Practical Stabilization for Nonholonomic Chained Systems with Fast Convergence, Pole-Placement and Robustness

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Abstract
In this paper, issues of practical stabilization problem for nonholonomic chained systems are discussed. A novel controller that can guarantee a fast convergence like the exponential convergence is proposed based on the switching controllers approach. The steady-state error can be specified in advance. Moreover, the convergent rate can be pre-assigned and the transient response can be improved using the pole-placement method. The proposed controller is shown to be robust w.r.t. some uncertainties in the model. Simulation results for a unicycle-modeled mobile robot that has a small bias in orientation, an unknown radius in the rear wheels and an unknown distance in between them are presented to validate the effectiveness of our approach.

Keywords: Pole-placement, practical stabilization, nonholonomic chained systems, robust stability.

1. Introduction
Many mechanical systems such as tractor-trailer systems and wheeled mobile robots etc., can be modeled by the chained system described in the following
\[ \dot{x}_n = u_n \]
\[ \dot{x} = u_n A x + B u_1, \]
where \( x = [x_0, x_1, \ldots, x_n]^T \), \( B = [0, 0, \ldots, 1]^T \) and \((A, B)\) is in the controllable canonical form (CCF) [8]. In present literature, to achieve a fast convergence, several stabilizers to guarantee the exponential convergence were proposed recently [1], [10].

On the other hand, the robustness issue for the chained system (1) was also attracted much attention recently [3], [5], [7]. A mobile robot of unicycle type with a small bias in orientation was proposed in [5] as a counterexample to show that the controller from the homogeneous feedback is not robust w.r.t. a small parametric uncertainty. In [3], an interesting exponentially convergent robust controller was proposed using the discontinuous feedback. Quite recently, a different approach was proposed in [7] using hybrid feedback. However, to the best of our knowledge, a globally exponential stability for a mobile robot of unicycle type with a small bias in orientation is still unsolved in present literature.

In this paper, instead of using an exponential stabilizer, a practical stabilization problem that can achieve a fast convergence like the exponential convergence will be studied. A novel stabilizer for the chained system (1) will be proposed using the switching controllers approach. The subsystem (1b) will be studied first and the closed-loop system will be shown to be exponentially stable under some extra conditions imposed on the controller \( u_n \). Due to this result, there will be several possibilities for the choice of \( u_n \) such that the extra conditions are satisfied and simultaneously, solutions of the closed-loop system of subsystem (1a) are still in a pre-assigned domain. The proposed practical stability has the same effect as the usual exponential stability on achieving the fast convergence, and the controllers will be very simple and thus easily implemented.

Furthermore, a bound relating to the measurement of the transient behavior of the closed-loop system will be given explicitly. A numerical method can be invoked to minimize the bound and thus, improve the transient response of the closed-loop system. This is achieved in present literature yet. Moreover, we will show that the proposed controller is robust w.r.t. some uncertainties in the model. When apply to the example given in [3], it will be shown that a practical stability with fast convergence result can be also guaranteed under the parametric uncertainties based on our approach.

Notations:
\[ |v| = \sqrt{v_0^2 + v_1^2 + \cdots + v_n^2}, \]
\[ \forall v = (v_0, v_1, \ldots, v_n) \in \mathbb{R}^n, \quad \|A\| = \sup_{|v| = 1} |Av|, \]
\[ B_m = \{x \in \mathbb{R} \mid |x| \leq m\} \text{ and } \text{sign}(x) = \begin{cases} 1, \forall x > 0 \\ -1, \forall x < 0 \end{cases}. \]

2. Simultaneous stabilization for controllable systems
2.1 Stability and pole-placement
Consider the following controllable linear time-invariant systems:
\[ \dot{x} = \alpha A x + B u, \quad \sigma \in [-1, 1], \]
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^n \) and \((A, B)\) is a controllable pair. Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq (-\infty, 0) \) be any pre-assigned set of eigenvalues. Then, there exist a matrix \( K \) and a
unique positive definite matrix $P$ such that the eigenvalues of $A + BK_o$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the following Ricatti equation holds:

$$P(A + BK_o)^T + (A + BK_o)^T P + 2PBB^T P = 0. \quad (3)$$

Now, for each $\sigma$, a stabilizing controller can be given in the following form:

$$u = \sigma (K_o + B^T P)x - B^T Px. \quad (4)$$

With the choice of controller $u$, the closed-loop system can be written into the following system:

$$\dot{x} = Ax, \quad (5)$$

where $A_o = \sigma (A + BK_o + BB^T P) - BB^T P$. Then, it can be seen that $A_o$ has the same eigenvalues with $A + BK_o$. Thus, the following result can be given.

**Lemma 1:** Consider the linear time-invariant systems (2). With the controller $u$ given in (4), the origin of the closed-loop system (5) is simultaneously exponentially stable and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are poles of the system. ■

### 2.2 Convergent rate and transient behavior for systems in CCF

In this subsection, the exact solution of the closed-loop system (5) will be studied when $(A, B)$ is in the controllable canonical form (CCF). Let us assume that $B = [0, 0, \ldots, 1]^T$ and $(A, B)$ is in the CCF. Let $K_o$ be given by

$$K_o = [-a_o, -a_o, \ldots, -a_o]. \quad (6)$$

Then, for $\sigma = 1$, the system matrix $A_o = A + BK_o$ can be written into the following companion form

$$A + BK_o = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_o & -a_o & \ldots & -a_o
\end{bmatrix}. \quad (7)$$

Consider the following Vandermonde matrix

$$V(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{-1} & \lambda_2^{-1} & \ldots & \lambda_n^{-1}
\end{bmatrix}. \quad (8)$$

Let

$$\lambda_o = \min_{1 \leq i \leq n} |\lambda_i|. \quad (9)$$

Then, every solution of $\dot{x} = (A + BK_o)x$ can be estimated by

$$|x(t)| \leq ||V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})\cdot V^{-1}||x(0)|, \quad 0 \leq \sigma \leq 1, \quad (10)$$

where $V = V(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_o$ is any pre-assigned eigenvalue of $A + BK_o$. Then, every solution $x(t)$ satisfies the following inequalities:

\begin{enumerate}
  \item $|x(t)| \leq \left\| V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \cdot V^{-1} \right\| x(0), \quad \forall 0 \leq \sigma \leq 1 \leq \lambda_o, \quad (11)$
  \item $|x(t)| \leq M_o \cdot |x(0)|, \quad \forall \sigma \leq 1 \leq \lambda_o, \quad (12)$
\end{enumerate}

Then it can be seen that for every solution $x(t)$ of the system (5), the following inequality holds:

$$\text{max} \left| x(t) \right| \leq M_o \cdot |x(t)|, \quad \forall \sigma \geq 0. \quad (13)$$

We summarize the previous discussion into the following lemma.

**Lemma 2:** Consider the system (5) where $B = [0, 0, \ldots, 1]^T$ and $(A, B)$ is in the CCF. Then every solution $x(t)$ satisfies the inequalities (10) and (12).

### 3. Main results

#### 3.1 The exponential stability for partial state

In the remainder of this paper, it is said that $\text{sign}(u_o)$ can be defined on an open interval $(a, b)$ if there exists a finite subset $\{c_1, c_2, \ldots, c_n\}$ of $(a, b)$ such that $u_o(t) \neq 0$ and $\text{sign}(u_o(t))$ is a constant function for all $t \in (a, b) - \{c_1, c_2, \ldots, c_n\}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be any pre-assigned eigenvalues contained in $(-\infty, 0), K_o$ is the matrix defined in (6) where $a_o, a_o, \ldots, a_o$ are the coefficients of the characteristic polynomial $\prod_{i=1}^{n} (\lambda - \lambda_i)$ and $P$ is the positive definite matrix solving the Ricatti equation (3). Choose the controller $u_o$ as

$$u_o = u_o(K_o + B^T P)x - u_o B^T Px. \quad (14)$$

Then, the following result can be derived in view of Lemmas 1 and 2. The proof is omitted due to a limited space.

**Proposition 1:** Consider the subsystem (1b). Suppose $u_o(t)$ is a continuous bounded function defined on an open interval $(a, b)$ and $\text{sign}(u_o)$ can be defined on $(a, b)$. Then, with the controller $u_o$ chosen as in the equation (13), every solution $x(t)$ satisfies the following inequalities:

\begin{enumerate}
  \item $|x(t)| \leq \left\| V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \cdot V^{-1} \right\| x(0), \quad 0 \leq \sigma \leq 1 \leq \lambda_o, \quad (15)$
  \item $|x(t)| \leq M_o \cdot |x(0)|, \quad \forall \sigma \leq 1 \leq \lambda_o, \quad (16)$
\end{enumerate}

In the following, let us define a function $M(t)$ by

$$M(t) = \left\| V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \cdot V^{-1} \right\|, \quad \forall t \geq 0. \quad (17)$$

and make the following condition for the controller $u_o$.

**C1** Let $t_0 = 0$ and $\beta(t, s) = \int_{t_0}^{t} u_o(\tau) d\tau, \quad \forall 0 \leq s \leq t$. Suppose there exist a positive constant $\gamma < 1$, a positive constant $T$ and a strictly increasing sequence $\{t_1, t_2, \ldots, t_i, \ldots\} \subseteq [0, \infty)$ with $\lim_{i \to \infty} t_i = \infty$ and $|t_{i+1} - t_i| \leq T, \forall i \in \mathbb{N}$, such that for each $i \in \mathbb{N}$, $u_o(t)$ is a continuous bounded function defined on $(t_{i-1}, t_i)$. 

$$M_o = \max_{t \in [0, T]} \left\| V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \cdot V^{-1} \right\| \geq 1. \quad (18)$$

In the following, let us define a function $M(t)$ by

$$M(t) = \left\| V \cdot \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \cdot V^{-1} \right\|, \quad \forall t \geq 0. \quad (19)$$

and make the following condition for the controller $u_o$.
sign\( (u_i) \) can be defined on \( (t_i, t) \) and 
\[ M(\beta(t_i, t)) \leq \gamma. \]

Using condition (C1) and Proposition 1, the following exponential convergence result for the partial state \( x \) can be given.

**Proposition 2:** Consider the subsystem (1b). Suppose the controller \( u_i \) has been chosen such that condition (C1) holds. Then, with the controller \( u_i \) chosen as in the equation (13) and every solution \( x(t) \) of the closed-loop system of (1b) satisfies the following inequalities:

(i) \[ |x(t)| \leq k_i e^{\sigma_i t} |x(0)|, \quad \forall t \geq 0, \quad (17) \]

where \( \sigma_i = -\frac{\ln(\gamma)}{t} \) and \( k_i = \frac{M_i^2}{\gamma} e^{\sigma n_i}. \)

(ii) \[ |x(t)| \leq M_o^2 |x(0)|, \quad \forall t \geq 0. \quad (18) \]

### 3.2 The practical stability for full state: fast convergence and transient response

In the following, a technique lemma that is relating to an estimation of the function \( M(t) \) is given.

**Lemma 3:** Let \( \delta, \gamma \) be two given positive constants, and \( \{d_i, d_2, \ldots, d_n\} \subseteq [1, \infty) \) be a given sequence of real number. For any \( \lambda > 0 \), let the pre-assigned eigenvalue \( \lambda_i = -\lambda \), and recursively define the other pre-assigned eigenvalues as \( \lambda_i = \lambda_{i-1} - d_i \), \( \forall 2 \leq i \leq n \). Then, there is a large enough constant \( \overline{T}_0 \) such that the following inequality holds:

\[ M(\overline{T}_0) \leq \gamma, \quad \forall \lambda \geq \overline{T}_0. \quad (19) \]

Let us construct the controller \( u_o \). Let \( \gamma < 1, \quad m, \quad \epsilon < m \) and \( \delta_i < m - \epsilon \) be four given positive constants. Then, there exists a set of pre-assigned eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that \( M(\delta_i) \leq \gamma \) by Lemma 3. Let \( T \) be any given positive constant and \( h(t) \) be any continuous periodic function defined on \( [0, \infty) \) with period \( T \), \( h(0) = 0 \) and \( h(t) \neq 0 \), \( \forall t \in (0, T) \). Choose a positive constant \( \sigma_o = -\frac{\ln(\gamma)}{T} \) that is equal to the constant \( \sigma_i \) defined in Proposition 2. Define a positive constant \( h_o = \int_0^T |h(t)| dt \) and a continuous function \( \tilde{t}_i : \mathbb{R} \rightarrow [0, \infty) \) as the following:

\[ \tilde{t}_i(x_i) = \begin{cases} \frac{1}{h_o \sigma_o} \ln \left[ \frac{\left[ x_i \right]}{\epsilon} \right], & \text{if } \left[ x_i \right] > \epsilon \\ 0, & \text{if } \left[ x_i \right] \leq \epsilon. \end{cases} \quad (20) \]

Let \( \left[ y \right] \) denote the least integer larger than or equal to a real number \( y \). For any initial state \( x_i(0) \), let \( \{t_i, t_2, \ldots, t_n, \ldots\} \) be a strictly increasing time sequence with \( t_i = T \left[ \tilde{t}_i(x_i(0)) \right] \) and \( t_i = (i - 1)T + t_i, \quad \forall i \in \mathbb{N} \). Note that \( \lim_{i \to \infty} t_i = \infty \) and \( \left| t_{i+1} - t_i \right| = T, \quad \forall i \in \mathbb{N} \). Let \( \mu_o \) be the positive constant such that \( \mu_o = \delta \sqrt{h_o} \).

Now, the controller \( u_o \) can be chosen as the following

\[ u_o = \begin{cases} -\sigma_o \left[ h(t) \right] x_i, & \text{if } \left[ x_i \right] > \epsilon \quad \text{and } 0 \leq t < t_i, \\ \mu_o (\delta - h(t)), & \text{if } t_i \leq t < t_{i+1}, \quad \text{for some } i \in \mathbb{N}. \end{cases} \quad (21) \]

Then, it can be derived that

\[ \beta(t_i, t) = \mu_o \int_{t_i}^t \left[ h(t) \right] dt = \mu_o \int_{t_i}^t \left[ h(t) \right] dt = \delta_i, \quad \forall i \in \mathbb{N}, \quad \text{by the definition of } \mu_o. \]

Thus, we have \( M(\beta(t_i, t)) = M(\delta_i) \leq \gamma \), \( \forall i \in \mathbb{N} \), by Lemma 3 and the choice of \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Note that

\[ \dot{x}_i(t) = -\sigma_o [h(t)] x_i(t), \quad (22) \]

when \( \left[ x_i(0) \right] > \epsilon \) and \( 0 \leq t < t_i \) in view of the equation (21). From the choice of controller \( u_o \), it is easy to see that condition (C1) holds.

In the following, let us study the closed-loop system of (1a). Let \( x_i(t) \) be any solution of the closed-loop system of (1a). Then, it can be described into the following equation:

\[ x_i(t) = \begin{cases} e^{-\epsilon} \int_{t_i}^t \left[ h(x_i(s)) \right] ds, & \text{if } t_i \leq t < t_{i+1}, \quad \text{for some } i \in \mathbb{N}. \end{cases} \quad (23) \]

From the equation (23) and the definition of \( t_i \), it can be checked that

\[ x_i(t) = e^{-\frac{\ln \left[ x_i(0) \right]}{T} \epsilon} \left[ x_i(0) \right] \leq \epsilon, \quad x_i(t_{i+1}) = x_i(t_i) \quad \text{and} \quad x_i(t_{i+1}) = x_i(t_i) \pm \delta_i, \quad \forall i \in \mathbb{N}. \]

This results in

\[ x_i(t) \leq \epsilon, \quad \text{if } t \geq t_i, \quad (24) \]

Thus, we have the following estimation for any solution \( x_i(t) \) of the closed-loop system of (1a):

\[ x_i(t) \leq \epsilon, \quad \text{if } t \geq t_i, \quad (24) \]

In particular, the following proposition can be given in view of Proposition 2.

**Proposition 3:** Consider the system (1). Let \( \gamma < 1, \quad m, \quad \epsilon < m, \quad \delta_i < m - \epsilon \) and \( T \) be five given positive constants. With the controllers \( u_i \) and \( u_o \) chosen as in the equation (13) and (21), every solution \( (x_i(t), x(t)) \) of the closed-loop system satisfies the inequalities (17)-(18) and (24).

In the remainder of this section, let us show that the exponential practical stability can be guaranteed based on Proposition 3. For simplicity, the function \( h(t) \) is
chosen as \( h(t) = 1 - \cos\left(\frac{2\pi t}{T}\right) \). Then, \( h_0 = T \) and 
\[
\mu_0 = \frac{\delta_1}{T}. 
\]
By the equation (17), we have \( \lim_{i \to \infty} x(t) = 0 \).
Thus, for any positive constant \( \varepsilon_i \), the constant 
\[
t_i = \min\left\{ \varepsilon_i, \frac{\ln\gamma}{\varepsilon_i} \right\}, \quad \forall i \in \mathbb{N} \]  
(25)
is well-defined. We modify the controller \( u_i \) in (21) by 
\[
u_i = \frac{\delta}{T} \left( 1 - \cos\left(\frac{2\pi t}{T}\right) \right), \quad \text{if } t_i \leq t < t_{i+1}, 
\]  
(26)
Note that we also have \( |x_i(t)| \leq \varepsilon \) by the third inequality in (24). In view of equation (13), \( u_i = 0 \) implies that \( u_i = 0 \). Thus, \( x_i(t) = x_i(t_i) \) and \( x(t) = x(t_i) \), \( \forall t \geq t_i \), in this case. Now, the following theorem is readable from Proposition 3.

**Theorem 1:** Consider the chained system (1). Let \( \gamma < 1 \), \( m > 0 \), \( \delta_i < m - \varepsilon \), \( T \) and \( \varepsilon_i \) be six given positive constants. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be a set of pre-designed eigenvalues such that \( M(\delta_i) \leq \gamma \).
Choosing the controllers \( u_i \) and \( u_0 \) as in equations (13) and (26), the closed-loop system is exponentially practically stable in the following sense. With 
\[
\sigma_i = \sigma_0 = -\frac{\ln\gamma}{T} \quad \text{and} \quad k_i = M_i^2 \text{e}^{\varepsilon_0}, \quad \text{every solution} 
\]
\((x_i(t), x(t))\) of the closed-loop system satisfies the following inequalities:

(i) \( |x(t)| \leq \begin{cases} k_1 e^{-\varepsilon_1 t} |x(0)| & \forall 0 \leq t < t_i, \\ e^{|x(0)|} & \forall t \geq t_i, \end{cases} \)

(ii) \( |x_i(t)| \leq m_i \) if \( i \leq t < t_i \), 

\[
|x_i(t)| \leq e^{|x(0)|} \quad \text{if } t_i \leq t < t_{i+1}, \quad \forall i \in \mathbb{N} \quad \text{with} \ t_i < t. \]  

(27)

(28)

3.3 A robust stability result

In this subsection, we want to show that the controller proposed in previous subsection has the robust property w.r.t. some model uncertainties. Consider the following chained system with parametric uncertainties:

\[
\dot{x} = u_i x + Bu_i \Delta A_i(\eta_i, \eta_i)x + \Delta B_i(\eta_i, \eta_i)u_i, \]  
(29)

where \((A, B)\) is in the CCF; \( \eta_i \) is an unknown constant and \( \eta_i \in \mathbb{R}^n \) is an unknown vector; \( \Delta A_i(\eta_i, \eta_i) \) and \( \Delta B_i(\eta_i, \eta_i) \) are both smooth matrix-valued functions with \( \Delta A(0,0) = 0 \) and \( \Delta B(0,0) = 0 \).

Without lose of generality, let us assume that \( \eta_i \in (-1, 1) \). This implies that \( 0 < 1 + \eta_i < 2 \). Choose the controller \( u_0 \) as in the first equation in (26). Then, for \( |x_i(0)| > \varepsilon \), an inequality like the first inequality in (28) holds with \( \sigma_i \) replaced by \( \sigma_i = -\frac{\ln\gamma}{T}(1 + \eta_i) \) in view of the first equation in (29). Then, we can modify the constant \( t_i \) as in the following

\[
t_i = T \min \left\{ \varepsilon \right\}, \quad \text{for all} \quad t_{i+1} = t_i + T, \quad \forall i \in \mathbb{N} \quad \text{and choose the controller} \quad u_i \quad \text{as in the first and second equations in (26).} \]  

Furthermore, let us choose a small positive constant \( \eta_i^0 < 1 \) such that \( \delta_i(1 + \eta_i^0) < m - \varepsilon \). Then, it can be directly checked that the second inequality in (28) also holds for any \( \eta_i \in (-\eta_i^0, \eta_i^0) \).

Choose the controller \( u_i \) as in (13). Let us show that the first inequality in (29) holds for small values of \( \eta_i \) and \( \eta_i \). Again using the time-scaling method, define \( \beta(t) = \beta(t, t_{i+1}) = \int_{t_i}^{t} |\mu_i(s)| ds \) for all \( t_{i+1} \leq t \leq t_i \), and all \( i \in \mathbb{N} \). Let \( \tilde{x}_i(t) = x(\beta^{-1}(t)) \), for all \( t \in [0, b] \) with \( b_i = \beta(t_i, t_{i+1}) = \int_{t_i}^{t} \mu_i(s) ds \).

\[
\tilde{x}_i = x + \Delta A_i(\eta_i, \eta_i) \tilde{x}_i, \]  
(31)

where \( A_i = \sigma(A + BK_i + BB^T P) - BB^T P \), \( \Delta A_i = \sigma(\Delta A + \Delta B(K_i + B^T P)) - \Delta B B^T P \) and \( \sigma = \left( \begin{array}{l} -\text{sgn}(x_i(0)), \text{if } i = 1 \text{ and } |x_i(0)| > \varepsilon \\ (-1)^{-i}, \text{otherwise} \end{array} \right) \).

Note that \( \Delta A_i \) is also a smooth function of \( \eta_i \) and \( \eta_i \) with \( \Delta A_i(0, 0) = 0 \). Let \( \tilde{x}(t) \) be a solution of \( \tilde{x} = A_i \tilde{x} \) with \( \tilde{x}(0) = \tilde{x}_i(0) \). Then, \( \tilde{x}(t) \) satisfies the inequalities (10) and (12) by Lemma 2. Let \( e(t) = \tilde{x}(t) - \tilde{x}(t) \). Then, \( e(t) = 0 \) and \( e(t) \) satisfies the following equation:

\[
\dot{e} = (A_i + \Delta A_i)e + \Delta A_i \tilde{x}, \]  
(32)

Since \( A_i \) is a stable matrix, there is a positive definite Lyapunov matrix \( Q_i \) [2] such that

\[
Q_i A_i + A_i^T Q_i = -I. \]  
(33)

Let \( \lambda_i \) be the constant defined in (9) and \( \lambda_i(\lambda_i) \) denote the minimum eigenvalue of \( Q_i \). Then, for any positive constant \( r_i \), there exist two small constants \( \eta_i^0 \) and \( \eta_i \) such that

\[
\|Q_i A_i(\eta_i, \eta_i)\| < \min \{1/4, r_i \sqrt{\lambda_i(\lambda_i)} \}, \]  
(34)
for all $|\eta_i| < \eta'_i$ and $|\eta| < \eta'_i$ in view of $\Delta A_x(0,0) = 0$.

Let $V_x = e^T P_x e$. Then, we have

$$
\dot{V}_x \leq 2\|Q A_x\| \|P_x\| \|\eta\| < \eta'_x, \forall |\eta| < \eta'_x. \hspace{1cm} (35)
$$

Note that $e(0) = x(0) - \bar{x}(0) = 0$ by definition.

Integrating the two side of (35) and using the inequalities (10) and (34), we have

$$
\lambda_x(Q_x)|e(\tau)|^2 \leq \int\left(2\|Q A_x\| \|P_x\| \|\eta\| + \|P_x\|^2\right) d\tau 
$$

$$
\leq 2\lambda_x \lambda_x(Q_x) \|P_x\|^2 \left(1 - e^{-2\lambda_x \tau}\right) 
$$

$$
\leq r_x^2 \lambda_x(Q_x) \|P_x\|^2, \forall \tau \in [0,b_x]. \hspace{1cm} (36)
$$

This implies that $e(\tau) \leq r_x \|P_x\|$ and $|e(\tau)| \leq |e(\tau)| + \|P_x\| \leq r_x \|P_x\| + \|P_x\|$. Note that by definitions, $\bar{x}(0) = x(0) = x(t_{s1})$. In view of the inequalities (11) and (13), the following inequalities

$$
|x(t)| \leq (r_x + M_x) \|P_x\| \leq (r_x + M_x) |x(t_{s1})| \hspace{1cm} (37)
$$

and

$$
|\dot{x}(t)| = |\dot{t}_{s1}| \leq (r_x + M_x) |x(t_{s1})| \hspace{1cm} (38)
$$

hold for all $t_{s1} \leq t \leq t_1$, $|\eta| < \eta'_x$ and $|\eta| < \eta'_x$. The inequality (37) under the condition $M(\delta_x) \leq \gamma$.

Choose a small positive constant $r_x$ such that $\gamma + r_x < 1$. Then it can be checked that Proposition 2 also holds for the system (29) where $M_x$ is replaced by $M_x + r_x$ and $\gamma$ is replaced by $\gamma + r_x$. Now, Choosing the time constant $t_{s1}$ as in the equation (25), Theorem 1 also holds. Let us summarize the previous discussion into the following theorem.

**Theorem 2:** Consider the chained system (29) with unknown parameters $\eta_0$ and $\eta_1$. Let $\gamma < 1$, $m$, $\epsilon < m$, $\delta_i < m - \epsilon$, $T$ and $\epsilon_i$ be six given positive constants. Let $r_x < 1 - \gamma$ be a positive constant and $\lambda_1, \lambda_2, \cdots, \lambda_{10}$ be a set of pre-designed eigenvalues such that $M(\delta) \leq \gamma$. Let $\eta'_0$ and $\eta'_1$ be two positive constants such that $\delta_i(1 + \eta'_1) < m - \epsilon$, $|\eta| < 1$ and the inequality (34) holds. Choose the controllers $u_i$ as in the equations (13) and (26). Then, every solution $(x_v(t), x(t))$ of the closed-loop system satisfies the inequalities (27) and (28) with

$$
\sigma = \frac{-\ln(\gamma + r_x)}{T}, \sigma_0 = \frac{-\ln(1 + \eta)}{T} \text{ and } k_i = \frac{(M_x + r_x)}{\gamma + r_x} e^{\sigma \tau}
$$

for all $|\eta| < \eta'_0$ and $|\eta| < \eta'_1$.

---

**4. Simulations and case study**

Let us apply Theorem 2 to study the parking problem of the following mobile robot model [3];

$$
\dot{x} = pv \cos(\theta + r) 
$$

$$
\dot{y} = pv \sin(\theta + r) 
$$

$$
\dot{\theta} = qw, 
$$

where $p$ and $q$ are both unknown parameters relating to the unknown radius of the rear wheels and the distance between them and $r$ denotes a small bias in orientation. We assume that $|\eta| \leq \Delta r$ with $\Delta r$ being a known bound, and $p$ and $q$ belong to a known interval $(p_{min}, p_{max})$ and $(q_{min}, q_{max})$, respectively.

Let $p_x = \frac{p_{max} + p_{min}}{2}$, $q_x = \frac{q_{max} + q_{min}}{2}$, $\Delta p_x = \frac{p_{max} - p_{min}}{2}$ and $\Delta q_x = \frac{q_{max} - q_{min}}{2}$. Define $\eta_x = \frac{q}{q_x} - 1 \in [-\Delta q_x, \Delta q_x]$, $p = \frac{p}{p_x}$, and $\eta_i = (r, \overline{r})$ with $|\eta| \leq \sqrt{(\Delta r)^2 + (\Delta p)^2}$. Then, $q = q_x(1 + \eta_i)$ and $p = p_x(1 + \overline{p})$. Define the new coordinates by [4]

$$
\begin{bmatrix}
\eta \\
\eta_i \\
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
\sin \theta & -\cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\eta_x \\
\eta_i \\
\end{bmatrix}
$$

In the new coordinate and by the definitions of $\eta_0$ and $\eta_1$, the mobile robot system can be transformed into a system of the form (29) where the matrix-valued functions $\Delta A(\eta_0, \eta_1)$ and $\Delta B(\eta_0, \eta_1)$ can be given in the following

$$
\Delta A = \begin{bmatrix}
-(1 + \overline{p}) \sin r & \eta_i \\
-\eta_i + \overline{p} \cos r - 1 + \cos r & 0 \\
\end{bmatrix}, \hspace{1cm} \Delta B = \begin{bmatrix}
-(1 + \overline{p}) \sin r \\
\overline{p} \cos r - (1 - \cos r) \\
\end{bmatrix}, \hspace{1cm} (41)
$$

Thus, Theorem 2 can be applied to guarantee a globally exponential practical robust stability for small values of $\Delta q_x$, $\Delta p_x$, $\Delta r$. This is achieved in present literature yet.

Note that a two-wheeled mobile robot (see Fig. 1) can be modeled as a system of the form (39). Indeed, let $w_i$ and $w_j$ be the angular velocities of the left wheel and the right wheel, respectively. Let $R$ and $L$ denote the unknown radius of the rear wheels and the distance between them, respectively. Define

$$
\omega = \frac{w_j - w_i}{2} \hspace{1cm} \text{and} \hspace{1cm} v = \frac{w_i + w_j}{2}.
$$

Then, it can be modeled as a system of the form (39) with $p = R$ and $q = \frac{2R}{L}$ [4]. For simulations, suppose $p = R = 0.11$ $(m)$ and $L = 0.5$ $(m)$. Then, $q_x = 0.44$. Let $(p_{min}, p_{max}) = (0.05, 0.33)$ and $(q_{min}, q_{max}) = (0.2, 0.5)$. Then, $p_x = 0.15$ and $q_x = 0.35$. Using the numerical
method, we find $\lambda_0 = 1.3$ and $\delta_0 = 1$ that determine an approximation optimal value $M_0 = 1.3141$.

**Table 1 The parameters used in the simulation.**

<table>
<thead>
<tr>
<th>m</th>
<th>$T$</th>
<th>$y$</th>
<th>$\varepsilon = \varepsilon_1$</th>
<th>$d_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>0.5</td>
<td>0.003</td>
<td>11.3</td>
</tr>
</tbody>
</table>

For simulation, let the initial condition be $x(0) = 0(m)$, $y(0) = 1(m)$, $\theta(0) = 0(rad)$. Simulation results for the case $r = 0$ (having no bias in orientation) and the case $r = \pi/6$ (having a small bias in orientation) are shown in Figs. 2-4. In each case, a satisfied result is achieved. It can be seen that the case of $r = \pi/6$ has a more bad transient behavior than the case of $r = 0$. This is due to the error in orientation.

### 5. Conclusion

A novel practical stabilizer was proposed for chained systems. The pole-placement method was used to improve the convergence rate and the transient response. From simulation study, the proposed exponential practical stabilizer has the same fast convergence behavior as the usual exponential stabilizer. Moreover, it was shown that the proposed controller is robust w.r.t. some uncertainties in the model.

**Acknowledgement:** This work was supported by the National Science Council, Taiwan, R.O.C., under contracts NSC-90-2213-E-159-003.

**References**


Fig. 1. A two-wheeled mobile robot.

Fig. 2. Time history of position and angle ($x$: solid line; $y$: dashed line). (a) $r = 0$, (b) $r = \pi/6$.

Fig. 3. Applied angular velocities (solid line: $w_r$, dashed line: $w_\theta$). (a) $r = 0$, (b) $r = \pi/6$.

Fig. 4. Moving trajectory of mobile robot. (a) $r = 0$, (b) $r = \pi/6$. 

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