( a_{ij} \) denotes the strength of the \( j \)th unit on the \( i \)th unit at time \( t \), \( b_{ij} \) denotes the strength of the \( j \)th unit on the \( i \)th unit at time \( t - \tau_j \). \( I_i \) denotes the input to the \( i \)th unit, \( \tau_j \) corresponds to the transmission delay along the axon of the \( j \)th unit and is a nonnegative constant, \( d_i \) represents the positive rate with which the \( i \)th unit will reset its potential to the resting state in isolation when disconnected from the network and the external inputs \( I_i \).

System (1) can be rewritten for \( u = (u_1, u_2, \ldots, u_n)^T \) as

\[ \frac{du(t)}{dt} = -Du(t) + Ag(u(t)) + Bg(u(t - \tau)) + I \]

where \( T \) denotes transpose, \( D = \text{diag}(d_1, d_2, \ldots, d_n), g(u) = (g_1(u_1), g_2(u_2), \ldots, g_n(u_n))^T, I = (I_1, I_2, \ldots, I_n)^T, \tau = (\tau_1, \tau_2, \ldots, \tau_n)^T, A = \{a_{ij}\} \) is the feedback matrix, \( B = \{b_{ij}\} \) is the delayed feedback matrix.

**Definition 1**

A real \( n \times n \) matrix \( A = \{a_{ij}\} \) is said to be an M-matrix if \( a_{ij} \leq 0, i, j = 1, 2, \ldots, n, i \neq j \), and all successive principle minors of \( A \) are positive.

**Definition 2**

A real square matrix \( A \) is said to be Lyapunov Diagonally Stable (LDS) if there exists a positive diagonal matrix \( P \) such that the symmetric part of \( PA, i.e., [PA]^T \), is positive definite.

**Definition 3**

A continuous function \( g : \mathbb{R}^n \to \mathbb{R}^n \) is of class \( G \) onto itself if \( H \in C^0, H \) is one-to-one, \( H \) is onto and the inverse map \( H^{-1} \in C^0 \).

**Definition 4**

A continuous function \( g : \mathbb{R}^n \to \mathbb{R}^n \) of the form \( g = (g_1, g_2, \ldots, g_n)^T \) is said to be of class \( G\{G_1, G_2, \ldots, G_n\} \), where \( G = \text{diag}(G_1, G_2, \ldots, G_n) \) with \( 0 < G_i < +\infty, i = 1, 2, \ldots, n \), if the function \( g(x) \) satisfies

\[ 0 \leq \frac{g_i(x) - g_i(y)}{x - y} \leq G_i \]

for each \( x, y \in \mathbb{R}, x \neq y \) and for \( i = 1, 2, \ldots, n \).

For system (2), Chen has proved the following global convergence theorem (see Ref. 5).

**Theorem A**

Suppose that \( g(x) \in G\{G_1, G_2, \ldots, G_n\} \) and \( DG^{-1} - |A| - |B| \) is an M-matrix, then any solution of (2) converges to a unique equilibrium \( u^* \) exponentially.

By use of the theory of the M-matrix,\(^{25}\) we can get many similar results ensuring the global stability of the delayed system, for example, see Refs. 4, 5 and 13.

Conditions in Theorem A and many related criteria are explicit and easily verified in practice. But they neglect the signs of entries in the connection matrices \( A \) and \( B \), and thus, the difference between excitatory and inhibitory effects might be ignored. Recently, Van Den Driessche and Zou,\(^{30}\) Arik and Tavsanoglu,\(^{3}\) Zhang,\(^{32}\) Liao and Wang,\(^{23}\) Joy\(^{17,18}\) attempted to overcome this disadvantage. But most of them only considered the asymptotic stability of the system with either bounded or strictly
monotonic increasing activation function. In the following section we will further these results with only the assumption that \( g(x) \in G\{G_1, G_2, \ldots, G_n\} \). Clearly, this output function is more general than the piecewise-linear function \( g_i(x) = \frac{1}{2}(\|x+1\| - \|x-1\|) \) used in the traditional DCNNs, which were investigated extensively by many researchers. Furthermore, we consider the exponential stability instead of global asymptotic stability.

Before we establish a criterion for the global exponential stability of system (2), we give some lemmas concerning the existence and uniqueness of the equilibrium point of (2).

**Lemma 1**

(Forti and Tesi\textsuperscript{14}) Suppose that \( g \in G\{G_1, G_2, \ldots, G_n\} \) and \( DG^{-1} - A \in LDS \), then we have

1. \( H(u) = -Du + Ag(u) + I \) is a homeomorphism of \( R^n \) onto itself
2. the system \( \frac{du(t)}{dt} = -Du(t) + Ag(u(t)) + I \) has a unique equilibrium point for each \( I \in R^n \).

**Lemma 2**

Suppose that \( g \in G\{G_1, G_2, \ldots, G_n\} \) and there exist positive diagonal matrices \( P \) and \( Q \) such that

\[
2PDG^{-1} - (PA + AT P) - PBQ^{-1}(PB)^T - Q > 0
\]

i.e., the left matrix is positive definite, then, for each \( I \in R^n \), system (2) has a unique equilibrium point.

**Proof**

From (4), we get

\[
2PDG^{-1} - (PA + AT P) > PBQ^{-1}(PB)^T + Q
\]

By the inequality \( [Q^{-\frac{1}{2}}(PB)^T - Q^\frac{1}{2}]^T[Q^{-\frac{1}{2}}(PB)^T - Q^\frac{1}{2}] \geq 0 \), we have

\[
PQ^{-1}(PB)^T + Q \geq PB + (PB)^T
\]

so (5) becomes

\[
2PDG^{-1} > P(A + B) + (A + B)^T P
\]

i.e.,

\[
(PDG^{-1} - (A + B))^S > 0
\]

which implies \( DG^{-1} - (A + B) \in LDS \) by Definition 2. From Lemma 1, \( H(u) = -Du + Ag(u) + Bg(u) + I \) is a homeomorphism of \( R^n \) onto itself and system (2) has a unique equilibrium point for each \( I \in R^n \). \( \Box \)

### 3. Global Exponential Stability Result

In this section, we will prove the following result.

**Theorem 1**

Suppose that \( g \in G\{G_1, G_2, \ldots, G_n\} \) and there exist positive diagonal matrices \( P \) and \( Q \) such that

\[
2PDG^{-1} - (PA + AT P) - PBQ^{-1}(PB)^T - Q > 0
\]

Then, for each \( I \in R^n \), system (2) has a unique equilibrium point which is globally exponentially stable, independent of the delays.

**Proof**

By Lemma 2, we know that for each \( I \in R^n \), system (2) has a unique equilibrium, namely, \( u^* \). By means of coordinate translation \( x(t) = u(t) - u^* \), (2) can be rewritten as

\[
\frac{dx(t)}{dt} = -Dx(t) + A\varphi(t) + B\varphi(t - \tau)
\]

where \( x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))^T, \varphi(\cdot) = (\varphi_1(x_1(\cdot)), \ldots, \varphi_n(x_n(\cdot))^T, \varphi_i(x_i(\cdot)) = g_i(x_i(\cdot) + u_i^*) - g_i(u_i^*) \) \( i = 1, 2, \ldots, n. \)

Clearly, \( u^* \) is globally exponentially stable for (2) if and only if the trivial solution of (10) is globally exponentially stable.

To analyze the global stability of the origin of (10), consider following Lyapunov functional

\[
V(x(t), t) = \sum_{i=1}^{n} x_i^2 e^{\epsilon t} + 2\alpha \sum_{i=1}^{n} P_i e^{\epsilon t} \int_{0}^{x} \varphi_i(\rho) d\rho + (\alpha + \beta) \sum_{i=1}^{n} Q_i 
\times \int_{t-\tau_i}^{t} \varphi_i^2(x_i(s)) e^{(s+\sigma_i)} ds
\]

where positive constants \( \alpha \) and \( \beta \) are to be decided and \( \epsilon > 0 \) is a small real number.
Differentiating $V$ along the solution of (10), we have
\[
\dot{V}(x(t), t) = \varepsilon e^{\varepsilon t} x^T x + 2e^{\varepsilon t} x^T (-Dx + A\varphi(t)) + B\varphi(t - \tau) + 2\alpha e^{\varepsilon t} \sum_{i=1}^{n} P_i \\
+ \int_{0}^{\tau} \varphi_i(p) dp + 2\alpha e^{\varepsilon t} \varphi^T(t) \\
\times [-PDx + PA\varphi(t) + PB\varphi(t - \tau)]
\]

Thus,
\[
\dot{V}(x(t), t) \leq e^{\varepsilon t} \left\{ 2x^T \left( D + \frac{1}{2} I + \frac{1}{2} \alpha PG \right) x + 2x^T A\varphi(t) + 2x^T B\varphi(t - \tau) + 2\alpha [\varphi(t)P \varphi(t)] + \varphi^T(t)PA\varphi(t) + \varphi^T(t)PB\varphi(t - \tau) \right\} + (\alpha + \beta)[\varphi^T(t)E^\tau Q \varphi(t)e^{\varepsilon t} + (\alpha + \beta)[\varphi^T(t)E^\tau Q \varphi(t)e^{\varepsilon t} + \varphi^T(t)E^\tau Q \varphi(t) - \varphi^T(t)Q \varphi(t - \tau)]
\]

Let $\varepsilon = \alpha \varepsilon$, then the above inequality changes to
\[
\dot{V}(x(t), t) \leq e^{\varepsilon t} \left\{ 2x^T \left( \frac{D}{2} + \frac{\varepsilon}{2\alpha} I + \frac{\varepsilon}{2} PG \right) x + 2x^T A\varphi(t) + 2x^T B\varphi(t - \tau) + 2\alpha [\varphi(t)P \varphi(t)] + \varphi^T(t)PA\varphi(t) + \varphi^T(t)PB\varphi(t - \tau) \right\} + (\alpha + \beta)[\varphi^T(t)E^\tau Q \varphi(t) - \varphi^T(t)Q \varphi(t - \tau)]
\]

Now, we will choose suitable parameters $\varepsilon$, $\alpha$, $\beta$ in order to prove $\dot{V}(x(t), t) \leq 0$.

Firstly, we choose a fixed positive $\beta$ such that
\[
\beta > \frac{||B||^2||D^{-1}||}{\min_i Q_i} \tag{17}
\]
where the matrix norm $|| \cdot ||$ is defined as $||X|| = (\lambda_{\max}(X^TX))^{\frac{1}{2}}$, $\lambda_{\max}$ is the largest eigenvalue of the matrix.

Secondly, we choose a sufficiently small $\varepsilon > 0$ and a sufficiently large $\alpha > 0$ such that
\[
D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG > 0 \tag{18}
\]
\[
\frac{\varepsilon}{2\alpha} ||D^{-1}|| + \frac{\varepsilon}{2} ||PGD^{-1}|| \leq 1 - \frac{||B||^2||D^{-1}||}{\min_i Q_i} \tag{19}
\]

Moreover, by the inequality (9), we have
\[
\alpha [2PGD^{-1} - (PA + A^T P) - PBQ^{-1}(PB)^T - E^{\frac{\varepsilon}{2}} Q] - \beta E^{\frac{\varepsilon}{2}} Q - A^T \\
\times \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} A > 0 \tag{20}
\]

for the sufficiently large $\alpha$, sufficiently small $\varepsilon$ and fixed $\beta$.

From Eq. (19), we also have
\[
\beta \geq \frac{\min_i Q_i (1 - \frac{\varepsilon}{2\alpha} ||D^{-1}|| + \frac{\varepsilon}{2} ||PGD^{-1}||)}{||B||^2 ||D^{-1}||} \tag{21}
\]

where we use the inequality (Ref. 15): $||(I-F)^{-1}|| \leq \frac{1}{1-||F||}$ for $||F|| \leq 1$. Thus,
\[
\beta \geq \frac{B^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} B}{\min_i Q_i} \tag{22}
\]

Up to now, we have chosen the parameters and got two inequalities (20) and (22).

Now, we are to prove $\dot{V}(x(t), t) \leq 0$. Notice that $D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG$ is positive definite from (18), we have
\[-x^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right) x + 2x^T A\varphi(t) = - \left[ \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{\frac{1}{2}} x - \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-\frac{1}{2}} A\varphi(t) \right]^T \times \left[ \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{\frac{1}{2}} x - \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-\frac{1}{2}} A\varphi(t) \right] + \varphi^T(t) A^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} A\varphi(t) \leq \varphi^T(t) A^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} A\varphi(t) \] (23)

Similarly, we have

\[-x^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right) x + 2x^T B\varphi(t - \tau) \leq \varphi^T(t - \tau) B^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} B\varphi(t - \tau) \] (24)

\[-\alpha \varphi^T(t - \tau) Q\varphi(t - \tau) + 2\alpha \varphi^T(t) PB\varphi(t - \tau) \leq \alpha \varphi^T(t) PBQ^{-1}B^T P\varphi(t) \] (25)

Therefore, from Eq. (16), we have

\[ \dot{V}(x(t), t) \leq \varepsilon^2 \dot{x}^T \left[ 2\alpha PDG^{-1} - \alpha(PA + A^T P) - \alpha PBQ^{-1}B^T P - (\alpha + \beta)E^{\frac{1}{2}}Q - A^T \right] \times \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} A \varphi(t) + \varphi^T(t - \tau) \left[ B^T \left( D - \frac{\varepsilon}{2\alpha} I - \frac{\varepsilon}{2} PG \right)^{-1} B - \beta Q \right] \varphi(t - \tau) \] (26)

By Eqs. (20) and (22), we obtain

\[ \dot{V}(x(t), t) \leq 0 \] (27)

Therefore, \( V(x(t), t) \leq V(x(0), 0) \). Thus, \( \sum_{i=1}^{n} x_i^2 \leq V(x(0), 0)e^{-\alpha t} \). This fact implies that \( x = 0 \) is globally exponentially stable for (10). The proof of Theorem 1 is completed. \( \square \)

We should notice that it is not easy to select \( P \) and \( Q \) directly since inequality (9) is a matrix operation. But we can transform it equivalently to a Linear Matrix Inequality (LMI). Then we can get the solution \( P \) and \( Q \) by use of the LMI Toolbox in Matlab. We illustrate it in details. Firstly, we have

**Lemma 3**

(Schur Complement, S. Boyd et al.) The following LMI

\[ \begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0 \] (28)

where \( Q(x) = Q^T(x) \), \( R(x) = R^T(x) \) and \( S(x) \) depend affinely on \( x \), is equivalent to

\[ R(x) > 0 \quad \text{and} \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \] (29)

From Lemma 3, we can easily get the following result

**Theorem 2**

Suppose that \( P = \text{diag}(P_1, \ldots, P_n) \), \( Q = \text{diag}(Q_1, \ldots, Q_n) \), \( P_i > 0, Q_i > 0, i = 1, 2, \ldots, n \), then

\[ 2PDG^{-1} - (PA + A^T P) - PBQ^{-1}(PB)^T - Q > 0 \] (30)

is equivalent to the following LMI

\[ \begin{pmatrix} 2PDG^{-1} - (PA + A^T P) - Q & PB \\ (PB)^T & Q \end{pmatrix} > 0 \] (31)

Therefore, the stability criterion has been transformed to the linear matrix inequality (31). We can get \( P \) and \( Q \) by use of the LMI Toolbox in Matlab.

In the following, we give a specific example to compute the matrices \( P \) and \( Q \).

Considering the two-dimension system:

\[
\begin{cases}
\dot{x}_1(t) = -9x_1(t) + 2g(x_1(t)) - g(x_2(t)) \\
\quad + 3g(x_1(t - \tau_{11}) + g(x_2(t - \tau_{21})) + I_1 \\
\dot{x}_2(t) = -9x_2(t) - 2g(x_1(t)) + 3g(x_2(t)) \\
\quad + \frac{1}{2}g(x_1(t - \tau_{11}) + 2g(x_2(t - \tau_{21})) + I_2 
\end{cases}
\] (32)
where $\tau_1 = 1$, $\tau_2 = 2$, $I_1 = 1$, $I_2 = 2$, $g(x) = \frac{1}{2}(|x + 1| - |x - 1|)$.

Thus, $D = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, $G = I$, $A = \begin{pmatrix} -2 & -1 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

From LMI (31), we get $P$ and $Q$ by the LMI Toolbox in Matlab:

$$P = \begin{pmatrix} 0.0912 & 0 \\ 0 & 0.1215 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.0912 & 0 \\ 0 & 0.1215 \end{pmatrix}.$$ 

Furthermore, we get

$$-2PDG^{-1} + (PA + A^TP) + PBQ^{-1}B^TP + Q = \begin{pmatrix} -0.2963 & 0.0304 \\ 0.0304 & -0.8102 \end{pmatrix}$$

whose eigenvalues are $-0.8120, -0.2945$.

We give a figure to describe the stability of system (32) as it is done in the paper of Qiao et al. We get the phase plots with initial functions $(x_1, x_2) = (\sin s, \cos s)$, $(2s, \exp s)$, $(\log (s + 3), s^2)$, $(s^3, \arctan s - 4)$, $(27, -45)$, $(\tan s - 3, -(s + 10)^4)$, $(81, 128)$, where $s \in [-2, 0]$.

**Remark 1**

There are many stability results for such delayed system in literatures. Here we give some comparisons with others.

In Joy, the similar model was considered and some interesting results were given. It should be noted that the theorem (Theorem 3.1, Ref. 18) proposed a similar condition to Theorem 1 if the diagonal matrix $Q$ is chosen to be the identity matrix $I_n$. However, the activation function in Ref. 18 was required to be either strictly monotonic increasing, or bounded and nondecreasing, in addition to $g \in G\{G_1, G_2, \ldots, G_n\}$. In our proof, we only assume that $g \in G\{G_1, G_2, \ldots, G_n\}$, which includes a class of more general functions. This extension is necessary specially when infinite intervals with zero slope (such as the piecewise linear models) are presented in unbounded activations. Moreover, in
in their results, there was an additional requirement following one dimension system:

\[ \text{ensure fast response in the network.} \]

ing piece-wise linear models: \( g_i(u_i(t)) = 0.5(|u_i(t) + 1| - |u_i(t) - 1|) \in [-1, +1] \) was assumed. Furthermore, in their results, there was an additional requirement that \(- (A + A^T)\) was positive definite. Obviously, it imposes more constraints on the connection matrix. Thus, Theorem 1 represents a generalization of some previous results on the global convergence in the literature.

Remark 2

In Arik’s 2002 paper (Ref. 1), the author proposed a Lyapunov functional to analyze the cellular neural networks. But there are two main differences between Arik’s and ours. Firstly, Arik’s Lyapunov functional is chosen to discuss the asymptotic stability, in our paper, we chose the functional to analyze the exponential stability. Secondly, in Arik’s paper, the author got two separate conditions by the Lyapunov functional. One is for connection matrix \( A \), the other is for connection matrix \( B \). But we got an integral condition concerning matrices \( A \) and \( B \). Here we give an example for which Arik’s methods fails and ours is successful. Considering the following one dimension system:

\[ \dot{x} = -x + \frac{1}{3} g(x) + 1 \]  

where \( g(x) = \frac{1}{2}(|x + 1| - |x - 1|) \). Thus, \( A = \frac{1}{3} \), \( B = 0 \), \( D = 1 \), \( G = 1 \).

Obviously, \(- (A + A^T + \beta I)\) is not positive definite. So Arik’s method fails to show the stability. In our paper, by choosing \( P = Q = 1 \), We have

\[ 2PDG^{-1} - (PA + A^T P) - PBQ^{-1}(PB)^T - Q = 2 - \frac{2}{3} - 1 = \frac{1}{3} > 0 \]  

It shows the exponential stability by our criterion.

Moreover, we can show that the criterion given in Arik’s paper is recovered from our theorem as a special case. Since the activation function \( g(x) = 0.5(|x + 1| - |x - 1|) \), we know \( G \) is identity matrix \( I \). Here the matrix \( D \) is also assumed to be \( I \). We restate their theorem as

**Proposition 1**

The equilibrium point of (10) is globally asymptotically stable if there exists a positive \( \beta \) such that

\[ -(A + A^T + \beta I) > 0 \]  

\[ \begin{cases} \|B\|_2 \leq \sqrt{2\beta} & \text{if } \beta \geq 1 \\ \|B\|_2 \leq \sqrt{1 + \beta} & \text{if } 0 < \beta \leq 1 \end{cases} \]

Now, we point out that Proposition 1 can be deduced from Theorem 1. In Theorem 1, (9) reduces to

\[ 2P - (PA + A^T P) - PBQ^{-1}(PB)^T - Q > 0 \]

We will consider the following two cases separately.

**Case 1**

\( \beta \geq 1 \). In this case, we have \(- (A + A^T + \beta I) > 0 \) and \( \|B\|_2 \leq \sqrt{2\beta} \), then

\[ 2I - (A + A^T) - \beta^{-1}BB^T - \beta I > 0 \]

which is just the (9), if \( P = I, Q = \beta I \), where \( I \) is identity matrix.

**Case 2**

\( 0 < \beta \leq 1 \). In this case, we have \(- (A + A^T + \beta I) > 0 \) and \( \|B\|_2 \leq \sqrt{1 + \beta} \), then, we have

\[ 2I - (A + A^T) - (1 + \beta)^{-1}BB^T - (1 + \beta)I > 0 \]

which is just the (9), if \( P = I, Q = (1 + \beta)I \).

If fact, from Theorem 1 in our paper we can get the following result:

**Proposition 2**

The equilibrium point of (10) is globally asymptotically stable if there exists a positive \( \beta \) such that

\[ -(A + A^T + \beta I) > 0 \]  

and

\[ \|B\|_2 \leq 1 + \frac{\beta}{2} \]
**Proof**

From \(\|B\|_2 \leq 1 + \frac{\beta}{2}\), we know \(\|BB^T\|_2 \leq (1 + \frac{\beta}{2})^2\). Let \(\alpha = 1 + \frac{\beta}{2}\), we have \((1 + \frac{\beta}{2})^2 = \alpha(2 + \beta - \alpha)\). Thus, \(\|BB^T\|_2 \leq \alpha(2 + \beta - \alpha)\). So we get \(\alpha^{-1}BB^T \leq (2 + \beta - \alpha)I\).

According to \(-\left(A + A^T + \beta I\right) > 0\), we know

\[-(A + A^T + \beta I) + (2 + \beta - \alpha)I - \alpha^{-1}BB^T > 0\]

(43)

that is

\[2I - (A + A^T) - \alpha^{-1}BB^T - \alpha I > 0\]  

(44)

This is exactly (9) when we choose \(P = I\) and \(Q = \alpha I\). Thus the global stability of the trivial solution of (10) follows from Theorem 1. Obviously, our result is more general than that of Proposition 1.

\(\qed\)

**Remark 3**

In Peng et al.’s 2002 paper (Ref. 27), the authors presented a new approach to stability analysis of Hopfield-type neural networks with time-varying delays. Here we extended the model and included the non-delay item. Peng et al. defined two nonlinear functions similar to the matrix norm and matrix measure. In fact, they gave the condition concerning the absolute value of entries in the connections matrix as it was done in Theorem A. Thus, the differences between excitatory and inhibitory effects of the networks are ignored. If some entries in the connection matrices are negative, there are some examples for which Theorem A in Sec. 2 fails to be true. However, Theorem 1 still holds. It illustrates that the hypothesis in Theorem A is conservative. This results from the ignoring differences between excitatory and inhibitory effects.

4. Further Result of Stability Analysis

In the previous section, we obtain the global exponential stability of system (2) if \(2PDG^{-1} - (PA + A^TP) - PBQ^{-1}(PB)^T - Q > 0\). Now, it is natural to ask: What will happen if the strict inequality is replaced by the following nonstrict inequality?

\[2PDG^{-1} - (PA + A^TP) - PBQ^{-1}(PB)^T - Q \geq 0\]  

(45)

In the present section, we give an affirmative answer when the activation functions are hyperbolic tangent. These functions are typically used in neural networks models.

Throughout this section, we assume that in system (1), \(g_i(x) = \tanh(G_i x)\), \(i = 1, 2, \ldots, n\). Constant \(G_i\) is positive. We have the following

**Theorem 3**

If there exist positive diagonal matrices \(P\) and \(Q\) such that

\[2PDG^{-1} - (PA + A^TP) - PBQ^{-1}(PB)^T - Q \geq 0\]  

(46)

Then system (1) has a unique equilibrium \(u^* = [u^*_1, u^*_2, \ldots, u^*_n]^T \in \mathbb{R}^n\) such that each solution of (1) satisfies \(\lim_{t \to +\infty} u(t) = u^*\).

Before the proof, we have to make some preparations.

Firstly, we prove (1) has at least one equilibrium \(u^*\). Defining the mapping:

\[H(u) = (E_n - \alpha D)u + \alpha(Ag(u) + Bg(u) + I)\]  

(47)

where \(E_n\) is identity matrix, \(D = \text{diag}(d_1, d_2, \ldots, d_n)\), \(\alpha\) is positive and to be selected.

Since \(g(u)\) is bounded, there exists a positive \(M_1\) such that

\[\|Ag(u) + Bg(u) + I\| \leq M_1\]  

(48)

There exist \(d\) and \(d\) such that \(0 < d \leq d_i < d\). We select a sufficiently small \(\alpha\) such that \(1 - \alpha d > 0\). Thus, \(1 - \alpha d > 0\).

Let \(M \geq \frac{1}{2}M_1\) and \(\Omega = \{x \in \mathbb{R}^n | \|x\| \leq M\}\).

Thus, for \(x \in \Omega\), we have

\[\|H(u)\| \leq (1 - \alpha d)M + \alpha M_1 \leq M\]  

(49)

Thus \(H(u)\) is a continuous mapping from \(\Omega\) to itself. Since \(\Omega\) is a convex and compact set, by Brouwer fixed point theory, there exists \(u^* \in \Omega\) such that \(H(u^*) = u^*\). From (46), we know this \(u^*\) satisfies

\[-Du^* + Ag(u^*) + Bg(u^*) + I = 0\]  

(50)

\(u^*\) is an equilibrium point of system (1).

Let \(x_i(t) = u_i(t) - u^*_i\), (1) can be written as

\[
\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} \varphi_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} \varphi_j(x_j(t - \tau_j))
\]  

(51)
where \( \varphi_i(x_i(\cdot)) = g_i(x_i(\cdot) + u_i^*) - g_i(u_i^*) \).

Thus, the uniqueness and asymptotic stability of (1) reduces to the global convergence of (50).

We also need the following properties for the sigmoidal function \( g \):

1. For any real number \( x \),
   \[
   |\varphi_j(x)| \leq G_j|x| \quad j = 1, 2, \ldots, n \tag{51}
   \]

2. Let \( E_1 > 0 \) be a fixed positive constant and \( |x| > E_1 \). Then there exist constants \( 0 < E_2 < 1, E_3 > 0 \) depending only on \( E_1 \) such that
   \[
   E_3 \leq |\varphi_j(x)| \leq E_2 G_j|x| \quad j = 1, 2, \ldots, n. \tag{52}
   \]

**Proof of Theorem 3**

Define function \( \|x(t)\|_2^2 = \sum_{i=1}^{n} |x_i(t)|^2 \), similar to Ref. 6, we prove

\[
\lim_{t \to +\infty} \| x(t) \|_2 = 0 \tag{53}
\]

Otherwise, there is a constant \( E_1 \) such that for any large \( T > 0 \), there is \( T' > T \) such that \( \|x(t)\|_2 > 2\sqrt{n}E_1 \). Since \( \|x(t)\|_2 \) is uniformly continuous, then there is such an interval \((a_T, b_T)\) with \( b_T - a_T \geq \delta \) that \( \|x(t)\|_2 > \sqrt{n}E_1 \) whenever \( t \in (a_T, b_T) \), where \( \delta \) is a fixed positive number. Thus, for each \( t \in (a_T, b_T) \), there is an index \( i_0 \in \{1, 2, \ldots, n\} \) such that \( |x_{i_0}(t)| > E_1 \). From Eq. (52), we also know that there are constants \( 0 < E_2 < 1, E_3 > 0 \), such that

\[
E_3 \leq |\varphi_{i_0}(x_{i_0}(t))| \leq E_2 G_{i_0}|x_{i_0}(t)| \tag{54}
\]

Now, we consider the following functional

\[
V(x(t)) = 2 \sum_{i=1}^{n} P_i \int_{0}^{x_i(t)} \varphi_i(s)ds + \sum_{i=1}^{n} Q_i \int_{t-\tau}^{t} \varphi_i^2(x_i(\xi))d\xi \tag{55}
\]

Calculating \( \dot{V}(x) \) along the solutions of Eq. (50), we get

\[
\dot{V}(x) = 2\varphi^T(t)[-PDx + PAP\varphi(t) + PB\varphi(t - \tau)] + \varphi^T(t)Q\varphi(t) - \varphi^T(t - \tau)Q\varphi(t - \tau)
= -2\varphi^T(t)PDx + 2\varphi^T(t)PA\varphi(t) + 2\varphi^T(t)PB\varphi(t - \tau) + \varphi^T(t)Q\varphi(t)
- \varphi^T(t - \tau)Q\varphi(t - \tau) \tag{56}
\]

We also have

\[
2\varphi^T(t)PDx + \varphi^T(t - \tau)Q\varphi(t - \tau) \leq \varphi^T(t)PBD^{-1}(PB)^TQ\varphi(t) \tag{57}
\]

From Eq. (56), we obtain

\[
\dot{V}(x) \leq -2\varphi^T(t)PDx + \varphi^T(t)((PA + AT)P) + PBQ^{-1}(PB)^T + Q|\varphi(t) \tag{58}
\]

Thus, by Eqs. (51) and (54), we have

\[
\dot{V}(x) \leq -2\varphi_{i_0}(t)P_{i_0}d_{i_0}\frac{1}{E_2} G_{i_0}\varphi_{i_0}(t)
- 2\sum_{i \neq i_0} \varphi_i(t)P_id_i x_i(t) + \varphi^T(t)
\times [(PA + AT)P + PBQ^{-1}(PB)^T + Q|\varphi(t)
\leq -2\left(\frac{1}{E_2} - 1\right) \varphi_{i_0}(t)P_{i_0}d_{i_0}\frac{1}{G_{i_0}}\varphi_{i_0}(t)
- 2\varphi_{i_0}(t)P_{i_0}d_{i_0}\frac{1}{G_{i_0}}\varphi_{i_0}(t) - 2\sum_{i \neq i_0} \varphi_i(t)P_id_i
\times \frac{1}{G_{i_0}}\varphi_i(t) + \varphi^T(t)[(PA + AT)P]
+ PBQ^{-1}(PB)^T + Q|\varphi(t)
= -2\left(\frac{1}{E_2} - 1\right) P_{i_0}d_{i_0}\frac{1}{G_{i_0}}\varphi_{i_0}^2(t) - \varphi^T(t)
\times [2PDG^{-1} - PA - AT]P
- PBQ^{-1}(PB)^T + Q|\varphi(t) \tag{59}
\]

According to the condition (45), we get

\[
\dot{V}(x) \leq -2\left(\frac{1}{E_2} - 1\right) P_{i_0}d_{i_0}\frac{1}{G_{i_0}}\varphi_{i_0}^2(t) \tag{60}
\]

From Eq. (54), we obtain

\[
\dot{V}(x) \leq -2\left(\frac{1}{E_2} - 1\right) P_{i_0}d_{i_0}\frac{1}{G_{i_0}}E_3^2 = -C \tag{61}
\]

Where \( C \) is a fixed positive number. Now, we claim that

\[
\lim_{t \to +\infty} V(t) = -\infty \tag{62}
\]

Let \((a_1, b_1)\) be such an interval with the length \( |b_1 - a_1| \geq \delta \) that \( \|x(t)\|_2 \geq 2\sqrt{n}E_1 \) for all \( t \in (a_1, b_1) \). If \( b_1 = +\infty \), then

\[
\lim_{t \to +\infty} |V(t) - V(0)| = \lim_{t \to +\infty} \int_{0}^{t} \dot{V}(s)ds
\leq \lim_{t \to +\infty} (-Ct) = -\infty \tag{63}
\]
If \( b_1 < +\infty \), we can find another interval \((a_2, b_2)\) with the same property \((a_1, b_1)\) possesses and \( a_2 \geq a_1 \). By induction, we can find a sequence of intervals \((a_i, b_i)\) with this property and \( a_{i+1} \geq b_i \). If for some \( i \), we have \( b_i = +\infty \), then by previous arguments, we conclude that \( V(t) \to -\infty \). Otherwise,

\[
\lim_{t \to +\infty} [V(t) - V(0)] \leq \lim_{i \to +\infty} \sum_{i=1}^{n} [V(b_i) - V(a_i)] \\
\leq \lim_{n \to +\infty} (-nC\delta) = -\infty \quad (64)
\]

In any case, \( V(t) \to -\infty \), which contradicts \( V(t) \geq 0 \). Thus, Theorem 3 is proved. \( \square \)

**Remark 1**

In Theorem 3, if the activation function \( g \in G\{G_1, G_2, \ldots, G_n\} \), then the conclusion of the global convergence of (1) is not valid any more. Following example verifies it. Let \( n = 1 \) and consider the differential equation

\[
\frac{du(t)}{dt} = -u(t) + g(u(t)) + 1 \quad (65)
\]

If \( g(u) = u \), then \( G = 1 \). Thus, \( DG^{-1} - A = 0 \). But, obviously, the above system has no equilibrium. However, when \( g(u) = \tanh(u) \), the system has a globally asymptotically stable equilibrium, as Theorem 3 has stated.

**Remark 2**

From the process of proving the above theorem, we can infer that the theorem is still valid for a class of sigmoidal functions, which is more general than \( \tanh(G, x) \).

5. Results for BAM Networks

In this section, we will investigate the existence and exponential stability of unique equilibrium point of bidirectional associative memory (BAM) neural network described by the following delayed model

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= -a_ix_i(t) + \sum_{j=1}^{n} b_{ij}f_j(y_j(t - \sigma_j)) + I_i \\
\frac{dy_i(t)}{dt} &= -c_iy_i(t) + \sum_{j=1}^{n} d_{ij}g_j(x_j(t - \tau_j)) + J_i
\end{align*}
\]

(66)

for \( i = 1, 2, \ldots, n \).

If we denote \( x = (x_1, x_2, \ldots, x_n)^T \), \( y = (y_1, y_2, \ldots, y_n)^T \), then Eq. (66) reduces to

\[
\begin{align*}
\frac{dx(t)}{dt} &= -Ax(t) + Bf(y(t - \sigma)) + I \\
\frac{dy(t)}{dt} &= -Cy(t) + Dg(x(t - \tau)) + J
\end{align*}
\]

(67)

System (66) consists of two sets of \( n \) neurons arranged on two layers, namely, \( I \)-layer and \( J \)-layer. \( x_i(\cdot) \) and \( y_i(\cdot) \) denote membrane potentials of \( i \)th neurons from the \( I \)-layer and \( J \)-layer, respectively; \( b_{ij}, d_{ij} \) correspond to synaptic connection matrices. \( I_i, J_i \) denote external inputs to the neurons introduced from outside the network; \( \sigma_j, \tau_j \) are time delays.

It is clear that Eq. (66) extended those models studied by Refs. 16, 19, 20, 21, 24. It was also discussed by Mohamad.26 But most of the works discuss the asymptotic stability and the conditions guaranteeing the convergence ignore the signs of entries in the connection matrices. In this section, we simplify the two-layer model and obtain the global exponential stability results with more general activation functions.

**Theorem 4**

Suppose that \( f \in G\{F_1, F_2, \ldots, F_n\}, g \in G\{G_1, G_2, \ldots, G_n\} \) and there exist positive diagonal matrices \( P_1, P_2, Q_1 \) and \( Q_2 \) such that

\[
2P_1AG^{-1} - P_1BQ_2^{-1}(P_1B)^T - Q_1 > 0 \quad (68)
\]

\[
2P_2CF^{-1} - P_2DQ_1^{-1}(P_2D)^T - Q_2 > 0 \quad (69)
\]

Then, for each \( I, J \in R^n \), system (66) has a unique equilibrium point, say, \( \left( \begin{array}{c} x^* \\ y^* \end{array} \right) \) which is globally exponentially stable, independent of the delays.

**Proof**

Firstly, we simplify the two-layer model and so, the bidirectional associative memory neural network can be regarded as a single-layer system, i.e., the delayed Hopfield neural network studied in the previous section.

We let \( w(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_n(t))^T \), then system (66) can be rewritten
as
\[
\frac{dw(t)}{dt} = -\left( \begin{array}{cc} A & 0 \\ 0 & C \end{array} \right) w(t)
+ \left( \begin{array}{cc} 0 & B \\ D & 0 \end{array} \right) h(w(t - \eta)) + \left( \begin{array}{c} I \\ J \end{array} \right)
\]

(70)

which is a Hopfield-type neural network. We note that there is only the delayed item and the delayed connection matrix is \( \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} \). And the activation function is \( h = (g_1, g_2, \ldots, g_n, f_1, f_2, \ldots, f_n)^T \). \( \eta = (\tau_1, \tau_2, \ldots, \tau_n, \sigma_1, \sigma_2, \ldots, \sigma_n)^T \) denotes the new time delays. Since \( f \in G\{F_1, F_2, \ldots, F_n\} \), \( g \in G\{G_1, G_2, \ldots, G_n\} \), we get \( h \in G\{G_1, G_2, \ldots, G_n, F_1, F_2, \ldots, F_n\} \).

From the conditions (68) and (69), after a direct calculation, we can obtain
\[
\begin{align*}
2 & \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & F \end{pmatrix}^{-1} \\
- & \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}^{-1} \\
\times & \begin{pmatrix} 0 & B^T \\ D & 0 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} > 0.
\end{align*}
\]

Therefore, by Theorem 1, system (70) has a unique equilibrium point, say, \( w^* \), or equivalent to \( \begin{pmatrix} x^* \\ y^* \end{pmatrix} \) for system (66), which is globally exponentially stable, independent of the delays. The proof is completed.

\[ \square \]

6. Conclusions

A sufficient condition for the global exponential stability independent of time delays is obtained for a class of delayed neural networks systems. We don’t assume the symmetry of the connection matrices. The boundedness and differentiability of the activation functions are not required, either. The criteria concerning the differences between excitatory and inhibitory effects on units extend some existing results in the literature. Furthermore, the condition can be relaxed in the case that the activation functions are a class of sigmoidal functions. We also give similar results concerning bidirectional associative memory (BAM) neural network.

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