Interaction Graphs: Exponentials

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Abstract
In two previous papers \cite{Sei12a, Sei12b}, we exposed a combinatorial approach to the program of Geometry of Interaction, a program initiated by Jean-Yves Girard \cite{Gir89b}. The strength of our approach lies in the fact that we interpret proofs by simpler structures — graphs — than Girard's constructions, while generalizing the latter since they can be recovered as special cases of our setting. This third paper tackles the complex issue of defining exponential connectives in this framework. In order to succeed in this, we consider a generalization of graphs named graphings, which is in some way a geometric realization of a graph. We explain how we can then define a GoI for Elementary Linear Logic (ELL), a sub-system of linear logic where representable functions are exactly the functions computable in elementary time. This construction is moreover parametrized by the choice of a map, giving rise to a whole family of models.

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1. Introduction

1.1. Geometry of Interaction

The program of geometry of interaction was introduced by Girard [Gir87b, Gir89b] after his discovery of linear logic [Gir87a]. In a first approximation, it aims at defining a semantics of proofs that accounts for the dynamics of cut-elimination. Namely, the geometry of interaction models differ from usual (denotational) semantics in that the interpretation of a proof $\pi$ and its normal form $\rho$ are not equal, but one has a way of computing the interpretation of the normal form $\rho$ from the interpretation of the proof $\pi$ (illustrated in Figure 1). As a consequence, a geometry of interaction models not only proofs — programs — but also their normalization — their execution. This semantical counterpart to the cut-elimination procedure was called the execution formula by Girard in his first papers about geometry of interaction [Gir89a, Gir88, Gir95a], and it is a way of computing the solution to the so-called feedback equation. This equation turned out to have a more general solution [Gir06], which lead Girard to the definition of a geometry of interaction in the hyperfinite factor [Gir11].

Geometry of Interaction, however, is not only about the interpretation of proofs and their dynamics, but also about reconstructing logic around this semantical counterpart to the cut-elimination procedure. This means that logic arises from the dynamics and interactions of proofs — programs —, as a syntactical description of the possible behaviors of proofs — programs. This aspect of the geometry of interaction program has been less studied than the proof interpretation part.

We must also point out that geometry of interaction has been successful in providing tools for the study of computational complexity. The fact that it models the execution of programs explains that it is well suited for the study of complexity classes in time [BP01, Lag09], as well as in space [AS12, AS13]. It was also used to explain [GAL92] Lamping’s optimal reduction of lambda-calculus [Lam90].

1.2. Interaction Graphs

Interaction Graphs were first introduced [Sei12b] to define a combinatorial approach to Girard’s geometry of interaction in the hyperfinite factor [Gir11].
The main idea was that the execution formula — the counterpart of the cut-elimination procedure — can be computed as the set of alternating paths between graphs, and that the measurement of interaction defined by Girard using the Fuglede-Kadison determinant [FK52] can be computed as a measurement of a set of cycles.

The setting was then extended to deal with additive connectives [Sei12a], showing by the way that the constructions were a combinatorial approach not only to Girard's hyperfinite GoI construction but also to all the earlier constructions [Gir87b, Gir89a, Gir88, Gir95a]. This result could be obtained by unveiling a single geometrical property, which we called the trefoil property, upon which all the constructions of geometry of interaction introduced by Girard are founded. This property, which can be understood as a sort of associativity, suggests that computation — as modeled by geometry of interaction — is closely related to algebraic topology.

This paper takes another direction though: based on ideas that appeared in the author's phd thesis [Sei12c], it extends the setting of graphs in order to deal with both exponential connectives and second order quantification. In order to do so, we generalize the previous interaction graphs settings in two different directions. First, we introduce the notion of thick graph, which is a generalization of the notion of sliced graphs introduced to deal with additive connectives. While sliced graphs were a simple superposition of graphs over the same set of vertices, thick graphs can possess edges between two different graphs of this superposition. This generalization is enough to implement contraction, as we explain in Section 4. However, we need to generalize the setting a bit further in order to define interesting exponential connectives. The idea is that an exponential should transform a thick graph into a graph. Since a thick graph is morally a superposition of a (finite) number of graphs — the slices, it can easily be seen as a graph in the usual sense. However, the number of slices can vary and this means that the number of vertices of the resulting graph is variable, which makes this construction unviable in the setting of thick graphs. In order to advert this issue, we generalize graphs in order to give vertices a size. This could be done syntactically, but we chose to take advantage of the tools of measure theory: a vertex will be a measurable set. It is then natural to interpret edges as measure-preserving transformations between two measurable sets. This motivates our use (and definition) of graphings, i.e. geometric realization of graphs as families of measure-preserving transformations between measurable sets.

1.3. Outline of the paper

In the first section, we recall some important definitions and properties on directed weighted graphs. We then recall some definitions and properties about the additive construction [Sei12a]. These properties are key to the understanding of the construction of the multiplicative-additive fragment of linear logic in the setting of interaction graphs.

In the fourth section, we define and study the notion of thick graphs, and show how it can be used to interpret the contraction $!A \to !A \otimes !A$. Even though the notion of thick graphs is enough to define contraction, we introduce in the
next section a generalization of graphs, namely the notion of graphings, that will be used to define an interesting notion of exponentials (with functorial promotion). After introducing the notion of graphing, we show how those can be used as graphs and one can consider paths and cycles. We also prove that for a family of well-suited map (called circuit-quantifying maps), the trefoil property — the keystone upon which the construction of multiplicative and additives are constructed — holds in this setting.

This leads to the definition of exponential connectives. We first exhibit a family of circuit-quantifying maps, and then explain how one can define an exponential connective. We show for this a result which allows us to encode any bijection over the natural numbers as a measure-preserving map over the unit interval of the real line. This result is then used to encode some combinatorics as measure-preserving maps and show that functorial promotion can be implemented for the exponential we defined. We then prove two soundness results for two variants (in Sections 7 and 8) of Elementary Linear Logic.

2. Interaction Graphs: Geometric Adjunction

2.1. Graphs

We will define the notion of a directed weighted graph. The weights considered will be taken in a set $\Omega$ closed under a binary operation that we will denote multiplicatively. In all our examples, we will use the set $[0,1]$ as $\Omega$, considered with the usual real multiplication.

**Definition 1.** A directed $(\Omega)$-weighted graph is a tuple $G = (V^G, E^G, s^G, t^G, \omega^G)$, where $V^G$ is the set of vertices, $E^G$ the set of edges, $s^G$ and $t^G$ are two maps from $E^G$ into $V^G$, the source and target functions respectively, and $\omega^G$ is a map: $E^G \to \Omega$, the weight map.

We will denote by $E^G(v,w)$ the set of edges $e \in E^G$ such that $s^G(e) = v$ and $t^G(e) = w$. Moreover, we will forget about the exponents when the context will be sufficiently clear.

We now define a construction on graphs which will allow us to consider, being given two graphs $G$ and $H$, the set of alternating paths between $G$ and $H$, i.e. the set of sequences of edges such that each edge in $G$ is followed by an edge in $H$ and conversely. This construction is fairly standard in the literature [AHS02, AJM94, dF09]. The first definition, the plugging of two graphs, is the keystone of the construction, and we will explain later how a slight modification can be introduced in order to define dialects, something which will be necessary to deal with additive and exponential connectives. Once we will be able to talk about paths and cycles that alternate between two graphs, we will show two very important results: we will define an associative operation called the execution of two graphs — an operation that will correspond to the cut-elimination procedure, and a geometric identity (Theorem 15) which is the key property upon which the interpretation of multiplicative and additive connectives lies [Sei12a].
In the following the symbol \$\sqcup\$ will be used to denote the disjoint union of sets. Given sets \$E, F\$ and \$X\$ together with functions \$f : E \rightarrow X\$ and \$g : F \rightarrow X\$, we will write \$f \sqcup g\$ the function \$E \sqcup F \rightarrow X\$ defined by the universal property of coproducts, the copairing of \$f\$ and \$g\$.

**Definition 2** (Union of graphs). Given two graphs \$G\$ and \$H\$, one defines the union \$G \cup H\$ of \$G\$ and \$H\$ by:

\[
(V^G \cup V^H, E^G \sqcup E^H, s^G \sqcup s^H, t^G \sqcup t^H, \omega^G \sqcup \omega^H)
\]

**Definition 3** (Plugging). Given two graphs \$G\$ and \$H\$, one defines the graph \$G \Box H\$ as the union of \$G\$ and \$H\$, considered with a coloring map \$\delta\$ from \$E^G \sqcup E^H\$ into \(\{0, 1\}\) and such that:

\[
\begin{align*}
\delta(x) &= 0 \text{ si } x \in E^G \\
\delta(x) &= 1 \text{ si } x \in E^H
\end{align*}
\]

We will refer to \$G \Box H\$ as the plugging of \$G\$ and \$H\$.

Figures 3 and 4 show two examples of plugging between the graphs \$F\$ and \$G\$ and the graphs \$F\$ and \$H\$ which appear in Figure 2. In these figures the colors of the edges are represented by the position of the edges: the edges represented above the set of vertices correspond to the edges colored 0, while the edges represented below the set of vertices correspond to the edges colored 1.
2.2. Cycles, Paths and Circuits

**Definition 4** (Paths). A path $\pi$ in a graph $G$ is a finite sequence of edges $(e_i)_{0 \leq i \leq n}$ in $E^G$ such that $s(e_{i+1}) = t(e_i)$ for all $0 \leq i < n - 1$. We will call the vertices $s(\pi) = s(e_0)$ and $t(\pi) = t(e_n)$ the source and target of the path respectively.

**Definition 5** (Alternating paths). Let $G$ and $H$ be two directed weighted graphs. We define the alternating paths between $G$ and $H$ as the paths $(e_i)$ in $G \Box H$ that satisfy:

$$\delta(e_i) \neq \delta(e_{i+1})$$

We will denote by $\text{Chem}(G,H)$ the set of alternating paths between $G$ and $H$, and we will denote by $\text{Chem}^{v,w}(G,H)$ the set of alternating paths between $G$ and $H$ of source $v$ and target $w$.

**Remark.** Keeping the notations of the preceding definition, if $V^G \cap V^H = \emptyset$, the set of alternating paths in $\text{Chem}(G,H)$ is reduced to the set $E^G \cup E^H$, modulo the identification between edges and paths of length 1.

**Definition 6** (Cycles and $k$-cycles). A path $\pi = (e_i)_{0 \leq i \leq n}$ is a cycle if $s(e_0) = t(e_n)$, i.e. if the source and target of $\pi$ are equal. If $\pi$ is a cycle, and $k$ is the greatest integer such that there exists a cycle $\rho$ with $\rho \equiv \pi \mod k$, we will say that $\pi$ is a $k$-cycle.

An alternating cycle in $G \Box H$ will be a cycle $(e_i)_{0 \leq i \leq n} \in \text{Chem}(G,H)$ such that $\delta(e_n) \neq \delta(e_0)$.

**Proposition 7.** Let $\rho = (e_i)_{0 \leq i \leq n-1}$ be a cycle, and $\sigma$ the permutation sending $i$ to $i+1$ ($i = 0, \ldots, n-2$) and $n-1$ to 0. One define the set:

$$\bar{\rho} = \{(e_{\sigma^k(i)})_{0 \leq i \leq n-1} \mid 0 \leq k \leq n-1\}$$

Then $\rho$ is a $k$-cycle if and only if the cardinality of the set $\bar{\rho}$ is equal to $n/k$. In the following, we will use the term $k$-circuit (or more generally circuit) to refer to such an equivalence class modulo cyclic permutations of a $k$-cycle.

---

2Here, $\rho^k$ denotes the concatenation of $k$ copies of $\rho$. 
\[ G \text{ is defined as the product of the weights of the edges that it is composed of:} \]
\[ \omega_G(n) = \prod_{i=0}^{n} \omega^G(e_i). \]

This last definition does not depend on the path in itself, but on the multiset of edges that it is composed of. This weight is therefore invariant modulo cyclic permutations and one can define the weight of a circuit as the weight of the cycles in the equivalence class.

**Definition 8.** The weight of a path \( \pi = (e_i)_{0 \leq i < n} \) in a directed weighted graph \( G \) is defined as the union graph \( F \) of \( G \) and \( \omega \), together with the coloring function \( \delta \):
\[
\begin{cases}
\delta(x) = (0, \delta^F(x)) & \text{if } x \in E^F \\
\delta(x) = (1, \delta^G(x)) & \text{if } x \in E^G
\end{cases}
\]

**Definition 9.** Let \( F, G \) be two graphs endowed with two coloring functions \( \delta^F, \delta^G \) on the edges. We extend the plugging operation as follows: the graph \( F \times G \) is defined as the union graph \( F \cup G \) of \( F \) and \( G \), together with the coloring function \( \delta \):
\[
\delta(x) = \begin{cases}
(0, \delta^F(x)) & \text{if } x \in E^F \\
(1, \delta^G(x)) & \text{if } x \in E^G
\end{cases}
\]

**Proposition 10.** The plugging operation is associative. Given three graphs \( F, G, \) and \( H \) with coloring functions \( \delta^i : E^i \rightarrow C^i \) \( (i = F, G, H) \):
\[
F \times (G \times H) \equiv (F \times G) \times H
\]
where the symbol \( \equiv \) means that the graphs are equal up to a renaming of colors.

**Proof.** The underlying graphs \( F \cup (G \cup H) \) and \( (F \cup G) \cup H \) are equal. The coloring function is thus the only thing that differs. Now, notice that if \( x \) is an edge in \( F \times (G \times H) \), we are in one of three following cases:
\[
\delta^{F \times (G \times H)}(x) = \begin{cases}
(0, \delta^F(x)) & \text{if } x \in E^F \\
(1, \delta^G(x)) & \text{if } x \in E^G
\end{cases}
\]

Similarly, if \( x \) is an edge in \( (F \times G) \times H \) we are in one of the three following cases:
\[
\delta^{(F \times G) \times H}(x) = \begin{cases}
(0, \delta^F(x)) & \text{if } x \in E^F \\
(1, \delta^G(x)) & \text{if } x \in E^G
\end{cases}
\]

We define the map \( \theta \) from \( [0,1] \times ([0,1] \times (C^F \cup C^G) \cup C^H) \) into \( [0,1] \times (C^F \cup [0,1] \times (C^G \cup C^H)) \) by:
\[
\begin{align*}
\theta(0,0,x) &= (0,x) \\
\theta(0,1,x) &= (1,0,x) \\
\theta(1,x) &= (1,1,x) \\
\theta(x) &= x \text{ otherwise}
\end{align*}
\]

One can then easily check that \( (F \times G) \times H \) together with the coloring function \( \theta \circ \delta^{(F \times G) \times H} \) is equal to the graph \( F \times (G \times H) \) together with the coloring function \( \delta^{F \times (G \times H)} \). \qed
Corollary 10.1. The set of alternating paths in \((F \Box G) \Box H\) is in bijection with the set of alternating paths in \(F \Box (G \Box H)\).

Corollary 10.2. The set of alternating cycles in \((F \Box G) \Box H\) is in bijection with the set of alternating cycles in \(F \Box (G \Box H)\).

Corollary 10.3. The set of alternating circuits in \((F \Box G) \Box H\) is in bijection with the set of alternating circuits in \(F \Box (G \Box H)\).

We will also need the following proposition later. Its proof is immediate. Indeed, renaming (bijectively) the vertices of two graphs \(F\) and \(G\) does not change the way they interact. This implies that the set of cycles are equal.

Proposition 11. Let \(F, G\) be two graphs, and \(\phi : V^F \cup V^G \to E\) a bijection. We denote by \(\phi(F)\) (resp. \(\phi(G)\)) the graph defined as \((\phi(V^F), E^F, s^F, t^F, \omega^F)\) (resp. \((\phi(V^G), E^G, s^G, t^G, \omega^G)\)). Then the set of alternating circuits in \(F \Box G\) is equal to the set of alternating circuits in \(\phi(F) \Box \phi(G)\).

2.3. Circuits Adjunction

As a direct consequence of the preceding results, one gets the geometric adjunction of circuits. We must first define the set of circuits and the operation of execution.

Definition 12 (Set of circuits). Let \(F, G\) be two graphs. We denote by \(\text{Circ}(F, G)\) the set of alternating circuits in \(F \Box G\).

We now define the execution of two graphs, which resembles the "composition and hiding" of strategies in game semantics. This operation is the counterpart of the cut-elimination procedure. If \(F\) and \(G\) are two graphs, their execution \(F \bowtie G\) will be defined as the set of alternating paths between \(F\) and \(G\) (the "composition" part) whose source and target are elements of the symmetric difference \(V^F \Delta V^G\) of \(V^F\) and \(V^G\) (the "hiding" part).

Definition 13 (Execution). Let \(G\) and \(H\) be two weighted directed graphs, and write \(G = (V^G, E^G, s^G, t^G, \omega^G)\) and \(H = (V^H, E^H, s^H, t^H, \omega^H)\). We define the execution of \(G\) and \(H\) as the graph \(G \bowtie H\) where:

\[
\begin{align*}
V^{G \bowtie H} &= V^G \Delta V^H \\
E^{G \bowtie H} &= \bigcup_{e, e' \in V^G \Delta V^H} \text{Chem}_{\text{G,H}}^e(e', e) \\
{s}^{G \bowtie H} &= (e_i)_{0 \leq i \leq n} \mapsto s^{G \bowtie H}(e_0) \\
{t}^{G \bowtie H} &= (e_i)_{0 \leq i \leq n} \mapsto t^{G \bowtie H}(e_n) \\
{\omega}^{G \bowtie H} &= (e_i)_{0 \leq i \leq n} \mapsto \omega^{G \bowtie H}(e_i)_{0 \leq i \leq n}
\end{align*}
\]

It is obviously necessary to choose a unique renaming for both graphs, otherwise it would be possible to make it so that the intersection between the sets of vertices of the renamings of \(F\) and \(G\) be empty while \(V^F \cap V^G \neq \emptyset\).
Remark. If the graphs $G, H$ have disjoint sets of vertices (i.e. if $V^G \cap V^H = \emptyset$), then $G \sqcap H$ is equal to $G \cup H$ up to a renaming of the set of edges (identifying the paths of length 1 with the edge it is composed of).

Figures 5 and 6 show the alternating paths in $F \square G$ and $F \square H$, where $F, G$ and $H$ are the graphs introduced in Figure 2. Notice the apparition of an internal cycle between $F$ and $H$, i.e. an alternating cycle between $F$ and $G$ disappearing during the computation of the execution $F \sqcap G$.

In the following proposition, one should understand the equality symbol as meaning “equal modulo a renaming of edges”. We chose to write the equality symbol since we will be working in the following with equivalence classes modulo such renamings. Since one can show that two graphs which are equal modulo a renaming of edges are universally equivalent (Definition 40 for the universal equivalence, and Proposition 41). We decided to push the discussion about universal equivalence to the end of the section about graphs in order to give a general definition that applies not only to directed weighted graphs, but also to the sliced, thick and sliced graphs that will be introduced later on.

**Theorem 14** (Associativity of Execution). Let $F, G, H$ be three graphs such that $V^F \cap V^G \cap V^H = \emptyset$. Then:

$$F \sqcap (G \sqcap H) = (F \sqcap G) \sqcap H$$
Proof. This is an immediate consequence of Corollary 10.3.

Remark. Execution is not a simple composition of functions and it is associative, as a consequence of locativity, only when some condition about how the sets of vertices (the locations) intersect is satisfied. To get a counter-example to associativity when this condition is not satisfied, it is sufficient to consider the three graphs $F, G, H$ with $V_F = V_G = V_H = \{1\}$ and such that $F, G$ have no edges and $H$ has a unique edge (whose source and target are necessarily the vertex 1): then $F \cap (G \cap H) = F$ and $(F \cap G) \cap H = H$.

Even though associativity is valid only when this condition is satisfied, the preceding theorem has as a corollary the associativity of composition in the category defined from our constructions, since the composition in this category will be defined after carefully renaming the sets of vertices [Sei12b, Sei12a].

Theorem 15 (Trefoil Property). Let $F, G$ and $H$ be three graphs such that $V_F \cap V_G \cap V_H = \emptyset$. Then:

$$\text{Circ}(F, G \cap H) \cup \text{Circ}(G, H) \equiv \text{Circ}(H, F \cap G) \cup \text{Circ}(F, G)$$

where $\equiv$ means that there exists an $\omega$-invariant bijection, i.e. a bijection that preserves the weights.

Proof. This is a direct consequence of Corollary 10.3. Indeed, the set of alternating circuits in $F \sqcap (G \sqcap H)$ is in bijection with the set $\text{Circ}(F, G \cap H) \cup \text{Circ}(G, H)$. Similarly, the set of alternating circuits in $(F \sqcap G) \sqcap H$ is in bijection with the set $\text{Circ}(F \cap G, H) \cup \text{Circ}(F, G)$.

Remark. This property is not satisfied if one considers the sets of cycles instead of the sets of circuits. An alternating circuit in $F \sqcap (G \sqcap H)$ which contains at least an edge of $F$ corresponds to a unique alternating circuit in $F \sqcap (G \cap H)$. However, the number of edges (the length) of these two circuits are not in general equal, and therefore there are no bijection between the set of alternating cycles in $F \sqcap (G \sqcap H)$ going through at least one edge in $F$ and the set of alternating cycles in $F \sqcap (G \cap H)$.

Corollary 15.1 (Adjunction). Let $F, G, H$ be three graphs such that $V^G \cap V^H = \emptyset$. Then:

$$\text{Circ}(F, G \cup H) \equiv \text{Circ}(H, F \cap G) \cup \text{Circ}(F, G)$$

3. Sliced Graphs and the Additive Construction

The trefoil property and the resulting adjunction we presented were stated as the existence of a bijection between sets of circuits. We will now explain how one can obtain quantitative versions of these properties. The interest in this

\[\text{Under the assumption that } V^F \cap V^G \cap V^H = \emptyset \text{ obviously.}\]
change of viewpoint is twofold: first of all, we will be able to obtain a more classical adjunction (with only two terms, as in the coherence spaces for instance) by introducing the notion of wager, and it will be used moreover to obtain a combinatorial construction of additives.

We will then explain how one can generalize the notion of graphs by adding slices, and then by considering thick graphs. These generalizations will be used to construct and interpret the additive and exponential connectives.

3.1. Numerical Adjunctions

Definition 16 (Quantification of cycles). A circuit-quantifying map is a function: \( \Omega \to \mathbb{R}_{\geq 0} \cup \{\infty\} \).

Given such a circuit-quantifying map \( m \), one defines a measurement of the interaction between two graphs \( F, G \) as:

\[
[F, G]_m = \sum_{\rho \in Cy(F, G)} m(\omega(\rho))
\]

Theorem 17 (Numerical Trefoil Property). Let \( F, G, \) and \( H \) be graphs such that \( V_F \cap V_G \cap V_H = \emptyset \). Then:

\[
[F, G :: H]_m + [G, H]_m = [F, G]_m + [F :: G, H]_m
\]

Proof. This is a direct consequence of the geometric trefoil property\(^{15}\) and the fact that the circuit-quantifying map takes only positive values which means that the series converge if and only if they are summable.

Theorem 18 (Numerical Adjunction). Let \( F, G, \) and \( H \) be graphs such that \( V_G \cap V_H = \emptyset \). Then:

\[
[F, G \cup H]_m = [F, G]_m + [F :: G, H]_m
\]

We will now explain how one can obtain, from the last stated property, a more standard adjunction that has the same format as the adjunctions one finds in linear logic denotational semantics. What one has to do is get rid of the additional term \( [F, G]_m \) that appears in the numerical adjunction. In order to do this, we will use the same method as Girard in his geometry of interaction in the hyperfinite factor, namely the introduction of wagers.

The idea behind the introduction of wagers is very simple. Instead of considering only graphs, we will associate to them a real number which will keep track of the eventual errors in the execution, i.e. which will capture the additional term of the adjunction when it is non-zero. We thus extend in the natural way the generalization of the notions of execution and measurement to couples of a real number and a graph.

We consider couples \((a, A)\), where \( a \) is an element in \( \text{im}(\cdot, \cdot)_m \), and \( A \) is a graph. We then define:

1. \( \ll (a, A), (b, B) \gg = a + b + [A, B]_m \);
2. \( (f, F) :: (a, A) = (a + f + \|F, A\|_m, F :: A) \).
Notice that when $V^F \cap V^A = \emptyset$, one has $(f, F) : a, A = (a + f, A \cup F)$, which will be denoted $(a, A) \cup (f, F)$.

**Proposition 19** (Trefoil Property). If $F, G, H$ are graphs such that $V^F \cap V^G \cap V^H = \emptyset$, and $f, g, h$ are elements in $\text{im}([\cdot, \cdot])$, then:

$$\ll (f, F), (g, G) \cup (h, H) \gg = \ll (f, F), (g, G) \cup (h, H) \gg = \ll (h, H), (f, F) : (gG) \gg$$

**Corollary 19.1** (Adjunction). If $F, G, H$ are graphs such that $V^G \cap V^H = \emptyset$, and $f, g, h$ are elements in $\text{im}([\cdot, \cdot])$, then:

$$\ll (f, F), (g, G) \cup (h, H) \gg = \ll (f, F), (g, G) \cup (h, H) \gg$$

### 3.2. Sliced Graphs

We introduced the notion of sliced graphs in order to deal with additive connectives [Sei12a]. These are juxtapositions of graphs, called slices in order to point out the connection with the notion of slices in Girard’s version of additive proof nets [Gir95b]. We will therefore consider finite families of graphs $G = \{G_i\}_{i \in I}$ sharing the same set of vertices $V^G$. Sliced graphs can also be represented as graphs over the set of vertices $V \times I$ such that the edges do not go from one slice to another: if $(v, i)$ and $(w, j)$ are respectively the source and target of an edge $e$, then $i = j$.

All the constructions introduced in the preceding chapter can be extended to sliced graphs by applying them slice by slice. However, this framework turns out to be insufficient for defining additive connectives correctly, and we will consider sliced graphs equipped with a weight map on slices. This generalization is indeed necessary in order to get enough counter-projects in the orthogonal of the additive connective $\&$.

We can now define the objects with which we will work. A directed weighted sliced graph $G$ of carrier $V^G$ is a finite family (that we chose to write as a formal sum) $\sum_{i \in I} a^G_i G_i$ where for all $i \in I^G$, $G_i$ is a directed weighted graph such that $V^G_i = V^G$, and $a^G_i \in \mathbb{R}$. The graph $G_i$ will be called the $i$-th slice of $G$, and we will use the following notation:

$$1_G = \sum_{i \in I^G} a^G_i$$

Figure 7 shows an example of sliced graph: it represents the graph $\frac{1}{2}F + \pi(G \cup H)$ where $F, G, H$ are the graphs represented in Figure 2. From now on, we will use the following graphical convention:

- The graphs will be once again represented with colored edges (each graph being associated with a single color), and delimited by hashed lines;
- The elements of the carrier $V^G$ will be represented on a horizontal scale, while elements of the index set $I^G$ will be represented on a vertical scale;
- Inside a given graph, slices are separated by a hashed line;
In a corner of each box representing a slice, we write the monomial of the sum $\sum_{i \in I} \alpha_i^G G_i$ which is contained in the box.

**Definition 20** (Execution). Let $F$ and $G$ be two sliced graphs. We define their execution:

$$\left( \sum_{i \in I_F} \alpha_i^F F_i \right) \cdot \left( \sum_{i \in I_G} \alpha_i^G G_i \right) = \sum_{(i, j) \in I_F \times I_G} \alpha_i^F \alpha_j^G F_i : G_j$$

When $V_F \cap V_G = \emptyset$, we will denote the execution of $F$ and $G$ as a union: $F \cup G$.

**Definition 21.** Let $F$ and $G$ be two sliced graphs. We define the measurement of their interaction:

$$\left\lfloor \sum_{i \in I_F} \alpha_i^F F_i, \sum_{i \in I_G} \alpha_i^G G_i \right\rfloor_m = \sum_{(i, j) \in I_F \times I_G} \alpha_i^F \alpha_j^G \left[ F_i, G_j \right]_m$$

When some of the terms $\left[ F_i, G_j \right]_m$ are equal to $\infty$, we define $\left[ F, G \right]_m$ to be equal to $\infty$.

A simple calculation allows one to show that the trefoil property holds in this generalized setting.

**Proposition 22** (Trefoil Property). Let $F$, $G$, $H$ be sliced graphs such that $V_F \cap V_G \cap V_H = \emptyset$,

$$[F, G : H]_m + 1_F[H, G]_m = [H, F : G]_m + 1_H[F, G]_m$$
Proof. This is simple calculation, using Proposition 19.

\[ \|F, G : H\|_m + 1_F [G, H]_m \]

\[ = \|F, G : H\|_m + \left( \sum_{i \in I^F} d^F(i) \sum_{j \in I^G} \sum_{k \in I^H} [G_j, H_k]_m \right) \]

\[ = \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) [F_i, G_j : H_k]_m \]

\[ + \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) [G_j, H_k]_m \]

\[ = \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) ([F_i, G_j : H_k]_m + [G_j, H_k]_m) \]

\[ = \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) ([F_i, G_j] + [F_i, G_j]_m) \]

\[ = \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) [F_i \cap G_j, H_k]_m \]

\[ + \sum_{i \in I^F} \sum_{j \in I^G} \sum_{k \in I^H} d^F(i) d^G(j) d^H(k) [F_i, G_j]_m \]

\[ = \|H, F : G\|_m + \left( \sum_{k \in I^H} d^H(k) \right) \sum_{i \in I^F} \sum_{j \in I^G} [F_i, G_j]_m \]

\[ = \|H, F : G\|_m + 1_H [F, G]_m \]

\[ \square \]

**Corollary 22.1** (Adjunction).

If \( F, G, H \) are sliced graphs such that \( V^F = V^G \cup V^H \) and \( V^G \cap V^H = \emptyset \),

\[ \|F, G \cup H\|_m = \|F \cap G, H\|_m + 1_H \|F, G\|_m \]

3.3. Construction of Additives

3.3.1. Wagers

As we explained at the beginning of this section, the addition of wagers allows one to keep track of the additional term of the adjunction and obtain a setting in which an "real" adjunction (without the additional term) holds. We follow the same pattern here and consider projects, which are couple \( a = (a, A) \) of a real number (eventually infinite) \( a \in \mathbb{R} \cup \{ \infty \} \) and a sliced graph \( A \). Then the execution and measurement can be extended from sliced graphs to projects naturally:

1. \( \langle (a, A), (b, B) \rangle = a 1_B + b 1_A + [A, B]_m \);
2. \( (f, F) : (a, A) = (a 1_F + f 1_A + [F, A]_m, F : A) \).

Notice that when \( V^F \cap V^A = \emptyset \), one has \( (f, F) : (a, A) = (a 1_F + f 1_A, A \cup F) \), which will be denoted \( (a, A) \cup (f, F) \).

We then obtain the following two results.

---

\(^5\)We use the convention that any arithmetical expression containing \( \infty \) is equal to \( \infty \).
Proposition 23 (Trefoil Property). If \( f = (f, F) \), \( g = (g, G) \) and \( h = (h, H) \) are projects such that \( V^F \cap V^G \cap V^H = \emptyset \), then:

\[
\ll (f, F), (g, G) \gg = \ll (g, G), (h, H) \gg = \ll (h, H), (f, F) \gg (gG) \gg
\]

Corollary 23.1 (Adjunction). If \( f = (f, F) \), \( g = (g, G) \) and \( h = (h, H) \) are projects such that \( V^G \cap V^H = \emptyset \), then:

\[
\ll (f, F), (g, G) \cup (h, H) \gg = \ll (f, F) \cup (g, G), (h, H) \gg
\]

3.3.2. Multiplicatives

The next step in the construction is to define a notion of orthogonality by saying that two projects are orthogonal when \( \ll a, b \gg \neq \emptyset \). The notion of orthogonality is a way of representing negation at the level of proofs and is strongly related to the correctness criterion for proof nets. The adjunction, as stated in the last corollary, insures that orthogonality (hence negation) and execution (hence linear implication) interact properly and can be restated as \( f \vdash a \otimes b \) if and only if \( f :: a \perp b \).

Orthogonality is used to define conducts — formulas — as types i.e. as a set of projects equal to its bi-orthogonal. One could equivalently define a conduct as a set of projects which is the orthogonal of a given set of projects — interpreted as tests. This leads to the definition of a model of Multiplicative Linear Logic by considering the constructions on conducts induced by the operations on projects:

\[
A \otimes B = \{ a \cup b \mid a \in A, b \in B \}_{\perp \perp}
\]

\[
A \rightarrow B = \{ f \mid \forall a \in A, f :: b \in B \}
\]

We refer to our previous papers \cite{Sei12b, Sei12a} for a detailed construction and a proof of the fact that this indeed defines an interesting model of MLL. Let us just stress that the equality \( (A \otimes B_{\perp})_{\perp} = A \rightarrow B \) is a simple consequence of the adjunction.

3.3.3. Additives

In order to obtain a good model of additive connectives, we restrict the notion of conduct and consider the better behaved notion of behavior. A behavior is a conduct \( A \) such that both \( A \) and \( A_{\perp} \) have the inflation property, i.e. for any \( a \in A \) (resp. \( a \in A_{\perp} \)) and any real number \( \lambda \), the project \( a + \lambda \circ \) is an element of \( A \) (resp. \( A_{\perp} \)). One then shows that the restriction of MLL constructions between behaviors produce behaviors, hence the restriction to behavior still is a suitable model of multiplicative linear logic\cite{Sei12a}.

The formal sum of sliced graphs can be naturally extended to projects. It can then be used to define additive connectives. The construction uses the formal

\[\text{More generally, one can define orthogonality as } \ll a, b \gg_m \in P \text{ for any subset } P \text{ of the compactified real line.}\]

\[\text{One however lose multiplicative units, which are not behaviors } \cite{Sei12a}.\]
sum to deal with additive contraction, as one can understand from the project used to implement distributivity.

**Proposition 24** (Distributivity). For any behaviors $A, B, C$, and any delocations $\phi, \psi, \theta, \rho$ of $A, A, B, C$ respectively, there is a project $\text{distr}$ in the behavior

$$((\phi(A) \rightarrow \theta(B)) \& (\psi(A) \rightarrow \rho(C)) \rightarrow (\sigma(A \& (B \& C)))$$

**Proof.** Define the projects:

$$f_1 = \overline{\delta} \delta \phi \otimes \overline{\delta} \delta \theta \otimes \psi(V_A) \cup \rho(V_C) \otimes 0$$

$$f_2 = \overline{\delta} \delta \psi \otimes \overline{\delta} \delta \rho \otimes \phi(V_A) \cup \theta(V_B) \otimes 0$$

$$\text{distr} = f_1 + f_2$$

Then $\text{distr} : A = f_1 + f_2 : g$ is the project we are looking for (a detailed proof can be found in the paper about the additive construction for Interaction Graphs [Sei12a]).

The additives are defined as follows on projects $a \& b = a + b$ while an element of $A \oplus B$ is either of the form $a \cup 0$ with $a \in A$ or of the form $0 \cup b$ with $b \in B$. It can be shown that this leads to the definition of an interesting model of MALL [Sei12a].

We now recall some propositions and theorems that are proved in our previous paper [Sei12a] and will be used (extensively for some of them) in the rest of this paper.

**Lemma 25** (Homothety). Conducts are closed under homothety: for all $a \in A$ and all $\lambda \in \mathbb{R}$ with $\lambda \neq 0, \lambda a \in A$.

**Proposition 26.** If $A$ is a non-empty set of projects of carrier $V$ such that $a \in A \Rightarrow a + \lambda 0 \in A$, then any project in $A^\perp$ is wager-free, i.e. if $(a, A) \in A^\perp$ then $a = 0$.

**Proposition 27.** If $A$ is a non-empty set of projects of same carrier $V^A$ such that $(a, A) \in A^\perp$ implies $a = 0$, then $b \in A^\perp$ implies $b + \lambda 0 \in A^\perp$ for all $\lambda \in \mathbb{R}$.

**Proposition 28.** We denote by $A \odot B$ the set $\{a \odot b \mid a \in A, b \in B\}$. Let $E, F$ be non-empty sets of projects of respective carriers $V, W$ with $V \cap W = \emptyset$. Then

$$(E \odot F)^\perp = (E^\perp \odot F^\perp)^\perp$$

**Proposition 29.** Let $A, B$ be conducts. Then:

$$((a \odot B \mid a \in A) \cup (a \odot b \mid b \in B))^\perp = A \oplus B$$

4. **Thick Graphs and Contraction**

4.1. **Thick Graphs**

**Definition 30.** Let $S^G$ and $D^G$ be finite sets. A directed weighted thick graph $G$ of carrier $S^G$ and dialect $D^G$ is a directed weighted graph over the set of vertices $S^G \times D^G$.

We will call slices the set of vertices $S^G \times \{d\}$ for $d \in D^G$. 

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Figure 8: Two thick graphs $G$ and $H$, both with dialect $\{1, 2\}$

Figure 8 shows two examples of thick graphs. Thick graphs will be represented following a graphical convention very close to the one we used for sliced graphs:

- Graphs are once again represented with colored edges and delimited by hashed lines;
- Elements of the carrier $S^G$ are represented on a horizontal scale, while elements of the dialect $D^G$ are represented on a vertical scale;
- Inside a given graph, slices are separated by a *dotted* line.

**Remark.** If $G = \sum_{i \in I^G} \alpha_i^G G_i$ is a sliced graph such that $\forall i \in I^G, \alpha_i^G = 1$, then $G$ can be identified with a thick graph of dialect $I^G$. Indeed, one can define the thick graph $(G)$ by:

\[
\begin{aligned}
V^{[G]} &= V^G \times I^G \\
E^{[G]} &= \bigcup_{i \in I^G} E_i^G \\
s^{[G]} &= e \in E_i^G \mapsto (s^G_i(e), i) \\
t^{[G]} &= e \in E_i^G \mapsto (t^G_i(e), i) \\
\omega^{[G]} &= e \in E_i^G \mapsto \omega^G(e)
\end{aligned}
\]

**Definition 31 (Variants).** Let $G$ be a thick graph and $\phi : D^G \to E$ a bijection. One defines $G^\phi$ as the graph:

\[
\begin{aligned}
V^{G^\phi} &= S^G \times E \\
E^{G^\phi} &= E^G \\
s^{G^\phi} &= (Id_{V^G} \times \phi) \circ s^G \\
t^{G^\phi} &= (Id_{V^G} \times \phi) \circ t^G \\
\omega^{G^\phi} &= \omega^G
\end{aligned}
\]
If $G$ and $H$ are two thick graphs such that $H = G^\phi$ for a bijection $\phi$, then $H$ is called a variant of $G$. The relation defined by $G \sim H$ if and only if $G$ is a variant of $H$ can easily be checked to be an equivalence relation.

**Definition 32 (Dialectal Interaction).** Let $G$ and $H$ be thick graphs.

1. We denote by $G^{\text{dp}}_{DH}$ the thick graph of dialect $D^G \times D^H$ defined as $\{ \sum_{i \in D^H} G \}$;
2. We denote by $H^{\text{dp}}_{DG}$ the thick graph of dialect $D^G \times D^H$ defined as $\{ \sum_{i \in D^G} H \}^{\tau}$ where $\tau$ is the natural bijection $D^H \times D^G \rightarrow D^G \times D^H, (a, b) \rightarrow (b, a)$.

We can then define the plugging $F \Box G$ of two thick graphs as the plugging of the graphs $F^{\text{dp}}_{DG}$ and $G^{\text{dp}}_{DH}$. Figure 10 shows the result of the plugging of $G$ and $H$, the thick graphs represented in Figure 8.

One can then define the execution $G \triangleright H$ of two thick graphs $G$ and $H$ as the execution of the graphs $G^{\text{dp}}_{DH}$ and $H^{\text{dp}}_{DG}$. Figure 11 shows the set of alternating paths in the plugging of the thick graphs $G$ and $H$ introduced in Figure 8. Figures 12 and 13 represent the result of the execution of these two thick graphs, the first is three-dimensional representation which can help make the connection with the set of alternating paths in Figure 11 while the second is a two-dimensional representation of the same graph. In a natural way, the measurement of the interaction between two thick graphs $G,H$ is defined as $[G^{\text{dp}}_{DH},H^{\text{dp}}_{DG}]_m$.

**Definition 33.** The execution $F \triangleright G$ of two thick graphs $F,G$ is the thick graph of carrier $S^F \Delta S^G$ and dialect $D^F \times D^G$ defined as $F^{\text{dp}}_{DG} \triangleright G^{\text{dp}}_{DF}$.

Since we only modified the graphs before plugging them together, we can make the following remark. Let $F, G, H$ be thick graphs. Then the thick graph

![Figure 9: The graphs $G^{\text{dp}}_{DH}$ and $H^{\text{dp}}_{DG}$](image-url)
Figure 10: Plugging of the thick graphs $G$ and $H$

Figure 11: Alternating paths in the plugging of thick graphs $G$ and $H$
Figure 12: Result of the execution of the thick graphs $G$ and $H$

Figure 13: The thick graph $G::H$ represented in two dimensions.
Definition 35. Let 

\[ F :: (G :: H) \]

be thick graphs. We define \( Cy^e(\cdot, \cdot) \) as the set of circuits in \( F^1 \phi \sqcap G^1 \phi \).

We also define, being given a dialect \( D^H \),

- the set \( Cy^e(\cdot, \cdot)^1 \phi \) of circuits in the graph \( (F^1 \phi \sqcap G^1 \phi)^1 \phi \)
- the set \( Cy^e(\cdot, \cdot)^1 \phi \) of circuits in the graph \( (F^1 \phi \sqcap G^1 \phi)^1 \phi \)

Proposition 36. Let \( F, G, H \) be thick graphs and \( \psi : D^H \rightarrow D \) a bijection. Then:

\[
\begin{align*}
Cy^e(F, H) &\equiv Cy^e(F, H^\psi) \\
Cy^e(F, G)^1 \phi &\equiv Cy^e(F, G)^1 \phi^1 \phi \\
Cy^e(F, G)^1 \phi &\equiv Cy^e(F, G)^1 \phi^1 \phi
\end{align*}
\]

Proof. These bijections are simple consequences of the property \[ [1] \]
Proposition 37 (Geometric Trefoil Property for Thick Graphs). If \( F, G, H \) are thick graphs such that \( S^F \cap S^G \cap S^H = \emptyset \), then:
\[
\text{Cy}^F(F, G :: H) \cup \text{Cy}^G(G, H) \uparrow^D \equiv \text{Cy}^F(F, H) \cup \text{Cy}^G(F, G) \uparrow^D
\]

Corollary 37.1 (Geometric Adjunction for Thick Graphs). If \( F, G, H \) are thick graphs such that \( S^G \cap S^H = \emptyset \), we have:
\[
\text{Cy}^F(F, G \cup H) \equiv \text{Cy}^F(F :: G, H) \cup \text{Cy}^F(F, G) \uparrow^D
\]

Definition 38. Being given a circuit quantifying map \( m \), one can define a measurement of the interaction between thick graphs. For every couple of thick graphs \( F, G \), it is defined as:
\[
[F, G]_m = \sum_{\pi \in \text{Cy}^F(F, G)} \frac{1}{\text{Card}(D^F \times D^G)} m(\omega(\pi))
\]

Proposition 39 (Numerical Trefoil Property for Thick Graphs). Let \( F, G, H \) be thick graphs such that \( S^F \cap S^G \cap S^H = \emptyset \). Then:
\[
[F, G :: H]_m + [G, H]_m = [H, F :: G]_m + [F, G]_m
\]

Proof. The proof is a simple calculation using the geometric trefoil property for thick graphs (Proposition 37). We denote by \( n^F \) (resp. \( n^G, n^H \)) the cardinality of the dialect \( D^F \) (resp. \( D^G, D^H \)).
\[
[F, G :: H]_m + [G, H]_m = \sum_{\pi \in \text{Cy}^F(F, G :: H)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^G(G, H)} \frac{1}{n^G n^H} m(\omega(\pi))
\]
\[
= \sum_{\pi \in \text{Cy}^F(F, G :: H) \cup \text{Cy}^G(G, H) \uparrow^D} \frac{1}{n^F n^G n^H} m(\omega(\pi))
\]
\[
= \sum_{\pi \in \text{Cy}^F(H, F :: G)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^G(F, G)} \frac{1}{n^F n^G n^H} m(\omega(\pi))
\]
\[
= [H, F :: G]_m + [F, G]_m
\]

Corollary 39.1 (Numerical Adjunction for Thick Graphs). Let \( F, G, H \) be thick graphs such that \( S^G \cap S^H = \emptyset \). Then:
\[
[F, G :: H]_m = [H, F :: G]_m + [F, G]_m
\]
The measurement of interaction we defined hides a convention: each slice of a thick graph \( F \) is considered as having a "weight" equal to \( 1/n_F \), so that the total weight of the set of all slices have weight 1. This convention corresponds to the choice of working with a normalized trace (such that \( tr(1) = 1 \)) on the idiom in Girard's hyperfinite geometry of interaction. It would have been possible to consider another convention which would impose that each slice have a weight equal to 1 (this would correspond to working with the usual trace on matrices in Girard's hyperfinite geometry of interaction). In this case, the measurement of the interaction between two thick graphs \( F, G \) is defined as:

\[
\langle F, G \rangle = \sum_{\pi \in Cy^c(F,G)} m(\omega(\pi))
\]

The numerical trefoil property is then stated differently: for all thick graphs \( F, G, H \) such that \( \mathcal{S}^F \cap \mathcal{S}^G \cap \mathcal{S}^H = \emptyset \), we have:

\[
\langle F, G |::| H \rangle + n^F \langle G, H \rangle = \langle H, F |::| G \rangle + n^H \langle F, G \rangle
\]

We stress the apparition of the terms \( n^F \) and \( n^H \) in this equality: their apparition corresponds exactly to the apparition of the terms \( 1_F \) and \( 1_H \) in the equality stated for the trefoil property for sliced graphs.

### 4.2. Sliced Thick Graphs

One can of course apply the additive construction presented in Section 3 in the case of thick graphs. A sliced thick graph \( G \) of carrier \( S^G \) is a finite family \( \sum_{i \in I^G} a^G_i G_i \) where, for all \( i \in I^G \), \( G_i \) is a thick graph such that \( S^{G_i} = S^G \), and \( a^G_i \in \mathbb{R} \). We define the dialect of \( G \) to be the set \( \omega_{i \in I^G} D^{G_i} \). We will abusively call a slice a couple \((i, d)\) where \( i \in I^G \) and \( d \in D_{G_i} \); we will say a graph \( G \) is a one-sliced graph when \( I^G = \{i\} \) and \( D_{G_i} = \{d\} \) are both one-element sets.

As in Section 3, one can extend the execution and the measurement of the interaction by applying the thick graphs constructions slice by slice:

\[
\left( \sum_{i \in I^F} a^F_i F_i \right) |::| \left( \sum_{i \in I^G} a^G_i G_i \right) = \sum_{(i,j) \in I^F \times I^G} a^F_i a^G_j F_i |::| G_j
\]

\[
\left\{ \sum_{i \in I^F} a^F_i F_i, \sum_{i \in I^G} a^G_i G_i \right\} m = \sum_{(i,j) \in I^F \times I^G} a^F_i a^G_j [F_i, G_j] m
\]

Figure 14 shows two examples of sliced thick graphs. The graphical convention we will follow for representing sliced and thick graphs corresponds to the graphical convention for sliced graphs, apart from the fact that the graphs contained in the slices are replaced by thick graphs. Thus, two slices are separated by a dashed line, two elements in the dialect of a thick graph (i.e. the graph contained in a slice) are separated by a dotted line.

One should however notice that some sliced thick graphs (for instance the graph \( F_a + F_b \) represented in red in Figure 14) can be considered both as a thick
Figure 14: Examples of sliced thick graphs: \( \frac{1}{2}F + 3G \) and \( F_a + F_b \)

graph — hence a sliced thick graph with a single slice — or as a sliced graph with two slices — hence a sliced thick graph with two slices. Indeed, consider the graphs:

\[
\begin{align*}
V_{F_a} &= \{1, 2\} & V_{F_b} &= \{1, 2\} & V_{F_c} &= (1, 2) \times \{a, b\} \\
E_{F_a} &= \{f, g\} & E_{F_b} &= \{f, g\} & E_{F_c} &= \{f_a, f_b, g_a, g_b\} \\
s_{F_a} &= \{f \mapsto 1\} & s_{F_b} &= \{f \mapsto 1\} & s_{F_c} &= \{f_i \mapsto s^{F_i}(f)\} \\
t_{F_a} &= \{f \mapsto 2\} & t_{F_b} &= \{f \mapsto 2\} & t_{F_c} &= \{f_i \mapsto t^{F_i}(f)\} \\
o_{F_a} &= 1 & o_{F_b} &= 1 & o_{F_c} &= 1
\end{align*}
\]

One can then define the two sliced thick graphs \( G_1 = F_c \) and \( G_2 = \frac{1}{2}F_a + \frac{1}{2}F_b \). These two graphs are represented in Figure 15. They are similar in a very precise sense: one can show that if \( H \) is any sliced thick graph, and \( m \) is any circuit-quantifying map, then \( [G_1, H]_m = [G_2, H]_m \). We say they are universally equivalent. Notice that this explains in a very formal way the remark about the convention on the measurement of interaction 4.1.

**Definition 40** (Universally equivalent graphs). Let \( F, G \) be two graphs. We say that \( F \) and \( G \) are universally equivalent (for the measurement \( [\cdot, \cdot]_m \)) — which will be denoted by \( F \simeq_u G \) — if for all graph \( H \):

\[
[F, H]_m = [G, H]_m
\]

The next proposition states that if \( F' \) is obtained from a graph \( F \) by a renaming of edges, then \( F \simeq_u F' \).
Proposition 41. Let $F, F'$ be two graphs such that $V^F = V^{F'}$, and $\phi$ a bijection $E^F \rightarrow E^{F'}$ such that:

$s^G \circ \phi = s^F$, \quad $t^G \circ \phi = t^F$, \quad $\omega^G \circ \phi = \omega^F$

Then $F \simeq_u F'$.

Proof. This is obvious. Indeed, the bijection $\phi$ induces, from the hypotheses in the source and target functions, a bijection between the sets of cycles $\mathrm{Cy}(F, H)$ and $\mathrm{Cy}(G, H)$. The condition on the weight map then insures us that this bijection is $\omega$-invariant, from which we deduce the proposition.

Proposition 42. Let $F, G$ be sliced graphs. If there exists a bijection $\phi : I^F \rightarrow I^G$ such that $F_i = G_{\phi(i)}$ and $a^F_i = a^G_{\phi(i)}$, then $F \simeq_u G$.

Proof. By definition:

$$[G, H]_m = \sum_{(i,j) \in I^F \times I^H} a^G_i a^H_j [G_i, H_j]_m$$

$$= \sum_{(i,j) \in I^F \times I^H} a^G_{\phi(i)} a^H_j [G_{\phi(i)}, H_j]_m$$

$$= \sum_{(i,j) \in I^F \times I^G} a^F_i a^G_j [F_i, G_j]_m$$

Thus $F$ and $G$ are universally equivalent.

Proposition 43. Let $F, G$ be thick graphs. If there exists a bijection $\phi : D^F \rightarrow D^G$ such that $G = F^\phi$, then $F \simeq_u G$.

Proof. Let $F, G$ be thick graphs such that $G = F^\phi$ for a bijection $\phi : D^G \rightarrow D^F$, and $H$ an arbitrary thick graph. Then the bijection $\phi \times \text{Id} : D^G \times D^H \rightarrow D^F \times D^H$ satisfies that $G^\phi_{\phi} = (F^\phi_{\phi})^{\phi \times \text{Id}}$. Using Proposition 11, one can state that the set
of alternating circuits in $F^\square H^I$ is the same as the set of alternating circuits in $(F^\uparrow)^{\phi \times 1^d} \square (H^\uparrow)^{\phi \times 1^d} = G^\square H^I$. Thus:

$$[F, H]_m = \sum_{\pi \in \text{Cy}(F, H)} m(\omega(\pi))$$

$$= \sum_{\pi \in \text{Cy}(G, H)} m(\omega(\pi))$$

$$= [G, H]_m$$

And finally $F$ and $G$ are universally equivalent. □

**Proposition 44.** Let $F = \sum_{i \in I^F} a^F_i F_i$ be a sliced thick graph, and let us define, for all $i \in I^F$, $n^F_i = \text{Card}(D^{F_i})$ and $n^F = \sum_{i \in I^F} n^F_i$. Suppose that there exists a scalar $\alpha$ such that for all $i \in I^F$, $a^F_i = \alpha \frac{n_F}{n^{F_i}}$. We then define the sliced thick graph with a single slice $aG$ of dialect $\cup D^{F_i} = \bigcup_{i \in I^F} D^{F_i} \times \{i\}$ and carrier $V^F$ by:

$$V^G = V^F \times \cup D^{F_i}$$

$$E^G = \cup_{i \in I^F} E^{F_i} \times \{i\}$$

$$s^G = (e, i) \mapsto (s^{F_i}(e), i)$$

$$t^G = (e, i) \mapsto (t^{F_i}(e), i)$$

$$\omega^G = (e, i) \mapsto \omega^{F_i}(e)$$

$$\left( (e, i) \Rightarrow (f, j) \Rightarrow (i \neq j) \lor (i = j \land e \neq F_i(f) \land f) \right)$$

Then $F$ and $G$ are universally equivalent.

**Proof.** Let $H$ be a sliced thick graph. Then:

$$[F, H]_m = \sum_{i \in I^H} \sum_{j \in I^F} a^H_i a^F_j [F_i, H_j]_m$$

$$= \sum_{i \in I^H} \sum_{j \in I^F} a^H_i a^F_j \frac{n^{F_i}}{n^{F_j}} [F_i, H_j]_m$$

$$= \sum_{i \in I^H} \sum_{j \in I^F} a^H_i \frac{n^{F_i}}{n^{F_j}} \frac{1}{n^{F_j}} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi))$$

$$= \sum_{i \in I^H} \sum_{j \in I^F} a^H_i \frac{1}{n^{F_j}} \sum_{j \in I^F} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi))$$

But one can notice that $\cup_{i \in I^F} \text{Cy}(F_i, H_j) = \text{Cy}(G, H)$. We thus get:

$$[F, H]_m = \sum_{i \in I^H} a^H_i \frac{1}{n^{F_j}} \sum_{j \in I^F} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi))$$

$$= \sum_{i \in I^H} a^H_i \frac{1}{n^{F_j}} \sum_{\pi \in \text{Cy}(F, H_j)} m(\omega(\pi))$$

$$= \| aG, H \|_m$$
Finally, we showed that $F$ and $aG$ are universally equivalent.

One of the consequences of the Propositions 41, 42 and 43 is that two graphs $F,G$ such that $G$ is obtained from $F$ by a renaming of the sets $E^F, I^F, D^F$ are universally equivalent. We will therefore work from now on with graphs modulo renaming of these sets.

4.3. Thick Graphs and Contraction

In this section, we will explain how the introduction of thick graphs allow the definition of contraction by using the fact that edges can go from a slice to another (contrarily to sliced graphs). In the following, we will be working with sliced thick graphs. The way contraction is dealt with by using slice-changing edges is quite simple, and the graph which will implement this transformation is essentially the same as the graph implementing additive contraction (i.e. the graph implementing distributivity — Proposition 24 — restricted to the location of contexts) modified with a change of slices.

The graph we obtain is then the superimposition of two $\mathcal{D}A$, but where one of them goes from one slice to the other.

**Definition 45** (Contraction). Let $\phi : V^A \rightarrow W_1$ and $\psi : V^A \rightarrow W_2$ be two bijections with $V^A \cap W_1 = V^A \cap W_2 = W_1 \cap W_2 = \emptyset$. We define the project $\mathcal{C}t_{\phi, \psi} = (0, \mathcal{C}t_{\phi, \psi})$, where...
where the graph $\text{Ctr}_\phi^\psi$ is defined by:

$$
\begin{align*}
V^\text{Ctr}_\phi^\psi &= V^A \cup W_1 \cup W_2 \\
D^\text{Ctr}_\phi^\psi &= \{1, 2\} \\
E^\text{Ctr}_\phi^\psi &= V^A \times \{1, 2\} \times \{i, o\} \\
\pi^\text{Ctr}_\phi^\psi &= \begin{cases} 
(v, 1, o) \rightarrow (\phi(v), 1) \\
(v, 1, i) \rightarrow (v, 1) \\
(v, 2, o) \rightarrow (\psi(v), 1) \\
(v, 2, i) \rightarrow (v, 2)
\end{cases} \\
\omega^\text{Ctr}_\phi^\psi &= 1
\end{align*}
$$

Figure 16 illustrates the graph of the project $\text{Ctr}_\phi^\psi$, where the functions are defined by $\phi : \{1, 2, 3\} \rightarrow \{4, 5, 6\}, x \mapsto x + 3$ and $\psi : \{1, 2, 3\} \rightarrow \{7, 8, 9\}, x \mapsto 10 - x$.

**Proposition 46.** Let $a = (0, A)$ be a project in a behavior $A$, such that $D^A \equiv \{1\}$. Let $\phi, \psi$ be two delocations $V^A \rightarrow W_1, V^A \rightarrow W_2$ of disjoint codomains. Then $\text{Ctr}_\phi^\psi : a \in \phi(A) \odot \psi(A)$.

**Proof.** We will denote by $\text{Ctr}$ the graph $\text{Ctr}_\phi^\psi$ to simplify the notations. We first compute $\text{Ctr}^\phi_{\text{Ctr}}$. We get $\text{Ctr}^\phi_{\text{Ctr}} = (V^A \times \{1, 2\}, E^A \times \{1, 2\}, S^A \times \text{Id}_{\{1, 2\}}, \text{Id}^\phi \circ \pi)$ where $\pi$ is the projection: $E^A \times \{1, 2\} \rightarrow E^A, (x, i) \mapsto x$. Moreover the graph $\text{Ctr}^\phi_{\text{Ctr}}$ is a variant of the graph $\text{Ctr}$ since $D^A \equiv \{1\}$. Here is what the plugging of $\text{Ctr}^\phi_{\text{Ctr}}$ with $A^\phi_{\phi_{\text{Ctr}}}$ looks like:

The result of the execution is therefore a two-sliced graph, i.e. a graph of dialect $D^A \times \{1, 2\} \equiv \{1, 2\}$, and which contains the graph $\phi(A) \cup \psi(A)$ in the slice numbered 1 and contains the empty graph in the slice numbered 2.

We deduce from this that $\text{Ctr}_\phi^\psi : a$ is universally equivalent (Definition 40) to the project $\frac{1}{2} \phi(a) \odot \psi(a) + \frac{1}{2} \psi$ from Proposition 44. Since $\phi(a) \odot \psi(a) \in \phi(A) \odot \psi(A)$, then the project $\frac{1}{2} (\phi(a) \odot \psi(a))$ is an element in $\phi(A) \odot \psi(A)$ by the homothety
Lemma 25. Moreover, $A$ is a behavior, hence $\phi(A) \otimes \psi(A)$ is also a behavior and we can deduce that $\frac{1}{2} \phi(a) \otimes \psi(a) + \frac{1}{2} \phi \otimes \psi$ is an element in $\phi(A) \otimes \psi(A)$. $\Box$

Figures 18, 19 and 20 illustrate the plugging and execution of a contraction with two graphs: the first — $A$ — having a single slice, and the other — $B$ — having two slices (the graphs are shown in Figure 17). One can see that the hypothesis $D^A \equiv \{1\}$ used in the preceding proposition is necessary, and that slice-changing edges allow to implement contraction of graphs with a single slice.

We will use the following direct corollary of Proposition 28.

Proposition 47. If $E$ is a non-empty set of project sharing the same carrier $V^E$, $F$ is a conduct and $\ell$ satisfies that $\forall \varepsilon \in E, \varepsilon \in F$, then $\ell \in E_{\perp \perp} \rightarrow F$.

This proposition insures us that if $A$ is a conduct such that there exists a set $E$ of one-sliced projects with $A = E_{\perp \perp}$, then the contraction project $\text{Ctr}_\psi^\phi \in A$ belongs to the conduct $A \rightarrow \phi(A) \otimes \psi(A)$.

We find here a geometrical explanation to the introduction of exponential connectives. Indeed, in order to use a contraction, we must be sure we are working with one-sliced graphs. We will therefore define, for all behavior $A$, a conduct $!A$ generated by a set of one-sliced graphs.

One should notice that a conduct $!A$ generated by a set of one-sliced projects cannot be a behavior: the projects $(a, \phi)$ necessarily belong to the orthogonal of
Figure 18: Plugging of Ctr$^\psi$ with the two graphs $A$ and $B$
(a) Result of the execution of $\text{Cnt}_A^\phi$ and $A$

(b) The graph of $\phi(a) \otimes \psi(a)$

Figure 19: Graphs of the projects $\text{Cnt}_A^\phi \vdash a$ and $\phi(a) \otimes \psi(a)$

!A. We will therefore introduce \textit{perennial} conducts as those conducts generated by a set of wager-free one-sliced projects. Dually, we introduce the \textit{co-perennial conducts} as the conducts that are the orthogonal of a perennial conduct.

But first, we will need a way to associate a wager-free one-sliced project to any wager-free project. In order to do so, we will introduce the notion of \textit{graphing}, a generalization of directed weighted graphs: a weighted graphing on a measured space $(X, \mathcal{B}, m)$ will be a directed weighted graph whose vertices are measurable sets, and whose edges are measure-preserving maps.

5. Graphings

5.1. Definitions

The idea is that a graphing is a sort of "geometric realization" of a graph: the vertices correspond to measurable subsets of a measured space, and edges correspond to measure-preserving maps from the source subset onto the target subset. Some difficulties arise when one wants to define a tractable notion of graphing. Indeed, a new phenomenon appears when vertices are measurable sets: what should one do when two vertices are neither disjoint or equal, i.e. when two vertices are not equal but their intersection is not of null measure? One solution would be to define graphings where vertices are disjoint subsets (i.e. their intersection is of null measure), but this makes the definition of execution extremely complex.

Let us consider for instance two graphings with a single edge each, and whose plugging is represented in Figure 21. To represent the set of alternating paths whose source and target are subsets of the symmetric difference of the carriers — the execution of the two graphs — we would need to decompose each
(a) Result of the execution of $\text{Ctr}_\psi^\phi$ and $B$

(b) Graph of the project $\phi(b) \otimes \psi(b)$

Figure 20: Graphs of the projects $\text{CTR}_\psi^\phi :: b$ and $\phi(b) \otimes \psi(b)$
of the measurable sets into a disjoint union of sets, each one corresponding to the source and/or target of a path. In the particular case we show in the figure, this operation is not that complicated: it is sufficient to consider the sets \((\phi\psi)^{-k}(U_t - V_s) \cap (U_s - V_t)\). However, the operation quickly becomes much more complicated as we add new edges and create cycles. Figure 22 represents the case of two graphings with two edges each. Defining the decomposition of the set of vertices induced by the execution is — already in this case — quite difficult to define. In particular, since the sets of vertices considered can be infinite (but countable), the number of cycles can be infinite, and the operation is then of an extreme complexity.

As a consequence, we have chosen to work with a different presentation of graphings, where two distinct vertices can have an intersection of strictly positive measure — they can even be equal. Even though we will be using graphings defined on the measured space \((\mathbb{R}, \mathcal{B}, \lambda)\) of the real line endowed with Lebesgue measure, we define a general notion of graphing on any measured space. This will be of use when we will need to introduce the sliced and/or thick graphings. We will now define the notion of graphing taking into account these remarks. The terminology is borrowed from a work of Damien Gaboriau [Gab00], in which the underlying notion of graphing (forgetting about the weights) is defined.

In the following, we will consider that we chose a set \(\Omega\) closed under multiplication which contains the possible weights of the edges. In all example, we will however use the set \(\Omega = ]0,1]\) endowed with the usual multiplication.

**Definition 48 (Weighted Graphings).** Let \(X = (X, \mathcal{B}, \lambda)\) be a measured space and
A graphing over $X$ of carrier $V^F$ is a countable family $F = \{(\omega^F_e, \phi^F_e : S^F_e \to T^F_e)_{e \in E^F}\}$, where, for all $e \in E^F$, $\omega^F_e$ is an element of $\Omega_e$ and $\phi^F_e$ is a measure-preserving transformation between the measurable sets $S^F_e \subset V^F$ and $T^F_e \subset V^F$.

We define the effective carrier of the graphing $F$ as the measurable set $V^F_{\text{eff}}$ defined as $\cup_{e \in E^F} S^F_e \cup T^F_e$, and which is by definition a subset of the carrier $V^F$ of $F$.

Remark. In particular, one can notice that if $F$ is a weighted graphing, then for all $e \in E^F$, $\lambda(S^F_e) = \lambda(T^F_e)$.

It is usual, when doing measure theory, to work modulo sets of null measure. Similarly, we will work with graphings modulo almost everywhere equality, a notion that we need to define first. Before giving the definition, we will define the useful notion of empty graphing. An empty graphing will be almost everywhere equal to the graphing without edges.

Definition 49 (Empty graphings). A graphing $F$ is said to be empty if its effective carrier is of null measure.

Definition 50 (Almost Everywhere Equality). Two graphings $F, G$ are almost everywhere equal if there exists two empty graphings $0_F, 0_G$ and a bijection $\theta : E^F \cup E^G \to E^F \cup E^G$ such that:

- for all $e \in E^F \cup E^G$, $\omega^F_{\theta(e)} = \omega^G_{\theta(e)}$;
- for all $e \in E^F \cup E^G$, $S^F_{\theta(e)} \Delta S^G_{\theta(e)}$ is of null measure;
- for all $e \in E^F \cup E^G$, $T^F_{\theta(e)} \Delta T^G_{\theta(e)}$ is of null measure;
- for all $e \in E^F \cup E^G$, $\phi^F_{\theta(e)}$ and $\phi^G_{\theta(e)}$ are equal almost everywhere on $S^G_{\theta(e)} \cap S^F_{\theta(e)}$.

Proposition 51. We define the relation $\sim_{a.e.}$ between weighted graphings:

$F \sim_{a.e.} G$ if and only if $F$ and $G$ are almost everywhere equal

This relation is an equivalence relation.

Proof. It is obvious that this relation is reflexive and symmetric (it suffices to take the bijection $\theta^{-1}$). We therefore only need to show transitivity. Let $F, G, H$ be three graphings such that $F \sim_{a.e.} G$ and $G \sim_{a.e.} H$. Therefore there exists four empty graphings $0_F, 0_G, 0_H, 0_H$ and two bijections $\theta_{F,G} : E^F \cup E^G \to E^G \cup E^0_{0_F}$ and $\theta_{G,H} : E^G \cup E^0_{0_H} \to E^H \cup E^0_{0_H}$ that satisfy the properties listed in the preceding definition. We notice that $0_F \cup 0_G \cup 0_H$ are empty graphings. One can then define $\theta_{F,H} = (\theta_{G,H} \cup \text{Id}_{E^0_{0_F}}) \circ (\text{Id}_{E^G} \cup \tau) \circ (\theta_{F,G} \cup \text{Id}_{E^0_{0_H}})$, where $\tau$ represents the symmetry $E^0_{0_F} \cup E^0_{0_H} \to E^0_{0_H} \cup E^0_{0_F}$;
It is then easy to verify that the three first properties of almost everywhere equality are satisfied. We will only detail the proof that the fourth property also holds. We will forget about the superscripts in order to simplify notations. We will moreover denote by \( \tilde{\theta}_{F,G} \) (resp. \( \tilde{\tau} \), resp. \( \tilde{\theta}_{G,H} \)) the function \( \theta_{F,G} \cup \text{Id}_{E^0_{H}} \) (resp. \( \text{Id}_{E^0_G} \cup \tau \), resp. \( \theta_{G,H} \cup \text{Id}_{E^0_G} \)).

Choose \( e \in E^F \cup E^0_F \cup E^0_{GH} \):

- if \( e \in E^0_{GH} \), then \( \tilde{\theta}_{F,G}(e) = e \), and \( \phi_{\tilde{\theta}(e)} = \phi(e) \);
- if \( e \in E^F \cup E^0_F \) then, by the definition of \( \tilde{\theta}_{F,G} \), \( \phi_{\tilde{\theta}(e)} \) is almost everywhere equal to \( \phi_e \) on \( S_e \cap S_{\tilde{\theta}(e)} \).

Thus \( \phi_{\tilde{\theta}(e)} \) and \( \phi_e \) are equal almost everywhere on \( S_e \cap S_{\tilde{\theta}(e)} \) in all cases. A similar reasoning shows that for all \( f \in E^G \cup E^0_H \cup E^G_F \), the functions \( \phi_{\theta_{G,H}(f)} \) and \( \phi_f \) are almost everywhere equal on \( S_{\theta_{G,H}(f)} \cap S_f \).

Moreover, \( \phi_{\tilde{\theta}_{G,F}(e)} \) and \( \phi_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \) are equal and have the same domain \( S_{\tilde{\theta}_{G,F}(e)} = S_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \). Thus \( \phi_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \) and \( \phi_e \) are almost everywhere equal on the intersection \( S_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \cap S_e \). Moreover, \( \phi_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \) and \( \phi_{\tilde{\theta}_{G,H}(\tilde{\theta}(\tilde{\theta}_{G,F}(e)))} \) are almost everywhere equal on the intersection \( S_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} \cap S_{\tilde{\theta}_{G,H}(\tilde{\theta}(\tilde{\theta}_{G,F}(e)))} \). We deduce from this that the functions \( \phi_e \) and \( \phi_{\tilde{\theta}_{G,H}(e)} \) are almost everywhere equal on

\[
S_e \cap S_{\theta_{G,H}(e)} \cap S_{\tilde{\theta}(\tilde{\theta}_{G,F}(e))} = S_e \cap S_{\theta_{G,H}(e)} \cap S_{\tilde{\theta}_{G,F}(e)}
\]

We denote by \( Z \) the set of null measure on which they differ. Since \( S_e \Delta S_{\tilde{\theta}_{G,F}(e)} \) of null measure, there exists two sets \( X,Y \) of null measure such that \( S_e \cup X = S_{\tilde{\theta}_{G,F}(e)} \cup Y \). We can then deduce that \( S_{\tilde{\theta}_{G,F}(e)} = S_e \cup X - Y \). Thus

\[
S_e \cap S_{\theta_{G,H}(e)} \cap S_{\tilde{\theta}_{G,F}(e)}
= S_e \cap S_{\tilde{\theta}_{G,F}(e)} \cap (S_e \cup X - Y)
= S_e \cap S_{\tilde{\theta}_{G,H}(e)} \cap S_e - Y
= S_e \cap S_{\tilde{\theta}_{G,H}(e)} - Y
\]

We then conclude that the functions \( \phi_e \) and \( \phi_{\tilde{\theta}_{G,H}(e)} \), restricted to \( S_e \cap S_{\tilde{\theta}_{G,H}(e)} \), are equal outside of \( Y \cup Z \) which is a set of null measure.

\[\text{One can chose } X \text{ in such a way so that } S_{\tilde{\theta}_{G,F}(e)} \cap Y = \varnothing.\]
5.2. Paths and Cycles

The choice we made to work with a notion of graphing where vertices may have non-trivial intersections makes the plugging operation easier to define. We will however need to define what is a path, since we won’t be able to work with the usual notion of a path in a graph. Obviously, a path will be a finite sequence of edges. We will replace the condition that the source of an edge be equal to the target of the preceding edge by the condition that the intersection of these source and target sets be of non-null measure.

**Definition 52** (Plugging). Being given two weighted graphings $F, G$, we define their plugging $F \square G$ as the weighted graphing $F \oplus G$ endowed with the coloring function $\delta : E^{F \oplus G} \to \{0, 1\}$ such that $\delta(e) = 1$ if and only if $e \in E^G$.

**Definition 53** (Alternating Paths). A path in a graphing $F$ is a finite sequence $(e_i)_{i=0}^n$ of elements of $E^F$ such that for all $0 \leq i < n - 1$, $E^F \cap S^F_{e_{i+1}}$ is of strictly positive measure.

An alternating path between two graphings $F, G$ is a path $(e_i)_{i=0}^n$ in the graphing $F \square G$ such that for all $0 \leq i < n - 1$, $\delta(e_i) \neq \delta(e_{i+1})$. We will denote by $\text{Ch}^n(F, G)$ the set of alternating paths in $F \square G$.

We also define the weight of a path $\pi = (e_i)_{i=0}^n$ in the graphing $F$ as the scalar $\omega^F_\pi = \prod_{i=0}^n \omega^F_{e_i}$.

Given a path $(e_i)_{i=0}^n$ in a graphing $F$, one can define a function $\phi^F_\pi$ as the partial transformation:

$$\phi^F_\pi = \phi^F_{e_0} \circ \chi_{T^F_{e_0} \cap S^F_{e_{-1}}} \circ \phi^F_{e_{-1}} \circ \chi_{T^F_{e_{-1}} \cap S^F_{e_{-2}}} \circ \cdots \circ \chi_{T^F_{e_1} \cap S^F_{e_0}} \circ \phi^F_{e_0}$$

where for all measurable set $A$, the function $\chi_A$ is the partial identity $A \rightarrow A$.

We denote by $S_\pi$ and $T_\pi$ respectively the domain and codomain of this partial transformation $S^F_{e_0} \rightarrow T^F_{e_n}$. It is then clear that the transformation $\phi^F_\pi : S_\pi \rightarrow T_\pi$ is measure-preserving.

**Definition 54** (Alternating Cycles). A cycle in a graphing $F$ is a path $(e_i)_{i=0}^n$ in $F$ such that $S^F_{e_0} \cap T^F_{e_n}$ is of strictly positive measure.

An alternating cycle between two weighted graphings $F, G$ is a cycle $(e_i)_{i=0}^n$ in $F \square G$ which is an alternating path and such that $\delta(e_0) \neq \delta(e_n)$. We will denote by $\text{Cy}^n(F, G)$ the set of alternating cycles between $F$ and $G$.

We now introduce the notion of carving of a graphing along a measurable set $C$. This operation will consists in replacing an edge by four disjoint edges whose source and target are either subsets of $C$ or subsets of the complementary set of $C$.

**Definition 55** (Carvings). Let $\phi : S \rightarrow T$ ba a measure-preserving transformation, $C$ a measurable set and $C^c$ its complementary set. We define the measure-
preserving transformations:

\[
\begin{align*}
[\phi]^i_A & = \phi_{|C\cap\phi^{-1}(C)} : A \cap C \cap \phi^{-1}(C) \to B \cap \phi(C) \cap C \\
[\phi]^o_A & = \phi_{|C\cap\phi^{-1}(C^c)} : A \cap C \cap \phi^{-1}(C^c) \to B \cap \phi(C^c) \cap C \\
[\phi]^i_o & = \phi_{|C^c \cap \phi^{-1}(C)} : A \cap C^c \cap \phi^{-1}(C) \to B \cap \phi(C) \cap C^c \\
[\phi]^o_o & = \phi_{|C^c \cap \phi^{-1}(C^c)} : A \cap C^c \cap \phi^{-1}(C^c) \to B \cap \phi(C^c) \cap C^c
\end{align*}
\]

We will denote by \([S]^b_{ab}, [T]^b_{ab}\) (\(a, b \in \{i, o\}\)) the domain and codomain of \([\phi]^b_{ab}\).

If \(F\) is a weighted graphing, define the carving of \(F\) along \(C\) as the weighted graphing \(F^{\cap C} = \{(o^F_e, (\phi^F_e)^b) | e \in E^F, a, b \in \{i, o\}\}\).

In some cases, the carving a graphing \(G\) along a measurable set \(C\) is almost the same as \(G\). Indeed, if each edge have its source and target (up to a null-measure set) either in \(C\) or in its complementary set, the graphing obtained from the carving operation is almost everywhere equal to \(G\).

**Definition 56.** Let \(A, B\) be two measurable sets. We say that \(A\) intersects \(B\) trivially if \(\lambda(A \cap B) = 0\) or \(\lambda(A \cap B^c) = 0\).

If \(F\) is a weighted graphing and \(e \in E^F, S^F_e\) and \(T^F_e\) intersect \(C\) trivially, then \(F\) will be said to be \(C\)-tough.

**Lemma 57.** Let \(F\) be a graphing and \(C\) be a measurable set. If \(F\) is \(C\)-tough, then \(F^{\cap C} \sim_{a.e.} F\).

**Proof.** Chose \(e \in E^F\). Since \(F\) is \(C\)-tough, we are in one of the four following cases:

- \(S^F_e \cap C\) and \(T^F_e \cap C\) are of null measure;
- \(S^F_e \cap C\) and \(T^F_e \cap C^c\) are of null measure;
- \(S^F_e \cap C^c\) and \(T^F_e \cap C\) are of null measure;
- \(S^F_e \cap C^c\) and \(T^F_e \cap C^c\) are of null measure;

These four cases are treated in a similar way. Indeed, among the functions \([\phi^F_e]^b_{ab}\), \(a, b \in \{i, o\}\), only one is of domain (and thus of codomain) a set of strictly positive measure. We thus define an empty graphing \(0_F\), with \(E^{0_F} = E^F \times \{1, 2, 3\}\), and a bijection \(E^{F^{\cap C}} \to E^F \cup E^{0_F}\) which associates to the element \((e, a, b) (e \in E^F, a, b \in \{i, o\})\) the element \(e \in E^{0_F}\) if the domain of \([\phi^F_e]^b_{ab}\) is of strictly positive measure, and one of the elements \((e, i) \in E^{0_F}\) otherwise. One can then easily show that this bijection satisfies all the necessary properties to conclude that \(F \sim_{a.e.} F^{\cap C}\).

Thanks to the carving operation, we are now able to define the execution of two weighted graphings \(F\) and \(G\): we consider the set of alternating paths between \(F\) and \(G\), and we then keep the part of each path which is external to the intersection \(C\) — the location of the cut — of the carriers of \(F\) and \(G\). The execution for weighted graphings is therefore the natural generalization of the execution we defined earlier on graphs.
Definition 58 (Execution). Let $F,G$ be two weighted graphings of respective carriers $V^F,V^G$ and let $C = V^F \cap V^G$. We define the graphing $F : \pi \to G$ as the family:
\[
\{(\omega^F, \phi^F) : (S^F)_{\pi} \to (T^G)_{\pi} \mid \pi \in \text{Ch}(F,G), \lambda((S^F)_{\pi}) \neq 0\}
\]

We now want to define the functions representing the circuits between graphings. This is where things get a little bit more complicated: if $\pi_1$ and $\pi_2$ are two cycles representing the same circuit (i.e. $\pi_1$ is a cyclic permutation of $\pi_2$), the functions $\phi_{\pi_1}$ and $\phi_{\pi_2}$ are not equal in general! We will however need to take this non-uniformity later, when defining the notion of circuit-quantifying maps (in the cases — which are those of interest — where these maps depend on the functions $\phi_{\pi_1}, \phi_{\pi_2}$ associated to the representatives of circuits).

Definition 59. Let $F,G$ be two weighted graphings. We denote by $\text{Cy}^m(F,G)$ the set of alternating paths between $F$ and $G$. A choice of representatives of circuits is a set $\text{Rep}(F,G)$ such that for all $\rho$ in $\text{Cy}^m(F,G)$ there exists a unique element $\pi$ in $\text{Rep}(F,G)$ such that $\tilde{\rho} = \tilde{\pi}$ (recall that $\tilde{\pi}$ is the equivalence class of $\pi$ modulo the action of cyclic permutations, see Proposition 7).

Definition 60 (Circuits and 1-circuits). If $F$ and $G$ are weighted graphings and $\text{Rep}(F,G)$ is a choice of representatives of circuits between $F$ and $G$, we define:
\[
\text{Circ}^m(F,G) = \{(\omega^F)_{\pi} \mid \pi \in \text{Rep}(F,G)\}
\]

Proposition 61. Let $F,F',G$ be weighted graphings such that $F \sim a.e. F'$. Then there exists a bijection
\[
\theta : \text{Ch}(F,G) \to \text{Ch}(F',G)
\]
such that $\phi_{\pi} = \phi_{\theta(\pi)}$ for all path $\pi$.

Proof. By definition, there exists two empty graphings $0_F,0_{F'}$ and a bijection $\theta : E^{F'} \cup E^{0_{F'}} \to E^F \cup E^{0_F}$ such that:
- for all $e \in E^{F'} \cup E^{0_{F'}}$, $\omega_e = \omega^F_{\theta(e)}$;
- for all $e \in E^{F'} \cup E^{0_{F'}}$, $S_e^{F'} \cup 0_{F'} \Delta S^0_{\theta(e)}$ is of null measure;
- for all $e \in E^{F'} \cup E^{0_{F'}}$, $T_e^{F'} \cup 0_{F'} \Delta T^0_{\theta(e)}$ is of null measure;
- for all $e \in E^{F'} \cup E^{0_{F'}}$, $\phi^0_{\theta(e)}$ and $\phi^F_{\theta(e)}$ are almost everywhere equal.

Let $\pi$ be an alternating path in $F \square G$. We treat the case $\pi = f_0 g_0 \ldots f_n g_n$ as an example, the other cases are dealt with in a similar way. We can define a path $\theta^{-1}(\pi)$ in $F \square G$ by $\pi = \theta^{-1}(f_0) g_0 \theta^{-1}(f_1) \ldots \theta^{-1}(f_n) g_n$. Indeed, since the sets $S_{f_i} \cap T_{g_i}$ (resp. $S_{g_i} \cap T_{f_i}$) are of strictly positive measure, then $\theta^{-1}(f_i+1)$ (resp. $\theta^{-1}(f_i)$) is of strictly positive measure (hence an element of $E^F$) and moreover satisfies that $S_{\theta^{-1}(f_i+1)} \cap T_{g_i}$ (resp. $S_{g_i} \cap T_{\theta^{-1}(f_i)}$) is of strictly positive measure.
Conversely, a path \( \pi' = e_0 g_0 \ldots e_n g_n \) in \( F' \Box G \) allows one to define a path 
\( \theta(\pi') = \theta(e_0) g_0 \theta(e_1) \ldots \theta(e_n) g_n \). It is clear that \( \theta(\theta^{-1}(\pi)) = \pi \) (resp. \( \theta^{-1}(\theta(\pi)) = \pi' \)) for all path \( \pi \) (resp. \( \pi' \)) in \( F \Box G \) (resp. \( F' \Box G \)).

We also need to check that the operation of execution is compatible with the notion of almost everywhere equality. Indeed, since we want to work with graphings considered up to almost everywhere equality, the result of the execution should not depend on the representative of the equivalence class considered.

**Corollary 61.1.** Let \( F, F', G \) be weighted graphings such that \( F \sim_{a.e.} F' \). Then \( F : m : G \sim_{a.e.} F' : m : G \).

**Proof.** Let \( \theta \) be the bijection defined in the statement of the preceding proposition. We notice that \( \omega_s = \omega_{\theta^{-1}(s)} \), and that \( \phi_s \) and \( \phi_{\theta^{-1}(s)} \) are almost everywhere equal as compositions of pairwise almost everywhere equal maps. In particular, their domain and codomain are equal up to a set of null measure. We can then conclude that \( \theta : E^{F : m : G} \to E^{F' : m : G} \) satisfies all the necessary properties: \( F' : m : G \) and \( F : m : G \) are almost everywhere equal.

**Corollary 61.2.** Let \( F, F', G \) be weighted graphings such that \( F \sim_{a.e.} F' \). Then \( Cy^m(F, G) \equiv Cy^m(F', G) \).

**Proof.** Let \( \theta \) be the bijection defined in the proof of Proposition 61. The functions \( \phi_s \) and \( \phi_{\theta^{-1}(s)} \) are almost everywhere equal and their domains and codomains are equal up to a set of null measure. We can deduce from this that \( [\phi_s]_i \) and \( [\phi_{\theta^{-1}(s)}]_i \) are almost everywhere equal, and their domains and codomains are equal up to a set of null measure.

### 5.3. Carvings, Cycles and Execution

We now show a technical result that will be useful later, and which gives better insights on the operation of execution between two graphings. The execution of the graphings \( F \) and \( G \) is defined as a restriction of the set of alternating paths between \( F \) and \( G \). One could have also considered the carvings of \( F \) and \( G \) along the intersection \( C \) of the carriers of \( F \) and \( G \), and then define the execution as the set of alternating paths whose source and target lie outside of the set \( C \). The technical lemma we now state and prove shows that these two operations are equivalent.

Let \( F, G \) be two weighted graphings and \( C = V^F \cap V^G \). One can notice that there should be a bijective correspondence between the edges of \( F : m : G \) and those of \( F^C : m : G^C \). Indeed, for two edges \( e, f \) to follow each other in a path, one should have that \( S_e \cap T_f \) is of strictly positive measure. But one can notice that, since \( S_e \cap T_f \) and \( S^f \cap S^e \) are subsets of \( C \), that the following expressions are equal:

\[
\chi_{S_e \cap T_f} \circ \phi_f^C \circ \chi_{S^f \cap S^e} \quad \text{and} \quad \chi_{S_e \cap T_f \cap C \cap \phi_f^C(C) \circ (\phi_f^C)^{-1}(C)} \circ \chi_{S_f \cap S_g \cap C \cap (\phi_f^C)^{-1}(C)}
\]

One can deduce from this the following equality:

\[
\chi_{S_e \cap T_f} \circ \phi_f^C \circ \chi_{S^f \cap S^e} = \chi_{S_e \cap T_f \cap C} \circ [\phi_f^C]^i \circ \chi_{S^f \cap S^e}
\]
One can obtain the following equalities in a similar manner:

\[
\begin{align*}
\chi^C \circ \phi_f \circ \chi_S \cap T_\delta &= \chi^C \circ [\phi_f]^1 \circ \chi (S_c \cap T_\delta) \\
\chi_S \cap T_\delta \circ \phi_f \circ \chi^C &= \chi (S_c \cap T_\delta) \circ [\phi_f]^1 \circ \chi^C \\
\chi^C \circ \phi_f \circ \chi^C &= \chi^C \circ [\phi_f]^1 \circ \chi^C
\end{align*}
\]

**Lemma 62.** Let \( F, G \) be two weighted graphings, \( V^F, V^G \) their carrier and \( C = V^F \cap V^G \). Then:

\[ F : m: G = F^{\wedge C} : m: G \]

**Proof.** By definition, the execution \( F : m: G \) is the graphing:

\[ ((\omega^F, \varphi^F) , (\omega^G, \varphi^G) : \{ S_\pi \} \mapsto [T_\pi]_0 \mid \pi \in \text{Ch}^m(F, G), \lambda([S_\pi])_0 \neq 0 \]

Similarly, the execution \( F^{\wedge C} : m: G \) is the graphing:

\[ ((\omega^{F^{\wedge C}}, \varphi^{F^{\wedge C}}) , (\omega^G, \varphi^G) : \{ S_\pi \} \mapsto [T_\pi]_0 \mid \pi \in \text{Ch}^m(F^{\wedge C}, G), \lambda([S_\pi])_0 \neq 0 \]

Let \( \pi \) be an alternating path in \( F \circ G \). Then \( \pi \) is an alternating sequence of elements in \( F^\wedge C \) and elements in \( F^G \). Suppose for instance \( \pi = f_0 g_0 1 \cdots f_k g_k \), and let us define \( \tilde{\pi} = [f_0]_1^0 g_0 1 \cdots [f_k]_1^0 g_k \). The function \([\varphi^{F^{\wedge C}}]_0 \) is equal to:

\[
\chi_{C \cap \varphi^{F^{\wedge C}}(C)} \circ \varphi^F_{i+1} \circ \chi_{S_{i+1} \cap T_{i+1}} \circ \varphi^G_{i+1} \circ \cdots \\
\cdots \circ \varphi^G_{i+1} \circ \chi_{S_{i+1} \cap T_{i+1}} \circ \varphi^F_{i+1} \circ \chi_{C \cap (\varphi^F_{i+1} \circ \varphi^G_{i+1} - 1(C))}
\]

From the remarks preceding the statement of the lemma, one can conclude that \([\varphi^{F^{\wedge C}}]_0 \) is equal to:

\[
\chi_{C \cap \varphi^{F^{\wedge C}}(C)} \circ [\phi^F_{i+1}]_0 \circ \chi_{S_{i+1} \cap T_{i+1}} \circ \varphi^G_{i+1} \circ \cdots \\
\cdots \circ \varphi^G_{i+1} \circ \chi_{S_{i+1} \cap T_{i+1}} \circ [\phi^F_{i+1}]_0 \circ \chi_{C \cap [\varphi^F_{i+1} \circ \varphi^G_{i+1} - 1(C)]}
\]

We therefore obtain that \([\varphi^{F^{\wedge C}}]_0 = [\varphi^{\wedge C \circ G}]_0 \). Conversely, each alternating path in \( F^{\wedge C} \circ G \) whose first and last edges are elements of \( F^{\wedge C} \) is necessarily of the form \([f_0]_1^0 g_0 f_1^1 \cdots [f_k]_1^0 g_k [f_{k+1}]_1^0 \) where the path \( f_0 g_0 f_1 \cdots g_k f_{k+1} \) is an alternating path in \( F \circ G \).

The other cases are treated in a similar way. \( \square \)

**Corollary 62.1.** Let \( F, G \) be two weighted graphings, \( V^F, V^G \) their carrier and \( C = V^F \cap V^G \). Then:

\[ F : m: G = F^{\wedge C} : m: G^{\wedge C} \]
We can also show the sets of cycles are equal.

**Lemma 63.** Let \( F, G \) be two weighted graphings, \( V^F, V^G \) their carrier, and \( C = V^F \cap V^G \). Then:

\[
\text{Cy}^m(F,G) = \text{Cy}^m(F^{\cap C},G)
\]

**Proof.** The argument is close to the one used in the preceding proof. Indeed, if \( \pi = e_0 \ldots e_n \) is an alternating cycle between \( F \) and \( G \), then we can associate it to the cycle \([\pi] = [e_0][e_1]^t \ldots [e_n]^t\). Conversely, if \( \pi' \) is a cycle in \( F^{\cap C} \cap G \), then each edge in \( \pi' \) necessarily is of the form \([e]^t\) for an element \( e \) in \( F \). Moreover, the associated functions are equal, i.e. \([\phi_{\pi}]^t = \phi_{[\pi]}\).

**Lemma 64.** Let \( F \) be a weighted graphing, and \( \pi = e_0 \ldots e_n \) a path in \( F \) such that \( S_\pi \) is of strictly positive measure. We define, for all couple of integers \( i < j \), \( \rho_i,j \) the path \( e_i e_{i+1} \ldots e_j \). Then:

- for all \( 0 < i < j \leq n \), \( S_{\rho_i,j} \cap T_{e_{i-1}} \) is of strictly positive measure;
- for all \( 0 < i < j < n \), \( T_{\rho_i,j} \cap S_{e_j+1} \) is of strictly positive measure.

**Proof.** Let us fix \( i,j \). We suppose that \( S_{\rho_i,j} \cap T_{e_{i-1}} \) is of null measure. Then for all \( x \in S_{p_i} \), \( \phi_{e_0 \ldots e_{i-1}} \) is defined at \( x \), and such that \( \phi_{e_0 \ldots e_{i-1}} \) is defined at \( \phi_{e_0 \ldots e_{i-1}}(x) \). In particular, \( \phi_{e_i \ldots e_j} \) is defined at \( \phi_{e_0 \ldots e_{i-1}}(x) \), i.e. \( \phi_{e_0 \ldots e_{i-1}}(x) \) is an element in \( S_{\rho_i,j} \). Moreover, by the definition, \( \phi_{e_0 \ldots e_{i-1}}(x) \) is an element of \( T_{e_{i-1}} \). Thus \( \phi_{e_0 \ldots e_{i-1}}(S_{\pi}) \subseteq S_{\rho_i,j} \cap T_{e_{i-1}} \). Since \( \phi_{e_0 \ldots e_{i-1}} \) is a measure-preserving transformation which is defined at all \( x \in S_{\pi} \), we deduce that \( \lambda(S_{\pi}) \leq \lambda(S_{\rho_i,j} \cap T_{e_{i-1}}) \). This lead us to a contradiction since this implies that \( \lambda(S_{\pi}) = 0 \).

A similar argument shows that \( T_{\rho_i,j} \cap S_{e_{j+1}} \) is of strictly positive measure.

**Remark.** In the preceding proof, we showed that \( \lambda(S_{\pi}) \leq \lambda(S_{\rho_i,j} \cap T_{e_{i-1}}) \) for all \( i,j \). In particular, \( \lambda(S_{\pi}) \leq \lambda(S_{e_i} \cap T_{e_{i-1}}) \), and therefore:

\[
\lambda(S_{\pi}) \leq \min(\lambda(S_{e_i} \cap T_{e_{i-1}}) | i = 1 \ldots n)
\]

As this was the case with the execution between graphs, we can show the associativity of execution under the hypothesis that the intersection of the carriers is of null measure.

**Proposition 65** (Associativity of Execution). Let \( F,G,H \) be three graphings such that \( \lambda(V^F \cap V^G \cap V^H) = 0 \). Then:

\[
F : m : (G : m : H) = (F : m : G) : m : H
\]

**Proof.** We can first suppose that \( F \) (resp. \( G \), resp. \( H \)) is \( C_F = V^F \cap (V^G \cup V^H) \)-tough (resp. \( C_G = V^G \cap (V^F \cup V^H) \)-tough, resp. \( C_H = V^H \cap (V^F \cup V^G) \)-tough). Indeed, if this was not the case, we can always consider the carving along the set \( C_F \) (resp. \( C_G \), resp. \( C_H \)). This simplifies the following argument since it allows us to consider paths instead of restrictions of paths. The proof then follows the proof of the associativity of the execution for directed weighted graphs (Theorem 14).
We can define the simultaneous plugging of the three weighted graphings $F,G,H$ as the weighted graphing $F\circ G\circ H$ endowed with a coloring map $\delta$ defined by $\delta(e) = 0$ when $e \in E^F$, $\delta(e) = 1$ when $e \in E^G$ and $\delta(e) = 2$ when $e \in E^H$. We can then define the set of 3-alternating paths between $F,G,H$ as the paths $e_0 e_1 \ldots e_n$ such that $\delta(e_i) \neq \delta(e_{i+1})$.

If $e_0 e_1 \ldots e_k$ is an alternating path in $F:G:H$, where every $e_i$ is an alternating path $e_i = g_i^0 h_i^1 \ldots g_i^n h_i^1$, then the sequence of edges obtained by replacing each $e_i$ by the associated sequence (and forgetting about parentheses) is a path. Indeed, we know that, for instance, $S_{e_i} \cap T_{f_{i-1}}$ is of strictly positive measure, and $S_{e_i} \subset S_g$, thus $S_g \cap T_{f_{i-1}}$ is of strictly positive measure. We therefore defined a 3-alternating path between $F,G,H$. The two paths define the same measure-preserving partial transformation, and have the same domains dans codomains.

Conversely, if $e_0 e_1 \ldots e_n$ is a 3-alternating path between $F,G,H$, then we can see it as an alternating sequence of edges in $F$ and alternating sequences between $G$ and $H$. Let $\pi = g_0 h_0 \ldots g_k h_k$ be the path defined by such a sequence appearing in the path $e_0 \ldots e_n$. We can use the preceding lemma to insure that $S_{e_i} \cap T_{e_j}$ is of strictly positive measure. Similarly, $T_{e_s} \cap S_{e_t}$ is of strictly positive measure. We thus showed that we had an edge in $F:G:H$. The two paths define the same partial measure-preserving transformation, and have the same domains.

5.4. Refinements

In order to deal with second order quantification, we will define the notion of refinement of a graphing. This a very natural operation of graphings to consider. A simple example of refinement is to consider a graphing $F$ and one of its edges $e \in E^F$: one can obtain a refinement of $F$ by replacing $e$ with two edges $f,f'$ such that $S_f \cup S_{f'} = S_e$ and $S_f \cap S_{f'}$ is of null measure (one should then define $T_f = \phi_e(S_f)$ and $T_{f'} = \phi_e(S_{f'})$ accordingly).

**Definition 66** (Refinements). Let $F,G$ be two weighted graphings. We will say that $F$ is a refinement of $G$ — denoted by $F \preceq G$ — if there exists a function $\theta : E^F \to E^G$ such that:

- for all $e,e' \in E^F$ such that $\theta(e) = \theta(e')$ and $e \neq e'$, $S_e^F \cap S_{e'}^F$ and $T_e^F \cap T_{e'}^F$ are of null measure;
- for all $e \in E^F$, $\omega^{G}_{\theta(e)} = \omega_{e}^F$;
- for all $f \in E^G$, $S_f^G$ and $\cup_{e\in\theta^{-1}(f)} S_e^F$ are equal up to a set of null measure;
- for all $f \in E^G$, $T_f^G$ and $\cup_{e\in\theta^{-1}(f)} T_{e}^F$ are equal up to a set of null measure;
- for all $e \in E^F$, $\phi^{G}_{\theta(e)}$ and $\phi_{e}^F$ are equal almost everywhere $S_{\theta(e)}^G \cap S_{e}^F$.

We will say that $F$ is a refinement of $G$ along $g \in E^G$ if there exists a set $D$ of elements of $E^F$ such that:
• \( \theta^{-1}(g) = D ; \)
• \( \theta\mid_{gF \rightarrow D} : E^F \rightarrow E^G \rightarrow (g) \) is bijective.

If \( D \) contains only two elements, we will say that \( F \) is a simple refinement along \( g \).

The refinements will sometimes be written \( (F, \theta) \) in order to precise the function \( \theta \).

**Proposition 67.** We define the relation \( \sim \) on the set of weighted graphings as follows:

\[
F \sim G \Leftrightarrow \exists H, (H \leq F) \land (H \leq G)
\]

This is an equivalence relation.

**Proof.** Reflexivity and symmetry are straightforward. We are therefore left with transitivity: let \( F, G, H \) be three weighted graphs such that \( F \sim G \) and \( G \sim H \). We will denote by \( (P_{F,G}, \theta) \) (resp. \( (P_{G,H}, \rho) \)) a common refinement of \( F \) and \( G \) (resp. of \( G \) and \( H \)). We will now define a weighted graphing \( P \) such that \( P \leq P_{F,G} \) and \( P \leq P_{G,H} \). Let us define:

- \( E^P = \{(e,f) \in E^{P_{F,G}} \times E^{P_{G,H}} \mid \theta(e) = \rho(f)\} \);
- \( S^P_{(e,f)} = S^P_{F,G} \cap S^P_{G,H} \) when \( \theta(e) = \rho(f) \);
- \( T^P_{(e,f)} = T^P_{F,G} \cap T^P_{G,H} \) when \( \theta(e) = \rho(f) \);
- \( \omega^P_{(e,f)} = \omega^P_{F,G} = \omega^P_{G,H} \);
- \( \phi^P_{(e,f)} \) is the restriction of \( \phi^P_{F,G} \) to \( S^P_{(e,f)} \);
- \( \mu_{F,G} : (e,f) \mapsto e \) and \( \mu_{G,H} : (e,f) \mapsto f \).

It is then easy to check that \( (P, \mu_{F,G}) \) (resp. \( (P, \mu_{G,H}) \)) is a refinement of \( P_{F,G} \) (resp. of \( P_{G,H} \)).

Since \( (P_{F,G}, \theta) \) is a refinement of \( F \) and \( (P, \mu_{F,G}) \) is a refinement of \( P_{F,G} \), it is clear that \( (P, \theta \circ \mu_{F,G}) \) is a refinement of \( F \). In a similar way, \( (P, \theta \circ \mu_{G,H}) \) is a refinement of \( H \). Finally, \( P \leq F \) and \( P \leq H \), which shows that \( F \sim H \).

**Proposition 68.** The relation \( \sim \) contains the relation \( \sim_{a.e.} \).

**Proof.** Let \( F, G \) be two graphings such that \( F \sim_{a.e.} G \). We will show that \( F \sim G \).

We will use the notations of Definition 50: \( 0_F, 0_G \) for the empty graphings and \( \theta \) for the bijection between the sets of vertices. First, we notice that \( F \leq F \cup \emptyset \) and \( G \leq G \cup \emptyset \). As a consequence, \( F \sim F \cup \emptyset \) and \( G \sim 0 \). Moreover, the bijection \( \theta : E^F \cup E^{0_F} \rightarrow E^G \cup E^{0_G} \) clearly satisfies the necessary conditions for \( (F \cup \emptyset, \theta) \) to be a refinement of \( G \cup \emptyset \), which implies that \( F \sim G \). Using the transitivity of \( \sim \), we can now conclude that \( F \sim G \).
Of course, the carving of a graphing $G$ along a measurable set $C$ defines a refinement of $G$ where each edge is replaced by exactly four disjoint edges. This will be of use to simplify some conditions later: a measure that is invariant under refinements will be invariant under carvings too.

**Proposition 69.** Let $G$ be a weighted graphing and $C$ a measurable set. The weighted graphing $G|_{C}$ is a refinement of $G$.

**Proof.** It is sufficient to verify that the function $\theta : E_{G}^{C} \rightarrow E_{G}^{C}$, $(e,a,b) \rightarrow e$ satisfies all the necessary conditions. Firstly, using the definition, the weights $\omega_{a,b}^{G}$ and $\omega_{e}^{G}$ are equal. Then, the sets $[S_{e}^{G}]_{a,b}, a,b \in (i,o)$ (resp. $[T_{e}^{G}]_{a,b}$) define a partition of $S_{e}^{G}$ (resp. $T_{e}^{G}$). Finally, using the definition again, $[\phi_{e}^{G}]_{a,b}$ is equal to $\phi_{e}^{G}$ on its domain. □

**Lemma 70.** Let $F, G$ be two graphings, $e \in F$ and $F(e)$ be a simple refinement of $F$ along $e$. Then $F_{e} : G$ is a refinement of $F_{\pi} : G$.

**Proof.** By definition,

$$F_{e} : G = ([\omega_{a}^{F,G}, \phi_{a}^{F,G}], [S_{a}^{F}]_{o} \rightarrow [T_{a}^{F}]_{e} | \pi \in Ch^{\pi}(G), \lambda((S_{a}^{F})_{o}) \neq 0]$$

Since $F(e)$ is a simple refinement of $F$, there exists a partition of $S_{e}^{F}$ in two sets $S_{1}^{F}$, $S_{2}^{F}$, and a partition of $T_{e}^{F}$ in two sets $T_{1}^{F}$, $T_{2}^{F}$ such that $\phi_{e}^{F} : (S_{1}^{F}) = T_{1}^{F}$. We can suppose, without loss of generality, that $S_{1}^{F} \cap S_{2}^{F} = \emptyset$ since there exists a weighted graphing which is almost everywhere equal to $F(e)$ and satisfies this additional condition, and since execution is compatible with almost everywhere equality. This additional assumption implies in particular that $T_{1}^{F} \cap T_{2}^{F} = \emptyset$.

To any element $\pi = \{e_{i}\}_{i=0}^{n}$ of $F(e)$, we associate the path $\theta(\pi) = \{\theta(a_{i})\}_{i=0}^{n}$. We now need to check that this is indeed a refinement. Let $\pi_{1}, \pi_{2}$ be two distinct paths such that $\theta(\pi_{1}) = \theta(\pi_{2})$. We want to show that $S_{\pi_{1}} \cap S_{\pi_{2}}$ is of null measure. Since $\pi_{1} = \{a_{i}\}_{i=0}^{n}$ and $\pi_{2} = \{c_{i}\}_{i=0}^{n}$ are distinct, they differ at least on one edge. Let $k$ be the smallest integer such that $a_{k} \neq c_{k}$. We can suppose without loss of generality that $a_{k} = f_{1}$ and $c_{k} = f_{2}$. If $x \in S_{\pi_{1}}$, then $x \in \phi_{f_{1}}^{-1}(S_{1})$. Similarly, if $x \in S_{\pi_{2}}$, then $x \in \phi_{f_{2}}^{-1}(S_{2})$. Since we supposed that $S_{\pi_{1}} \cap S_{\pi_{2}} = \emptyset$, we deduce that $S_{\pi_{1}} \cap S_{\pi_{2}} = \emptyset$.

By definition, the weight of a path $\pi$ is equal to the weight of every path $\pi'$ such that $\theta(\pi') = \pi$. Moreover, the functions $\phi_{\pi'}$ and $\phi_{\theta(\pi')}$ are by definition almost everywhere equal on the intersection of their domain since every $\phi_{e}$ is almost everywhere equal to $\phi_{\theta(e)}$.

We are now left to show $S_{\pi} = \cup \pi' \in \theta^{-1}(\pi) S_{\pi'}$ (the result concerning $T_{\pi}$ is then obvious). It is clear that $S_{\pi'} \subset S_{\pi}$ when $\theta(\pi') = \pi$, and it is therefore enough to show one inclusion: that for all $x \in S_{\pi}$ there exists a $\pi'$ with $\theta(\pi') = \pi$ such that $x \in S_{\pi'}$. Let $\pi = \pi_{0} \cdots \pi_{n}$ be a sequence of measurable sets where $\pi_{0} \in \pi_{0} \cdots \pi_{n}$. Now choose $x \in S_{\pi}$. Then for all $i = 0, \ldots, n$, $\phi_{\pi_{0} \cdots \pi_{i}}(x) \in S_{e_{i}}$ thus $\phi_{\pi_{0} \cdots \pi_{i}}(x)$ is either in $S_{1}$ or in $S_{2}$. We obtain in this way a sequence $a_{0}, \ldots, a_{n}$ in $(1,2)^{n}$. It is then easy to see that $x \in S_{\pi'}$ where $\pi' = \pi_{0} \cdots \pi_{n}$. □

\[44\]
Lemma 71. Let $F,G$ be two weighted graphings, $e \in E^F$ and $(F',\theta)$ a refinement of $F$ along $e$. Then $F':=G$ is a refinement of $F:=G$.

Proof. This is a simple adaptation of the proof of the preceding lemma. Let $D$ be the set of elements such that $\theta^{-1}(e) = D$; we can suppose, modulo considering an almost everywhere equal graphing, that the sets $S_d$ ($d \in D$) are pairwise disjoint. To every path $\pi = (f_i)_{i=0}^n$ in $F':=G$, we associate $\tilde{\theta}(\pi) = (\theta(f_i))_{i=0}^n$. Conversely, a path $\pi = (g_i)_{i=0}^n$ in $F:=G$ defines a countable set of paths:

$$C_\pi = \{(f_i)_{i=0}^n \mid \theta(f_i) = g_i\}$$

We are left with the task of checking that $\tilde{\theta} : \pi \mapsto \tilde{\theta}(\pi)$ is a refinement. For this, we consider two paths $\pi_1$ and $\pi_2$ such that $\tilde{\theta}(\pi_1) = \tilde{\theta}(\pi_2)$. Using the same argument as in the preceding proof, we show that $S_{\pi_1} \cap S_{\pi_2}$ is of null measure. The verification concerning the weights is straightforward, as is the fact that the functions are almost everywhere equal. The last thing left to show is that $S_\pi = \bigcup_{\pi' \in C_\pi} S_{\pi'}$. Here, the argument is again the same as in the preceding proof: an element $x \in S_\pi$ is in the domain of one and only one $S_{\pi'}$ for $\pi' \in C_\pi$. \qed

Theorem 72. Let $F,G$ be weighted graphings and $(F',\theta)$ be a refinement of $F$. Then $F':=G$ is a refinement of $F:=G$.

Proof. If $\pi$ is an alternating path $f_0g_0f_1\ldots f_ng_n$ between $F'$ and $G$, we define $\theta(\pi) = \theta(f_0)g_0\ldots \theta(f_n)$, $\theta(f_i)g_i\theta(f_{i-1})$. This defines a path, since $S_{f_i} \subset S_{\theta(f_i)}$ (resp. $T_{f_i} \subset T_{\theta(f_i)}$) and $\pi$ is itself a path.

Let us denote by $f_0,\ldots f_n,\ldots$ the edges of $F$. We define the graphings $F^n$ as the following restrictions of $F$: $(\omega_{f_i}^F,\phi_{f_i}^F)_{i=0}^n$. We define the corresponding restrictions of $F'$ as the graphings $(F')^n = (\omega_{f_i}^{F'},\phi_{f_i}^{F'})_{i=0}^n$. By an iterated use of the preceding lemma, we obtain that $(F')^n := G, \theta)$ is a refinement of $F^n := G$ for every integer $n$. It is then easy to see that $(F':=G,\theta) = (\bigcup_{n \geq 0} (F')^n := G, \theta)$ is a refinement of $\bigcup_{n \geq 0} F^n := G$, i.e. of $F:=G$. \qed

5.5. Measurement of circuits

We would like to define a measurement of circuits between two graphings $F$ and $G$ in such a way that if $(F',\theta)$ is a refinement of $F$, the measurements $[F,G]_m$ and $[F',G]_m$ are equal. Firstly, one should be aware that to define the notion of circuit-quantifying maps one should take into account the fact that if $\pi_1,\pi_2$ are two representatives of a given circuit, the functions $\phi_{\pi_1}$ and $\phi_{\pi_2}$ are not equal in general.

Secondly, suppose that we obtained such a map $q$ (which does not depend on the choice of representatives), that $\pi$ is an alternating cycle between $F$ and $G$ and that $(F',\theta)$ is a refinement of $F$. We will try to understand what the set of circuits induced by $\pi$ in $F'\boxdot G$ looks like. We first notice that the cycle $\pi$ corresponds to a family $E_\pi$ of alternating cycles between $F'$ and $G$. If for instance $\pi = f_0g_0\ldots f_ng_n$, one should consider the set of sequences $\{f'_0g_0\ldots f'_ng_n \mid \forall i, f'_i \in \theta^{-1}(f_i)\}$. However, each of these sequences does not necessarily define a path: it is possible that
\( S_{f_i} \cap T_{f_i}' \) (or \( S_{f_i}' \cap T_{f_i} \)) is of null measure. It is even possible that such a sequence will be a path without being a cycle, and that a cycle of length \( l \), once decomposed along the refinement, becomes a cycle of length \( m \times l \), where \( m \) is an arbitrary integer. Figure 23 shows how a cycle of length 2 can induce either a cycle of length 4 or a set of two cycles of length 2 after a refinement. However, a cycle of length 4 could very well be induced by the cycle \( \pi^2 \) if the latter is an element of \( \text{Cy}^m(F, G) \). The following definition takes all these remarks into account.

**Definition 73.** Let \( \pi \) be a cycle between two weighted graphings \( F, G \), and \( \pi^o = \{ \pi^k \mid k \in \mathbb{N} \} \cap \text{Cy}^m(F, G) \). Let \((F', \theta)\) be a refinement of \( F \). We fix \( \text{Rep}(F', G) \) a choice of representatives of circuits, and we write

\[
E_{\pi}^{(F', \theta)} = \{ \rho = f'_0 g_0 f'_1 g_1 \cdots f'_n g_n \in \text{Rep}(F', G) \mid \exists k \in \mathbb{N}, \theta(\rho) = \pi^k \}
\]

A function \( q \) from the set \( \pi^o \) of cycles into \( \mathbb{R}_{\geq 0} \) is **refinement-invariant** if for all graphings \( F, G \) and simple refinement \((F', \theta)\) of \( F \), the following equality holds:

\[
\sum_{\rho \in \pi^o} q(\rho) = \sum_{\rho \in E_{\pi}^{(F', \theta)}} q(\rho)
\]

This is the most general notion one could state and it allows one to define what it means to be invariant under refinement in the general setting of circuits.

---

9As in the graph setting, we work modulo a the renaming of edges. As a consequence, if the class of cycles is not a priori a set, the function \( q \) will not depend on the name of edges, but only on the weight and the transformation associated to the cycle. We can therefore define \( q \) as a function on the set of equivalence classes of cycles modulo renamings of edges.
In the particular case we will be interested in, i.e. the case where one considers only the set of 1-circuits, we can notice that the definition becomes much simpler. Indeed, the set \( \pi^\omega \) is reduced to the singleton \( \{ \pi \} \), and the equality that should be verified becomes

\[
q(\pi) = \sum_{\rho \in E^F_e} q(\rho)
\]

**Definition 74 (Circuit-Quantifying Maps).** A map \( q \) from the set of cycles into \( \mathbb{R}_{\geq 0} \) is a circuit-quantifying map if:

1. for all representatives \( \pi_1, \pi_2 \) of a circuit \( \pi \), \( q(\pi_1) = q(\pi_2) \);
2. \( q \) is refinement-invariant.

A circuit-quantifying map should therefore meet quite complex conditions and it is a very natural question to ask whether such maps exist. We will define in the next section a family of circuit-quantifying maps for the set of 1-circuits, answering this question positively.

We will now define the measurement associated to a circuit-quantifying map. If the formal definition depends on the choice of a family of representatives of circuits, the result \( [F, G]_m \) is obviously not dependent on this choice, a consequence of the first condition in the definition of circuit-quantifying maps.

**Definition 75 (Measure).** Let \( q \) be a circuit-quantifying map. We define the associated measure of interaction as the function \( [\cdot, \cdot]_m \) which associates, to all couple of weighted graphings \( F, G \), the quantity:

\[
[F, G]_m = \sum_{\pi \in \text{Circ}^m(F, G)} q(\pi)
\]

Where \( \text{Circ}^m(F, G) \) depends on a choice of a set of representatives of circuits.

**Lemma 76.** Let \( F, G \) be two weighted graphings, \( e \in E^F \), and \( F^{(e)} \) a simple refinement along \( e \) of \( F \). Then:

\[
[F, G]_m = [F^{(e)}, G]_m
\]

**Proof.** We write \( \theta : E^{F^{(e)}} \to E^F \) and \( (f, f') = \theta^{-1}(e) \). We will use the notations introduced in Definition [73]

We will also denote by \( O(\{e\}, G) \) the set of 1-circuits in \( \text{Cy}^m(\{e\}, G) \). Then the family \( \{\pi^\omega\}_{\pi \in O(\{e\}, G)} \) is a partition of \( \text{Cy}^m(\{e\}, G) \): it is clear that if \( \pi, \pi' \) are two distinct elements of \( O(\{e\}, G) \), \( \pi^\omega \) and \( (\pi')^\omega \) are disjoint, and it is equally obvious that \( \text{Cy}^m(\{e\}, G) = \bigcup_{\pi \in O(\{e\}, G)} \pi^\omega \) since the fact that \( \pi^k \in \text{Cy}^m(F, G) \) implies that \( \pi \in \text{Cy}^m(F, G) \).
By definition, one has:

\[
[F(e), G]_m = \sum_{\pi \in Cy^m(F, G)} q(\pi)
\]

\[
= \sum_{\pi \in Cy^m(F(e), G)} q(\pi) + \sum_{\pi \in Cy^m(\{e, f\}', G)} q(\pi)
\]

\[
= \sum_{\pi \in Cy^m(F-e, G)} q(\pi) + \sum_{\pi \in Cy^m(\{e\}, G) \cap E^m} q(\rho)
\]

\[
= \sum_{\pi \in Cy^m(F-e, G)} q(\pi) + \sum_{\pi \in Cy^m(\{e\}, G) \cap E^m} q(\rho)
\]

\[
= \sum_{\pi \in Cy^m(F, G)} q(\pi)
\]

Which shows that \([F(e), G]_m = [F, G]_m\). □

**Theorem 77.** Let \(F, G\) be weighted graphings and \((F', \theta)\) a refinement of \(F\). Then:

\([F, G]_m = [F', G]_m\)

**Proof.** The argument is now usual. We first enumerate the edges of \(F\), and denote them by \(f_0, \ldots, f_n, \ldots\). We then define:

\[
F^n = ((\omega_{f_i}, \theta_{f_i})_{i=0}^n
\]

\[
(F')^n = ((\omega_{f_i}, \theta_{f_i}) | \theta(e) = f_i)_{i=0}^n
\]

Then \(((F')^n, \theta)\) is a refinement of \(F^n\), and an iterated use of the preceding lemma shows that:

\([[(F')^n, G]_m = [F^n, G]_m\]

Then:

\[
[F', G]_m = \sum_{\pi \in Cy^m(F', G)} q(\pi)
\]

\[
= \lim_{n \to \infty} \sum_{\pi \in Cy^m((F')^n, G)} q(\pi)
\]

\[
= \lim_{n \to \infty} \|[F', G]_m
\]

\[
= \lim_{n \to \infty} [F^n, G]_m
\]

\[
= \lim_{n \to \infty} \sum_{\pi \in Cy^m(F^n, G)} q(\pi)
\]

\[
= \sum_{\pi \in Cy^m(F, G)} q(\pi)
\]

Finally, we showed that \([F', G]_m = [F, G]_m\). □
Proposition 78. Let $F, G, H$ be graphings such that $\lambda(V^F \cap V^G \cap V^H) = 0$. Then:

$$[F, G, H]_m + [G, H]_m = [H, F]_m + [H, F]_m$$

Proof. We consider the expression $[F, G, H]_m + [G, H]_m$. We can suppose without loss of generality that $F$ (resp. $G$, resp $H$) is $V^F \cap (V^G \cup V^H)$-tough (resp. $V^G \cap (V^F \cup V^H)$-tough, resp. $V^H \cap (V^F \cup V^G)$-tough). Indeed, if this is not the case the preceding proposition allows us to replace $F, G, H$ by the adequate carvings without changing the measure of interaction.

The end of the proof is now very similar to the proof of the trefoil property for graphs.

Let $\pi$ be an element in $\text{Circ}^m(F, G, H)$. Then $\pi$ is an alternating path between $F$ and $G, H$, for instance $\pi = f_0 \rho_0 f_1 \ldots f_n \rho_n$. Now, each $\rho_i$ is an alternating path between $G$ and $H$. Either each $\rho_i$ is an element of $G$ in which case $\pi$ is an alternating path between $F$ and $G, H$, and therefore corresponds to an element in $\text{Circ}^m(F, G)$, either at least one of the $\rho_i$ contains an edge of $H$. In this case, it is clear that $\pi$ is an element of $\text{Cy}^m(G, H)$ (we use Lemma 64) to insure that alternating paths between $F$ and $G$ that appear as part of $\pi$ have a domain —hence a codomain — of strictly positive measure. Similarly, an element of $\text{Cy}^m(G, H)$ is an element of $\text{Cy}^m(F, G, H)$.

We now will work with equivalence classes of weighted graphings for the equivalence relation $\sim_\epsilon$. Theorems 77 and 72 insure us that the operations of plugging and execution are well defined in this setting. If the fact that we will work with graphings considered up to the $\sim_{a.e.}$ equivalence (which is contained in $\sim_\epsilon$, Proposition 69) is quite natural, it can seem strange to work with graphings up to the equivalence relation $\sim_\epsilon$. Our reasons for this should be clarified when we will define second order quantification: working modulo $\sim_\epsilon$ allows to define universal quantification as an intersection in a very natural way.

5.6. Sliced Thick Graphings

The sliced graphings are obtained from graphings in the same way we defined sliced thick graphs from directed weighted graphs: we consider formal weighted sums $F = \sum_{i \in I^F} a_i^F F_i$, where the $F_i$ are graphings of carrier $V^{F_i}$. We define the carrier of $F$ as the measurable set $\cup_{i \in I^F} V^{F_i}$. The various constructions are then extended in the same way we explained in Section 3 the execution and the measurement are defined as:

$$\lambda(\sum_{i \in I^F} a_i^F F_i : \sum_{i \in I^G} a_i^G G_i) = \sum_{(i, j) \in I^F \times I^G} a_i^F a_j^G F_i : G_j$$

$$\lambda(\sum_{i \in I^F} a_i^F F_i, \sum_{i \in I^G} a_i^G G_i)_m = \sum_{(i, j) \in I^F \times I^G} a_i^F a_j^G [F_i : G_j]_m$$

The trefoil property and the adjunction are then easily obtained through the same computations as in the proofs of Propositions 23 and 23.1.

We will now consider the most general notion of thick graphing one can define. As it was the case in the setting of graphs, a thick graphing is a graphing whose
carrier has the form $V \times D$. The main difference between graphings and thick graphings really comes from the way two such objects interact.

**Definition 79.** Let $(X, \mathcal{B}, \lambda)$ be a measured space and $(D, \mathcal{D}, \mu)$ a probability space (a measured space such that $\mu(D) = 1$). A thick graphing of carrier $V \in \mathcal{B}$ and dialect $D$ is a graphing on $X \times D$ of carrier $V \times D$.

**Definition 80** (Dialectal Interaction). Let $(X, \mathcal{B}, \lambda)$ be a measured space and $(D, \mathcal{D}, \mu), (E, \mathcal{E}, \nu)$ two probability spaces. Let $F, G$ be thick graphings of respective carriers $V^F, V^G \in \mathcal{B}$ and respective dialects $D, E$. We define the graphings $F^{\tau_k}$ and $G^{\tau_0}$ as the graphs of respective carriers $V^F, V^G$ and dialects $E \times F$:

$$F^{\tau_k} = \{ (\omega^F_e, \phi^F_e \times \text{Id}_E : S^F_e \times D \times E \rightarrow T^F_e \times D \times E) \}_{e \in E^F}$$

$$G^{\tau_0} = \{ (\omega^G_e, \text{Id}_X \times (\tau \circ (\phi^G_e \times \text{Id}_D) \circ \tau^{-1}) : S^G_e \times D \times E \rightarrow T^G_e \times D \times E) \}_{e \in E^G}$$

where $\tau$ is the natural symmetry: $E \times D \rightarrow D \times E$.

**Definition 81** (Plugging). The plugging $F :: G$ of two thick graphings of respective dialects $D^F, D^G$ is defined as $F^{\tau_k} \sqcap G^{\tau_0}$.

**Definition 82** (Execution). Let $F, G$ be two thick graphings of respective dialects $D^F, D^G$. Their execution is equal to $F^{\tau_k} \sqcap_m G^{\tau_0}$.

**Definition 83** (Measurement). Let $F, G$ be two thick graphings of respective dialects $D^F, D^G$, and $q$ a circuit-quantifying map. The corresponding measurement of the interaction between $F$ and $G$ is equal to $\|F^{\tau_k} \sqcap_m G^{\tau_0}\|_m$.

As in the setting of graphs, one can show that all the fundamental properties are preserved when we generalize from graphings to thick graphings.

**Proposition 84.** Let $F, G, H$ be thick graphings such that $V^F \cap V^G \cap V^H$ is of null measure. Then:

$$F :: (G :: H) = (F :: G) :: H$$

$$\|F, G :: H\|_m + \|G, H\|_m = \|G, H :: F\|_m + \|H, F\|_m$$

In a similar way, the extension from thick graphings to sliced thick graphings should now be quite clear. One extends all operations by “linearity” to formal weighted sums of thick graphings, and one obtains, when $F, G, H$ are sliced thick graphings such that $V^F \cap V^G \cap V^H$ is of null measure:

$$F :: (G :: H) = (F :: G) :: H$$

$$\|F, G :: H\|_m + 1_F[G, H]\|_m = \|G, H :: F\|_m + 1_G[H, F]\|_m$$

6. Construction of an Exponential Connective

As we pointed out earlier, the existence of circuit-quantifying maps is not obvious. We will now fix the measured space $(\mathbb{R}, \mathcal{B}, \lambda)$ of the real line endowed with the $\sigma$-algebra of Borel sets and Lebesgue measure. We will also consider from now only the sets of 1-circuits on sliced thick graphings, and we will show that there exists a family of circuit-quantifying maps in this setting.
6.1. Quantifying circuits

**Definition 85.** Let \( \phi : X \to Y \) be a measure-preserving transformation. We define the measurable set:

\[
\{ \phi \} = \bigcap_{n \in \mathbb{N}} \phi^n(X) \cap \phi^{-n}(Y)
\]

**Theorem 86.** Let \( \phi : X \to X \) be a measure-preserving transformation, where \( X \subset \mathbb{R} \) is a measurable set of finite measure. Then the following function is a measurable map:

\[
\rho_\phi : \begin{cases} 
X & \to \mathbb{N} \cup \{\infty\} \\
x & \mapsto \inf \{ n \in \mathbb{N} \mid \phi^n(x) = x \} \\
x & \mapsto \infty \quad \text{otherwise}
\end{cases}
\]

**Proof.** We first show that the sets \( X_i \) are measurable. We can notice that \( \mathbb{R} \) can be written as the countable union \( \bigcup_{i \in \mathbb{Z}} [\frac{i}{2^n}, \frac{i+1}{2^n}] \) for every integer \( k \). We define \( I^k_i = \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right] \). Then, for all integer \( k \), one can define the family \( (X^k_i)_{i \in \mathbb{Z}} \) where \( X^k_i = I^k_i \cap X \). By definition, the sets \( X^k_i \) are measurable sets. Since \( X \) is of finite measure and \( \phi \) is measure-preserving, we can define the measurable map \( \rho^k_i : X^k_i \to \mathbb{N} \cup \{\infty\} \) which associates to each \( x \in X^k_i \) the integer \( \min \{ n \in \mathbb{N} \mid \phi^n(x) \in X^k_i \} \) when this expression is defined, and \( \infty \) otherwise. Thus, for each integer \( k \in \mathbb{N} \), we obtain a measurable map \( \rho^k : X \to \mathbb{N} \cup \{\infty\} \) defined as the union of the functions \( \rho^k_i \). One can easily notice that, for a fixed element \( x \in X \), the sequence \( \phi^k(x) \) is an increasing sequence in \( \mathbb{N} \cup \{\infty\} \), and thus admits a limit. We therefore define the function \( \rho : X \to \mathbb{N} \cup \{\infty\} \) as the point-wise limit of the functions \( \rho^k \). As a point-wise limit of measurable maps, this function is a measurable map. It only remains to check that this function coincides with the function defined in the statement of the theorem.

**Remark.** Poincaré’s recurrence theorem insures us that the set of points \( x \) such that \( \rho^k(x) = \infty \) is of null measure, and we could therefore forget about those points. However, it is the limit \( \rho(x) \) of the functions \( \rho^k(x) \) when \( k \) goes to infinity that will be of interest, and the set of points \( x \) such that \( \rho(x) = \infty \) need not be of null measure. It is even a possibility that this set has the same measure as \( X \): if \( \phi \) is an aperiodic transformation, then \( \lambda(\rho^{-1}(\infty)) = \lambda(X) \).

**Definition 87.** Let \( \pi \) be a cycle in the weighted graphing \( F \). Then the map \( \phi_\pi \) restricted to \( X = \{\phi_\pi\} \) is a measure-preserving transformation \( X \to X \). We can then define the map \( \rho_\phi \) on \( X \), which we will abusively denote by \( \rho_\pi \) from now on. We define the support \( \text{supp}(\pi) \) of \( \pi \) as the set \( \rho_\pi^{-1}(\mathbb{N}) \).

**Definition 88.** Let \( m \) be a measurable map \( \Omega \to \mathbb{R} \cup \{\infty\} \). We define \( q_m \) as the function:

\[
q_m : \pi \mapsto \int_{\text{supp}(\pi)} \frac{m(\omega(\pi))^{\rho_\phi(x)}}{\rho_\phi(x)} d\lambda(x)
\]
Lemma 89. For all measurable map \( m \), the function \( q_m \) has a constant value on the equivalence classes of cycles modulo the action of cyclic permutations.

Proof. Let \( \pi = e_0 e_1 \ldots e_n \) be a cycle, \( \text{supp}(\pi) \) its support. For all \( i \in \mathbb{N} \), we write \( \text{supp}(\pi)_i = \rho_{\pi}^{-1}(i) \). Consider now \( \pi' = e_1 e_2 \ldots e_n e_0 \), and \( \text{supp}(\pi') \) its support. We define \( \text{supp}(\pi')_i = \rho_{\pi'}^{-1}(i) \). We will first show that \( \text{supp}(\pi')_i = \phi_{e_0}(\text{supp}(\pi)_i) \) for all integer \( i \).

Let us now pick \( x \in \text{supp}(\pi')_i \), which means that \( x \in \text{supp}(\pi') \) and \( \phi_i(x) = x \). Since \( \phi_{\pi'}(x) = \phi_{e_0}(\phi_{e_1 \ldots e_n}(x)) \), we have \( x = \phi_{e_0}(\phi_{e_1 \ldots e_n}(\phi_{\pi'}^{-1}(x))) \). We now define \( y = \phi_{e_1 \ldots e_n}(\phi_{\pi'}^{-1}(x)) \) and we will show that \( y \in \text{supp}(\pi)_i \). Since \( \phi_{e_0}(y) \in \text{supp}(\pi') \), we have \( \phi_{e_0} \in S_{e_1 \ldots e_n} \), and therefore \( y \in S_{\pi} \). Moreover,

\[
\phi_{\pi'}^k(y) = \phi_i^k(\phi_{e_1 \ldots e_n}(\phi_{\pi'}^{-1}(x)))
\]

Thus \( y \) is an element in \( \text{supp}(\pi') \), and more precisely an element in \( \text{supp}(\pi')_i \). We therefore showed that \( \text{supp}(\pi')_i = \phi_{e_0}(\text{supp}(\pi)_i) \).

To show the converse inclusion, we take \( x = \phi_{e_0}(y) \) with \( y \in \text{supp}(\pi)_i \). Then \( y \in S_{\pi'} \) and therefore \( y \in S_{e_0 e_1 \ldots e_n e_0} \). Finally \( \phi_{e_0}(y) \in S_{\pi'} \). Moreover, we have:

\[
\phi_{\pi'}^k(x) = \phi_i^k(\phi_{e_0}(y)) = \phi_{e_0}(\phi_{\pi'}^k(y)) = \phi_{e_0}(y) = x
\]

As a consequence, \( x \) is an element in \( \text{supp}(\pi)_i \), which shows the converse inclusion.

One can now compute:

\[
q_m(\pi) = \int_{\text{supp}(\pi)} \frac{m(\omega_{\phi_{e_0}}(x))}{\rho_{\phi_{e_0}}(x)} d\lambda(x)
= \sum_{i \in \mathbb{N}} \int_{\text{supp}(\pi)_i} \frac{m(\omega_{\phi_{e_0}})}{i} d\lambda(x)
= \sum_{i \in \mathbb{N}} \int_{\text{supp}(\pi')_i} \frac{m(\omega_{\phi_{e_0}})}{i} d\lambda(x)
= \int_{\text{supp}(\pi')} \frac{m(\omega_{\phi_{e_0}}(x))}{\rho_{\phi_{e_0}}(x)} d\lambda(x)
\]

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Thus \( q_m(\pi) = q_m(\pi') \). We can now conclude that \( q_m \) takes a constant value over the equivalence classes of cycles modulo the action of cyclic permutations. □

Since the map \( q_m \) does not depend on the choice of the representative of a circuit, one can define \( q_m(\tilde{\pi}) \) for any 1-circuit \( \tilde{\pi} \). We will abusively write:

\[
\int_{\text{supp}(\tilde{\pi})} \frac{m(\omega^{\rho_1}(x))}{\rho_\pi(x)} d\lambda(x)
\]

where \( \tilde{\pi} \) is a 1-circuit.

**Lemma 90.** For all measurable map \( m \), the function \( q_m \) is refinement-invariant.

**Proof.** Let \( F, G \) be weighted graphings, and \( F^{(e)} \) a simple refinement of \( F \) along \( e \in E' \). We will denote by \( f, f' \) the two elements of \( F^{(e)} \) which are the decompositions of \( e \). Up to almost everywhere equality, one can suppose that \( S_f \cap S_{f'} = \emptyset \).

Let us now choose \( \pi \) a representative of a 1-circuit \( \tilde{\pi} \). Since we are working with 1-circuits, the set \( \pi^0 \) is equal to \( \{\pi\} \). We suppose that \( \pi \) contains occurrences of \( e \), and write \( \pi = \rho_0 e_0 \rho_1 e_1 \cdots e_{i_n} \rho_n \) where for all \( j \), \( e_j = e \) and \( \rho_j \) is a path (where the paths \( \rho_0 \) and \( \rho_n \) may be empty). We denote by \( E_\pi \) the set of 1-cycles \( \mu = \rho_0 e_0 \rho_1 e_1 \cdots e_{i_{l-1}} \rho_l \rho_1 e_1 \cdots e_{i_{k-1}} \rho_k \cdots e_{i_m} \rho_m \) where \( k \in N \) — which we will denote by \( l g(\mu) \), and where for all values of \( l, m \), \( e_{i_l} = e \) or equal to \( f \). We will denote by \( \tilde{E}_\pi \) the set of 1-circuits in \( E_\pi \), i.e. the set \( E^{(F, 0)}_\pi \) introduced in Definition[73]

Let us pick \( x \in \text{supp}(\pi) - \rho_\pi^{-1}(\infty) \). Then \( x \in (\text{supp}(\pi))_k \) for a given value \( k \) in \( N \), i.e. \( \phi^k_\pi(x) = x \). Since \( S_e = S_f \cup S_{f'} \), we have, for each occurrence \( e_{i_p} \) of \( e \) and each integer \( l \):

\[
\phi^k_\pi = \phi_\pi^l \circ \phi_{\rho_p e_{i_{l-1}} e_{i_l} \cdots e_{i_n} \rho_n \circ \phi_{e_{i_p}} \circ \phi_{\rho_0 e_0 \rho_1 e_1 \cdots e_{i_{l-1}} \rho_l} \circ \phi^{k-l-1}_\pi
\]

Then \( \phi_{\rho_0 e_{i_p}} \circ \phi^{k-l-1}_\pi(x) \) is either an element in \( S_f \) or an element in \( S_{f'} \). For each occurrence \( e_i \) of \( e \), we will write \( d_{i_p, l} = f \) or \( d_{i_p, l} = f' \) according to wether \( \phi_{\rho_0 e_{i_p} \cdots e_{i_{l-1}}} \circ \phi^{k-l-1}_\pi(x) \) is an element in \( S_f \) or an element in \( S_{f'} \). Then we obtain, for all integer \( 0 \leq l \leq k \), paths \( v_1 = \rho_0 d_{i_p, l} d_{i_{l+1}} \cdots d_{i_{k-1}} \rho_n \). By concatenation, we can define a cycle \( \nu = v_0 v_1 \cdots v_k \). This cycle is a \( d \)-cycle for a given integer \( d \), i.e. \( v = \pi^d \) where \( \pi \) is a 1-cycle in \( E_\pi \). It is clear from the definition of \( \pi \) that \( x \in \text{supp}(\tilde{\pi}) \) and that, for all 1-cycle \( \mu \) in \( E_\pi \), \( x \notin \text{supp}(\mu) \) when \( \mu \neq \tilde{\pi} \).

Moreover, it is clear that if \( x \in \text{supp}(\mu) \) for a given 1-cycle \( \mu \in E_\pi \), then one necessarily has \( x \in \text{supp}(\mu) \). We deduce from this that the family \( (\text{supp}(\mu))_{\mu \in E_\pi} \) is a partition of the set \( \text{supp}(\pi) \). Notice that \( \omega_\mu = \omega_\mu^{lg(\mu)} \). Moreover, for all \( x \in \text{supp}(\mu) \), one has \( \rho_{\phi_\pi(x)}(x) = \rho_\pi(x) \), and therefore \( \omega^{\rho_\pi(x)} = \omega^{\rho_\pi(x)}_\mu \).

We now notice that if \( \mu = 1 \cdots \mu_{lg(\mu)} \in E_\pi \), and if \( \sigma \) is the cyclic permutations over \( \{1, \ldots, lg(\mu)\} \) such that \( \sigma(i) = i + 1 \), then the 1-cycles

\[
\mu_{\sigma^k} = \mu \cdot \mu_{\sigma^k}(1) \cdot \mu_{\sigma^k}(2) \cdots \mu_{\sigma^k(lg(\mu))}
\]

for \( 0 \leq k \leq lg(\mu) - 1 \) are pairwise disjoint elements in \( E_\pi \). Indeed, these are 1-cycles since \( \mu \) is a 1-cycle, and they are pairwise disjoint because if \( \mu_{\sigma^k} = \mu_{\sigma^k'} \)
(supposing that $k > k'$), we can show that $\mu_{o(k-k')} = \mu$ and that $k - k'$ divides $lg(\mu)$, which contradicts the fact that $\mu$ is a 1-cycle.

We can now deduce that:

$$q_m(\pi) = \int_{\text{supp}(\pi)} m(\omega_\mu(x)) \frac{\rho_{\phi_\pi}(x)}{\rho_\pi(x)} d\lambda(x)$$

$$= \sum_{\mu \in E} \int_{\text{supp}(\mu)} m(\omega_\mu(x)) \frac{\rho_{\phi_\pi}(x)}{\rho_\pi(x)} d\lambda(x)$$

$$= \sum_{\mu \in E} \int_{\text{supp}(\mu)} l g(\bar{\mu}) m(\omega_\mu(x)) \frac{\rho_{\phi_\pi}(x)}{\rho_\pi(x)} d\lambda(x)$$

$$= \sum_{\mu \in E} \int_{\text{supp}(\mu)} l g(\bar{\mu}) m(\omega_\mu(x)) \rho_{\phi_\pi}(x) \times l g(\bar{\mu}) d\lambda(x)$$

$$= \sum_{\mu \in E} \int_{\text{supp}(\mu)} m(\omega_\mu(x)) \frac{\rho_{\phi_\pi}(x)}{\rho_\pi(x)} d\lambda(x)$$

Which shows that $q_m$ is refinement-invariant.

The two preceding lemmas have as a direct consequence the following theorem which shows that we defined a family of circuit-quantifying maps.

**Theorem 91.** For all measurable map $m : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$, the function $q_m$ is a circuit-quantifying map.

6.2. Perennial and Co-perennial conducts

Since we are working with sliced thick graphs, we can follow the constructions of multiplicative and additive connectives as they are studied in the author’s second paper on interaction graphs [Sei12a] and which were quickly recalled in Section 3.

**Definition 92 (Projects).** A project is a couple $a = (a, A)$ together with a support $V^A$ where:

- $a \in \mathbb{R} \cup \{\infty\}$ is called the wager;

- $A$ is a sliced and thick weighted graphing of carrier $V^A$, of dialect $D^A$ a discrete probability space, and index $I^A$ a finite set.

**Remark.** We made here the choice to stay close to the hyperfinite geometry of interaction defined by Girard [Gir11]. This is why we restrict to discrete probability spaces as dialects, a restriction that corresponds to the restriction to finite von
Neumann algebras of type I as idioms in Girard’s setting. However, the results of the preceding section about execution and measurement, and the definition of the family of circuit-quantifying maps do not rely on this hypothesis. It should therefore be possible to consider a more general set of project where the dialects may eventually be continuous. It may turn out that this generalization could be used to define more expressive exponential connectives than the one defined in this paper, such as the usual exponentials of linear logic (recall that the exponentials defined here are the exponentials of Elementary Linear Logic).

As we explained at the end of Section 4, we will need to consider a particular kind of conducts which are the kind of conducts obtained from the application of the exponential modality to a conduct and which are unfortunately not behaviors. We now study these types of conducts.

**Definition 93 (Perennialisation).** A Perennialisation is a function that associates a one-sliced weighted graphing to any sliced and thick weighted graphing.

**Definition 94 (Exponentials).** Let $\mathcal{A}$ be a conduct, and $\Omega$ a perennialisation. We define the $!_{\Omega}\mathcal{A}$ as the bi-orthogonal closure of the following set of projects:

$$\#_{\Omega}\mathcal{A} = \{!a = (0, \Omega(A)) \mid a = (0, \mathcal{A}) \in \mathcal{A}, I^\mathcal{A} \equiv \{1\}\}$$

The dual connective is of course defined as $?_{\Omega}\mathcal{A} = (\#_{\Omega}\mathcal{A}^\bot)^\bot$.

**Definition 95.** A conduct $\mathcal{A}$ is a perennial conduct when there exists a set $\mathcal{A}$ of projects such that:

1. $\mathcal{A} = \mathcal{A}^\bot^\bot$;
2. for all $a = (a, \mathcal{A}) \in \mathcal{A}$, $a = 0$ and $\mathcal{A}$ is a one-sliced graphing.

A co-perennial conduct is a conduct $\mathcal{B} = \mathcal{A}^\bot$ where $\mathcal{A}$ is a perennial conduct.

**Proposition 96.** A co-perennial conduct $\mathcal{B}$ satisfies the inflation property: for all $\lambda \in \mathbb{R}$, $b \in \mathcal{B}$, $\lambda b \in \mathcal{B}$.

*Proof.* The conduct $\mathcal{A} = \mathcal{B}^\bot$ being perennial, there exists a set $\mathcal{A}$ of one-sliced wager-free projects such that $\mathcal{A} = \mathcal{A}^\bot^\bot$. If $\mathcal{A}$ is non-empty, the result is a direct consequence of Proposition 27. If $\mathcal{A}$ is empty, then $\mathcal{B} = \mathcal{A}^\bot = \mathcal{A}^\bot$ is the full behavior $T_{V^A}$ which obviously satisfies the inflation property.

**Proposition 97.** A co-perennial conduct is non-empty.

*Proof.* Suppose that $\mathcal{A}^\bot$ is a co-perennial conduct of carrier $V^A$. Then there exists a set $\mathcal{A}$ of one-sliced wager-free projects such that $\mathcal{A} = \mathcal{A}^\bot^\bot$. If $\mathcal{A}$ is empty, then $\mathcal{A}^\bot = \mathcal{A}^\bot$ is the behavior $T_{V^A}$. If $\mathcal{A}$ is non-empty, then one can easily check that for all real number $\lambda \neq 0$, the project $\Sigma a \lambda = (\lambda, (V^A, \emptyset))$ is an element of $\mathcal{A}^\bot = \mathcal{A}^\bot$.

**Corollary 97.1.** Let $\mathcal{A}$ be a perennial conduct. Then $a = (a, \mathcal{A}) \in \mathcal{A} \Rightarrow a = 0$.  

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Proof. Since \( A \downarrow \) is co-perennial, it is a non-empty set of projects with the same carrier which satisfies the inflation property. The result is then obtained by applying Proposition 27.

Proposition 98. If \( A \) is a co-perennial conduct, then for all \( a \neq 0 \), the project \( \mathcal{D}a_{a} = (a,(V^{\bar{A}},\emptyset)) \) is an element of \( A \).

Proof. We write \( B \) the set of one-sliced wager-free projects such that \( B \downarrow = A \). Then for all element \( b \in B \), we have that \( 1_{B} = 1 \), from which we conclude that \( \ll b, \mathcal{D}a_{a} \gg_{m} = a1_{B} = a \). Thus \( \mathcal{D}a_{a} \in B \downarrow = A \) for all \( a \neq 0 \).

Proposition 99. The tensor product of perennial conducts is a perennial conduct.

Proof. Let \( A,B \) be perennial conducts. Then there exists two sets of one-sliced wager-free projects \( E,F \) such that \( A = E \downarrow \downarrow \) and \( B = F \downarrow \downarrow \). Using Proposition 28 we know that \( A \otimes B = (E \otimes F) \downarrow \downarrow \). But, by definition, \( E \otimes F \) contains only projects of the form \( c \otimes f \), where \( c,f \) are one-sliced and wager-free. Thus \( E \otimes F \) contains only one-sliced wager-free projects and \( A \otimes B \) is therefore a perennial conduct.

Proposition 100. If \( A,B \) are perennial conducts, then \( A \otimes B \) is a perennial conduct.

Proof. This is a consequence of Proposition 29.

Proposition 101. If \( A \) is a perennial conduct and \( B \) is a co-perennial conduct, then \( A \rightarrow B \) is a co-perennial conduct.

Proof. We recall that \( A \rightarrow B = (A \otimes B^{\perp})^{\perp} \). Since \( A \) and \( B^{\perp} \) are perennial conducts, \( A \otimes B^{\perp} \) is a perennial conduct, and therefore \( A \rightarrow B \) is a co-perennial conduct. In particular, \( A \rightarrow B \) is non-empty and satisfies the inflation property.

Proposition 102. If \( A \) is a perennial conduct and \( B \) is a behavior, then \( A \otimes B \) is a behavior.

Proof. If \( A = 0_{V^{A}} \) with \( B = 0_{V^{B}} \), then \( A \otimes B = 0_{V^{A} \cup V^{B}} \) which is a behavior.

Let \( A \) be the set of one-sliced wager-free projects such that \( A = A \downarrow \downarrow \). We have that \( A \otimes B = (A \otimes B)^{\perp} \) by Proposition 28. If \( B = 0_{V^{B}} \) and \( A \neq 0 \), then \( A \otimes B \) is non-empty and contains only wager-free projects. Thus \( (A \otimes B)^{\perp} \) satisfies the inflation property by Proposition 27.

Now suppose there exists \( \mathcal{f} = (f,F) \in (A \otimes B)^{\perp} \) such that \( f \neq 0 \). Then for all \( a \in A \) and \( b \in B \), \( \ll \mathcal{f}, a \otimes b \gg_{m} \neq 0,\infty \). But, since \( a \) is wager-free and \( 1_{A} = 1 \), \( \ll \mathcal{f}, a \otimes b \gg_{m} = f1_{B} + b1_{F} + [F,A \cup B]_{m} \). We can then define \( \mu = (-[F,A \cup B]_{m} - b1_{F}/f - 1_{B} \). Since \( B \) is a behavior, \( b + \mu \in B \). However:

\[
\ll \mathcal{f}, a \otimes (b + \mu) \gg_{m} = f(1_{B} + \mu) + b1_{F} + [F,A \cup (B + \mu0)]_{m}
\]
\[
= f(1_{B} + \mu) + b1_{F} + [F,A \cup B]_{m}
\]
\[
= f(1_{B} + (-[F,A \cup B]_{m} - b1_{F}/f - 1_{B}) + b1_{F} + [F,A \cup B]_{m}
\]
\[
= -[F,A \cup B]_{m} - b1_{F} + b1_{F} + [F,A \cup B]_{m}
\]
\[
= 0
\]
But this is a contradiction. Therefore the elements in \((A \otimes B)\perp\) are wager-free.

If \((A \otimes B)\perp\) is non-empty, it is a non-empty conduct containing only wager-free projects and satisfying the inflation property; it is therefore a (proper) behavior.

The only case left to treat is when \((A \otimes B)\perp\) is empty, but then \(A \otimes B = T_{V^A \cup V^B}\) is clearly a behavior.

\[ \text{Corollary 102.1. If } A \text{ is perennial and } B \text{ is a behavior, then } A \rightarrow B \text{ is a behavior.} \]

\[ \text{Proof. We recall that } A \rightarrow B = (A \otimes B)\perp. \text{ Using the preceding proposition, the conduct } A \otimes B\perp \text{ is a behavior since } A \text{ is a perennial conduct and } B\perp \text{ is a behavior. Therefore } A \rightarrow B \text{ is a behavior since it is the orthogonal of a behavior.} \]

\[ \text{Corollary 102.2. If } A \text{ is a behavior and } B \text{ is a co-perennial conduct, then } A \rightarrow B \text{ is a behavior.} \]

\[ \text{Proof. One just has to write } A \rightarrow B = (A \otimes B)\perp. \text{ Since } A \otimes B\perp \text{ is the tensor product of a behavior with a perennial conduct, it is a behavior. The result then follows from the fact that the orthogonal of a behavior is a behavior.} \]

\[ \text{Proposition 103. The weakening (on the left) of perennial conducts holds.} \]

\[ \text{Proof. Let } A, B \text{ be conducts, and } N \text{ be a perennial conduct. Chose } f \in A \rightarrow B. \text{ We will show that } f \circ \circ V_N \text{ is a project in } A \otimes N \rightarrow B. \text{ For this, we pick } a \in A \text{ and } n \in N \text{ (recall that } n \text{ is necessarily wager-free). Then for all } b' \in B\perp, \]

\[ \langle\langle f \circ \circ : (a \otimes n), b' \rangle \rangle \]

\[ = \langle\langle f \circ \circ, (a \otimes n) \otimes b \rangle \rangle \]

\[ = \langle\langle f \circ \circ, (a \otimes b') \otimes n \rangle \rangle \]

\[ = 1_F(1_A 1_B^n + 1_N 1_A b' + 1_N 1_B a) + 1_1 1_A 1_B f + [F \cup 0, A \cup B' \cup N]_m \]

\[ = 1_F(1_N 1_A b' + 1_N 1_B a) + 1_N 1_A 1_B f + [F \cup 0, A \cup B' \cup N]_m \]

\[ = 1_N(1_F(1_A b' + 1_B a) + 1_A 1_B f) + 1_N[F, A \cup B']_m \]

\[ = 1_N \langle\langle f, a \otimes b' \rangle \rangle \]

Since \(1_N \neq 0, \langle\langle f \circ \circ : (a \otimes n), b' \rangle \rangle \neq 0, \infty \) if and only if \(\langle\langle f : a, b' \rangle \rangle \neq 0, \infty. \)

Thus for all \(a \otimes n \in A \otimes N, (f \circ \circ : (a \otimes n)) \in B. \) This shows that \(f \circ \circ \in A \otimes N \rightarrow B \) by Proposition 17. \]

6.3. Second Order

We now define localized second order quantification and show the duality between second order universal quantification and second order existential quantification. We will not dwell on the strange properties that are due to localization, which are more or less the same as quantification in Ludics [Gir01], since we will not be talking about second order quantification in the rest of the paper. Indeed, we will prove soundness for Elementary Linear Logic without quantifiers. However, we give in the concluding section some ideas on how a soundness result for Elementary Linear Logic with second order quantification might be obtained. Obtaining this result would mean a slight change in the definition of graphings
so as to allow transports of measure as edges instead of restricting to measure-preserving maps. We believe that this modification can be performed without loosing any of the properties needed to define the connectives of linear logic, but this would make the setting we are working with even more complex and we believe these modifications extend beyond the scope of this paper.

**Definition 104.** We define the localized second order quantification as:

\[
\forall_L X F(X) = \bigcap_{A, V^A = L} F(A)
\]

\[
\exists_L X F(X) = \left( \bigcup_{A, V^A = L} F(A) \right)^\perp
\]

**Proposition 105.**

\[
(\forall_L X F(X))^\perp = \exists_L X (F(X))^\perp
\]

**Proof.** The proof is straightforward. Using the definitions:

\[
(\forall_L X F(X))^\perp = \left( \bigcap_{A, V^A = L} F(A) \right)^\perp
\]

\[
= \left( \bigcup_{A, V^A = L} (F(A))^\perp \right)^\perp
\]

\[
= \exists_L X F^\perp(X)
\]

Where we used the fact that taking the orthogonal turns an intersection into a union (see the proof of Proposition 29).

We must stress the fact that it is possible to restrain all the constructions we have defined until now to sub-classes of the class of graphings. Indeed, any subset of the set of measure-preserving transformations on \( \mathbb{R} \) which is closed under composition defines a sub-class of graphings. One could therefore define the class of *interval graphs* as the class of weighted graphings whose edges are translations on the real line. This leads to a simple extension of the sliced graphs in which one can define both the MALL connectives and localized second order quantification (though we would not know how to define exponential connectives in this setting).

### 6.4. A Construction of Exponentials

We will begin by showing a technical result that will allow us to define measure preserving transformations from bijections of the set of integers. This result will be used to show that functorial promotion can be implemented for our exponential modality.

**Definition 106.** Let \( \phi : \mathbb{N} \to \mathbb{N} \) be a bijection and \( b \) an integer \( \geq 2 \). Then \( \phi \) induces a transformation \( T^b_{\phi} : [0, 1] \to [0, 1] \) defined by \( \sum_{i \geq 0} a_k 2^{-k} \mapsto \sum_{i \geq 0} a_{\phi^{-1}(k)} 2^{-k} \).
**Remark.** Suppose that \( \sum_{i \geq 0} a_i b^{-i} \) and \( \sum_{i \geq 0} a'_i b^{-i} \) are two distinct representations of a real number \( r \). Let us fix \( i_0 \) to be the smallest integer such that \( a_{i_0} \neq a'_{i_0} \). We first notice that the absolute value of the difference between these digits has to be equal to 1: \( |a_{i_0} - a'_{i_0}| = 1 \). Indeed, if this was not the case, i.e. if \( |a_{i_0} - a'_{i_0}| \geq 2 \), the distance between \( \sum_{i \geq 0} a_i b^{-i} \) and \( \sum_{i \geq 0} a'_i b^{-i} \) would be greater than \( b^{-i_0} \), which contradicts the fact that both sums are equal to \( r \). Let us now suppose, without loss of generality, that \( a_{i_0} = a'_{i_0} + 1 \). Then \( a_j = 0 \) for all \( j > i_0 \) since if there existed an integer \( j > i_0 \) such that \( a_j \neq 0 \), the distance between the sums \( \sum_{i \geq 0} a_i b^{-i} \) and \( \sum_{i \geq 0} a'_i b^{-i} \) would be greater than \( b^{-j} \), which would again be a contradiction. Moreover, \( a'_j = b - 1 \) for all \( j > i_0 \); if this was not the case, one could show in a similar way that the difference between the two sums would be strictly greater than 0. We can deduce from this that only the reals with a finite representation in base \( b \) have two distinct representations.

Since the set of such elements is of null measure, the transformation \( T_\phi \) associated to a bijection \( \phi \) of \( \mathbb{N} \) is well defined as we can define \( T_\phi \) only on the set of reals that have a unique representation. We can however chose to deal with this in another way: choosing between the two possible representations, by excluding for instance the representations as sequences that are almost everywhere equal to zero. Then \( T_\phi \) is defined at all points and bijective. We chose in the following to follow this second approach since it will allow to prove more easily that \( T_\phi \) is measure-preserving. However, this choice is not relevant for the rest of the construction since both transformations are almost everywhere equal.

**Lemma 107.** Let \( T \) be a transformation of \( [0,1] \) such that for all interval \( [a,b] \), \( \lambda(T([a,b])) = \lambda([a,b]) \). Then \( T \) is measure-preserving on \( [0,1] \).

*Proof.* A classical result of measure theory states that if \( T \) is a transformation of a measured space \( (X, \mathcal{B}, \lambda) \), that \( \mathcal{B} \) is generated by \( \mathcal{A} \), and that for all \( A \in \mathcal{A} \), \( \lambda(T(A)) = \lambda(A) \), then \( T \) preserves the measure \( \lambda \) on \( X \). Applying this result with \( X = [0,1] \) and \( \mathcal{A} \) as the set of intervals \( [a,b] \subset [0,1] \), we obtain the result. \( \square \)

**Lemma 108.** Let \( T \) be a bijective transformation of \( [0,1] \) that preserves the measure on all interval \( I \) of the shape \( [\sum_{k=1}^p a_k b^{-k}, \sum_{k=1}^p a_k b^{-k} + b^{-p}] \). Then \( T \) is measure-preserving on \( [0,1] \).

*Proof.* Chose \( [a,b] \subset [0,1] \). One can write \( [a,b] \) as a union \( \bigcup_{i=0}^\infty [a_i, a_{i+1}] \), where for all \( i \geq 0 \), \( a_{i+1} = a_i + b^{-k_i} \). We then obtain, using the hypotheses of the state-
ment and the $\sigma$-additivity of the measure $\lambda$:

$$
\lambda(T([a,b])) = \lambda(T(\cup_{i=0}^{\infty}[a_i,a_{i+1}]))
= \lambda(\cup_{i=0}^{\infty}T([a_i,a_{i+1}]))
= \sum_{i=0}^{\infty} \lambda([a_i,a_{i+1}])
= \sum_{i=0}^{\infty} \lambda([a_i,a_{i+1}])
= \lambda(\cup_{i=0}^{\infty}[a_i,a_{i+1}])
= \lambda([a,b])
$$

We now conclude by using the preceding lemma.

[Proof]

**Theorem 109.** Let $\phi : N \rightarrow N$ be a bijection and $b \geq 2$ an integer. Then the transformation $T_h^b$ is measure-preserving.

**Proof.** We recall first that the transformation $T_h^b$ is indeed bijective (see Remark 6.3).

By using the preceding lemma, it suffices to show that $T_h^b$ preserves the measure on intervals of the shape $I = [a, a + b^{-k}]$ with $a = \sum_{i=0}^{k} a_i b^{-i}$. Let us define $N = \max(\phi(i) \mid 0 \leq i \leq k)$. We then write $[0,1]$ as the union of intervals $A_i = [i, b^{-N}, (i+1) \times b^{-N}]$ where $0 \leq i \leq b^N - 1$.

Then the image if $I$ by $T_h^b$ is equal to the union of the $A_i$ for $i \times b^{-N} = \sum_{i=0}^{N} x_i b^{-i}$, where $x_{\phi(j)} = a_j$ for all $0 \leq j \leq k$. The number of such $A_j$ is equal to the number of sequences $[0, \ldots, b-1]$ of length $N-k$, i.e. $b^{N-k}$. Since each $A_j$ has a measure equal to $b^{-N}$, the image of $I$ by $T_h^b$ is of measure $b^{-N} b^{N-k} = b^{-k}$, which is equal to the measure of $I$ since $\lambda(I) = b^{-k}$.

**Remark.** The preceding theorem can be easily generalized to bijections $N + \cdots + N \rightarrow N$ (the domain being the disjoint union of $k$ copies of $N$, $k \in N$) which induce measure-preserving bijections from $[0,1]^k$ onto $[0,1]$. The particular case $N+N \rightarrow N$, $(n,i) \rightarrow 2n+i$ defines the well-known measure-preserving bijection between the unit square and the interval $[0,1]$:

$$
(\sum_{i=0} a_i 2^{-i}, \sum_{i=0} b_i 2^{-i}) \mapsto \sum_{i=0} a_i 2^{-2i} + b_{2i+1} 2^{-2i-1}
$$

Let us now define the bijection:

$$
\psi : N + N \rightarrow N, \quad (x,i) \rightarrow 3x + i
$$

We also define the injections $i_i$ ($i = 0,1,2$):

$$
i_i : N \rightarrow N + N, \quad x \mapsto (x,i)
$$

We will denote by $\psi_i$ the composite $\psi \circ i_i : N \rightarrow N$.
Definition 110. Let $A \subset \mathbb{N} + \mathbb{N} + \mathbb{N}$ be a finite set. We write $A$ as $A_0 + A_1 + A_2$, and define, for $i = 0, 1, 2$, $n_i$ to be the cardinality of $A_i$ if $A_i \neq$ and $n_i = 1$ otherwise. We then define a partition of $[0, 1]$, denoted by $\mathcal{P}_A = \{P_{i, j}^{A} \mid \forall k \in [0, 1], 0 \leq i_k \leq n_i - 1\}$, by:

$$P_{A}^{i_1, i_2, i_3} = \left\{ \sum_{j=1}^{n_k} a_j 2^{-j} \mid \forall k \in [0, 1], \frac{i_k}{n_k} \leq \sum_{j=1}^{k} a_j 2^{-j} \leq \frac{i_k + 1}{n_k} \right\}$$

When $A_k$ is empty or of cardinality 1, we will not write the corresponding $i_k$ in the triple $(i_1, i_2, i_3)$ since it is necessarily equal to 0.

Proposition 111. Let us keep the notations of the preceding proposition and let $X = P_{A}^{i_1, i_2, i_3}$ and $Y = P_{A}^{i_1, i_2, i_3}$ be two elements of the partition $\mathcal{P}_A$. For all $x = \sum_{j=1}^{n_k} a_j 2^{-j}$, we define $T_{i_1, i_2, i_3}(x) = \sum_{j=1}^{n_k} b_j 2^{-j}$ where the sequence $(b_i)$ is defined by:

$$\forall k \in [0, 1], \sum_{j=1}^{n_k} b_j 2^{-j} = \sum_{j=1}^{m_i} a_j 2^{-j} + j_k - i_k$$

Then $T_{i_1, i_2, i_3}: X \to Y$ is a measure-preserving bijection.

Proof. For $k = 0, 1, 2$, we will denote by $(m_k^k)$ the sequence such that $j_k - i_k = \sum_{l=1}^{n_k} m_k^l 2^{-l}$. We can define the real number $t = \sum_{l=1}^{n_k} \sum_{k=0, 1, 2} m_k^l 2^{-3j + k}$. It is then sufficient to check that $T_{i_1, i_2, i_3}(x) = x + t$. Since $T_{i_1, i_2, i_3}$ is a translation, it is a measure-preserving bijection. \qed

Definition 112. Let $A \subset \mathbb{N}$ be a finite set endowed with the normalized — i.e. such that $A$ has measure 1 — counting measure, and $X \subset \mathcal{B}(\mathbb{R} \times A)$ be a measurable set. We define the measurable subset $\Gamma A \subset [0, 1]$: \n
$$\Gamma A = \{ (x, y) \mid \exists z \in A, (x, z) \in X, y \in P_{A}^z \}$$

We will write $\mathcal{P}_A^{-1}: [0, 1] \to A$ the map that associates to each $x$ the element $z \in A$ such that $x \in P_{A}^z$.

Proposition 113. Let $D^{A} \subset \mathbb{N}$ be a finite set endowed with the normalized counting measure $\mu$ (i.e. such that $\mu(A) = 1$), $S, T \in \mathcal{B}(\mathbb{R} \times D^{A})$ be measurable sets, and $\phi: S \to T$ a measure-preserving transformation. We define $\Gamma \phi^{-1}: \Gamma S \to \Gamma T$ by:

$$\Gamma \phi^{-1}(x, y) \rightarrow (x', y') \quad \phi(x, \mathcal{P}_A^{-1}(y)) = (x', z), \quad y' = T_{z}^z$$

Then $\Gamma \phi^{-1}$ is a measure-preserving bijection.

Proof. For all $(a, b) \in D^{A}$ we define the set $S_{a, b} = X \cap \mathbb{R} \times \{a\} \cap \phi^{-1}(Y \cap \mathbb{R} \times \{b\})$. The family $(S_{a, b})_{a, b \in D^{A}}$ is a partition of $S$, and the family $(\Gamma S_{a, b})_{a, b \in D^{A}}$ is a partition of $\Gamma A$. The restriction of $\Gamma \phi^{-1}$ to $\Gamma S_{a, b}$ can then be defined as the composite $T_{a} \circ \phi_{1}$ with:

$$\phi_{1} = (\pi_{1} \circ \phi) \times \text{Id}$$

$$T_{a} = \text{Id} \times T_{b}$$
These bijections induce bijections of $N$ which is an element of $P$ and we have finished the proof.

**Definition 114.** Let $A$ be a thick graphing, i.e. of support $V^A \subset R \times D^A$ measurable, where $D^A$ is a finite subset of $N$ endowed with the normalized counting measure. We define the graphing:

$\gamma A = \{ (\omega'_B \gamma, \gamma_{S^A_B} \gamma : \gamma T^A_{B} \gamma) \}_e \in E^A$

**Definition 115.** Let $A$ be a thick graphing of dialect $D^A$, and $\Omega : R \times [0,1] \rightarrow R$ an isomorphism of measured spaces. We define the graphing $!_{\Omega} A$ by:

$!_{\Omega} A = \{ (\omega'_B \Omega \circ \phi^A_{B} \gamma, \Omega^{-1} : \Omega(T^A_{B} \gamma) \} _{e \in E^A}$

**Definition 116.** A project $a$ is balanced if $a = (0,A)$ where $A$ is a thick graphing, i.e. $I^A$ is a one-element set, for instance $I^A = \{ 1 \}$, and $a_1^A = 1$.

**Definition 117.** Let $a$ be a balanced project. We define $!_{\Omega} a = (0,1_{\Omega} A)$. If $A$ is a conduct, we define:

$!_{\Omega} A = \{ !_{\Omega} a | a = (0,A) \in A, a \text{ a balanced}\}$

We will now show that it is possible to implement the functorial promotion. In order to do this, we define the bijections $\tau, \theta : N + N + N \rightarrow N + N + N$:

$$
\tau : \begin{cases} (x,0) & \rightarrow (x,1) \\ (x,1) & \rightarrow (x,0) \\ (x,2) & \rightarrow (x,2) \\
\end{cases}
$$

$$
\theta : \begin{cases} (x,0) & \rightarrow (2x,0) \\ (x,1) & \rightarrow (2x+1,1) \\ (2x,2) & \rightarrow (x,1) \\ (2x+1,2) & \rightarrow (x,2) \\
\end{cases}
$$

These bijections induce bijections of $N$ onto $N$ through $\psi : (x,i) \rightarrow 3x + i$. We will abusively denote by $T_{\tau} = T_{\psi \tau \psi^{-1}}$ and $T_{\theta} = T_{\psi \theta \psi^{-1}}$ the induced measure-preserving transformations $[0,1] \rightarrow [0,1]$.

Pick $a \in \mathcal{Z}(A)$ and $f \in \mathcal{Z}(A \rightarrow B)$, where $\phi$ is a delocation. By definition, $a = (0, \Omega(\gamma A \gamma))$ and $f = (0, \Omega(\gamma F \gamma))$ where $A,F$ are graphings of respective dialects $D^A, D^F$. We define the graphing $T = (1, \Omega(\text{Id} \times T_{\tau})),(1, \Omega(\text{Id} \times T_{\theta}))^{-1})$ of carrier $V^A \cup V^A$, and denote by $t, t^*$ the two edges in $E^T$. We fix $(x,y)$ an element of $V^B$ and we will try to understand the action of the path $f_0 t_{a_0^t} f_1 ... t_{a_{k-1}^t} t_{a_k^t} f_k$.

We fix the partition $\mathcal{A}_{D^F + D^A}$ of $[0,1]$, and denote by $(i,j)$ the integers such that $y \in \mathcal{A}_{D^F + D^A}$. By definition of $\gamma F \gamma$, the map $\phi_{(i,j)}$ sends this element to $(x_1, y_1)$ which is an element of $\mathcal{A}_{D^F + D^A}$. Then, the function $\phi_I$ sends this element on $(x_2, y_2)$, where $x_2 = x_1$ and $y_2$ is an element of $\mathcal{A}_{D^F + D^A}$. The function
We define the graphings \( T \) and \( V \). We define \( \phi \), \( \psi \), and \( \theta \) such that \( \phi \), \( \psi \), and \( \theta \) have pairwise disjoint carriers, there exists a project \( \text{prom} \) in the conduct

\[ \left[ \phi(A) \otimes \left( \left( A \to B \right) \to \psi(B) \right) \right] \]

**Proof.** Let \( f \in A \to B \) be a balanced project, \( \phi, \psi \), and \( \theta \) are delocations of \( A \) and \( B \) respectively. We define the graphings \( T = \{(1, \Omega(\text{Id} \times T_\tau)), (1, (\Omega(\text{Id} \times T_\theta))^{-1})\} \) of carrier \( V^{\phi(A)} \cup V^A \) and \( P = \{(1, \Omega(\text{Id} \times T_\theta)), (1, (\Omega(\text{Id} \times T_\theta))^{-1})\} \) of carrier \( V^B \cup V^{\psi(B)} \). We define \( t = (0, T) \) and \( p = (0, P) \), and the project:

\[ \text{prom} = (0, T \cup P) = t \otimes p \]

We will now show that \( \text{prom} \) is an element in \( \left( \exists \phi(A) \otimes \left( \left( A \to B \right) \to \psi(B) \right) \right) \).

We can suppose, up to choosing refinements of \( A \) and \( F \), that for all \( e \in E^A \cup E^F \), \( S_e \) and \( T_e \) are one-elements sets\(^{10}\).

\(^{10}\)The sets \( S_e \) and \( T_e \) being subsets of a product, we write \( (S_e)_2 \) (resp. \( (T_e)_2 \)) the result of their projection on the second component.
Pick $a \in \mathcal{A}$ and $f \in \mathcal{A} \rightarrow \mathcal{B}$. Then, by definition $a = (0, \Omega(\mathcal{A}))$ and $f = (0, \Omega(\mathcal{F}))$ where $A, F$ are graphings of dialects $\mathcal{D}A, \mathcal{D}F$. We get that $a \otimes f : \text{prom} = ((\alpha \times : \text{prom} : \phi) \circ : \text{p} \text{from the associativity and commutativity of } \otimes$: (recall that $a \otimes f = a \times f$).

We show that $\mathcal{A} \rightarrow \mathcal{F}$ is the graphings composed of the !$f_\alpha$ for $a \in \mathcal{E}$, where !$f_\alpha$ is defined by:

$$!f_\alpha : (x, y) \mapsto (x', y'), \quad \phi_\alpha(x, \mathcal{D}^{-1}_A + \mathcal{D}A(y)) = (x', z), \quad y' = T^{(e, 1)}_{\mathcal{D}^{-1}_A + \mathcal{D}A}(y)$$

This is almost straightforward. An element in $\mathcal{A} \rightarrow \mathcal{F}$ is a path of the form $tat^*$. It is therefore the function $\phi_\alpha \circ !f_\alpha \circ \phi_\alpha^{-1}$. By definition,

$$!f_\alpha : (x, y) \mapsto (x', y'), \quad n = \mathcal{D}^{-1}_A(y), \quad \phi_\alpha(x, n) = (x', k), \quad y' = T^k_n(y)$$

But $\phi_\alpha : \text{Id} \times T_\theta$ and $T_\theta$ is a bijection from $\mathcal{D}A(y)$ to $\mathcal{D}A(1, y)$.

We now describe the graphing $G = (\mathcal{A} \rightarrow \mathcal{F}) : (\mathcal{F})$. It is composed of the paths of the shape $\rho = f_0(ta_0t^*)f_1(ta_1t^*)f_2 \ldots f_{n-1}(ta_{n-1}t^*)f_n$. The associated function is therefore:

$$\phi_{\rho} = \phi_{f_0} \circ \phi_{f_1} \circ \cdots \circ \phi_{f_{n-1}} \circ \phi_{f_n}^{-1}$$

Let $\pi = f_0a_0f_1 \ldots f_{n-1}a_{n-1}f_n$ be the corresponding path in $\mathcal{F} : \mathcal{A}$. The function $\phi_\pi$ has, by definition, as domain and codomain measurable subsets of $\mathcal{R} \times \mathcal{D}F \times \mathcal{D}A$.

We define, for such a function, the function !$i f_\pi$ by:

$$!i f_\pi : (x, y) \mapsto (x', y')$$

$$(n, m) = \mathcal{D}^{-1}_F + \mathcal{D}A(y), \quad \phi_\pi(x, n, m) = (x', k, l), \quad y' = T^{(k, l)}_{\mathcal{D}F + \mathcal{D}A}(y)$$

One can then check that !$i f_\pi = \phi_\pi$.

Finally, $G : (\mathcal{F})$ is the graphing composed of paths that have the shape $ppp^*$ where $p$ is a path in $G$. But $\phi_p : \text{Id} \times T_\theta$ applies a bijection, for all couple $(k, l) \in \mathcal{D}F \times \mathcal{D}A$, from the set $\mathcal{D}F \times \mathcal{D}A$ to the set $\mathcal{D}F \times \mathcal{D}A$ where:

$$\theta(\mathcal{D}F + \mathcal{D}A) = \{ \theta(f, a) \mid f \in \mathcal{D}F, a \in \mathcal{D}A \}$$

We deduce that:

$$\phi_{ppp^*} : (x, y) \mapsto (x', y')$$

$$n = \theta(k, l) = \mathcal{D}^{-1}_F \mathcal{D}A(\theta(y'), \phi_{ppp^*}(x, k, l) = (x', k', l'), \quad y' = T^{(k', l')}_{\mathcal{D}F + \mathcal{D}A}(y)$$

Modulo the bijection $\mu : \mathcal{D}F \times \mathcal{D}A \rightarrow \theta(\mathcal{D}F + \mathcal{D}A) \subset \mathcal{N}$, we get that $G : (\mathcal{F})$ is the delocation (along $\psi$) of the graphing !$(\mathcal{F} : \mathcal{A})$.

Therefore, for all $a, f$ in $\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B}$ respectively there exists a project $b$ in $\mathcal{B}$ such that prom : (a $\otimes$ f) = b. We showed that for all $g \in \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}$, one has prom : = g $\otimes$ b, and thus prom is an element in $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}$ by Proposition 47. But $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}$ by Proposition 28. □
Figure 24: Functorial Promotion
In the setting of its hyperfinite geometry of interaction [Gir11], Girard shows how one can obtain the exponentials isomorphism as an equality between the conducts \(!A \otimes !B\) and \(!A \& !B\). Things are however quite different here. Indeed, if the introduction of behaviors in place of Girard’s negative/positive conducts is very interesting when one is interested in the additive connectives, this leads to a (small) complication when dealing with exponentials. The first thing to notice is that the proof of the implication \(!A \otimes !B \rightarrow !(A \& B)\) in a sequent calculus with functorial promotion and without dereliction and digging rules cannot be written if the weakening rule is restrained to the formulas of the form \(\vdash A\): 

\[
\begin{array}{c}
\vdash A, A \downarrow \downarrow \text{ax} \\
\vdash A, B \downarrow \downarrow, A \downarrow \downarrow \text{weak} \\
\vdash B \downarrow \downarrow, A \downarrow \downarrow \& \text{weak} \\
\vdash B \downarrow \downarrow, A \downarrow \downarrow, A \downarrow \downarrow, B \downarrow \downarrow \vdash !(A \& B) \\
\vdash !(A \& B) \vdash !(A \& B)
\end{array}
\]

In Girard’s setting, weakening is available for all positive conducts (the conducts on which one can apply the ? modality), something which is coherent with the fact that the inclusion \(!A \otimes !B \subset !(A \& B)\) is satisfied. In our setting, however, weakening is never available for behaviors and we think the latter inclusion is therefore not satisfied. This question stays however open.

Concerning the converse inclusion, it does not seem clear at first that it is satisfied in our setting either. This issue comes from the contraction rule. Indeed, since the latter does not seem to be satisfied in full generality (see Remark 119), one could think the inclusion \(!!(A \& B) \subset !(A \otimes B)\) is not satisfied either. We will show however in Section 7.1 through the introduction of alternative “additive connectives”, that it does hold (a result that will not be used until the last section).

**Proposition 119.** The conduct \(1\) is a perennial conduct, equal to \(!T\).

**Proof.** By definition, \(1 = ((0, \emptyset) \downarrow \downarrow)\) is a perennial conduct. Moreover, the balanced projects in \(T\) are the projects of the shape \(t_D = (0, \emptyset)\) with dialects \(D \subset N\). Each of these satisfy \(!t_D = (0, \emptyset)\). Thus \(\Downarrow T = ((0, \emptyset))\) and \(\vdash !T = 1\). □

**Corollary 119.1.** The conduct \(\bot\) is a co-perennial conduct, equal to \(\vdash 0\).

**Proof.** This is straightforward:

\[
\bot = 1 \downarrow = (!T) \downarrow = (\Downarrow T) \downarrow \downarrow = (\Downarrow T) \downarrow = (\Downarrow 0) \downarrow = 0
\]

7. Soundness for Behaviors

7.1. Sequent Calculus

To deal with the three kinds of conducts we are working with (behaviors, perennial and co-perennial conducts), we introduce three types of formulas.
Definition 120. We define three types of formulas, (B)ehaviors, (N)egative — perennial, and (P)ositive — co-perennial, inductively defined by the following grammar:

\[
\begin{align*}
B & := T \mid 0 \mid X \mid X \downarrow \mid B \otimes B \mid B \boxplus B \mid B \& B \mid N \otimes B \mid P \boxdot B \\
N & := 1 \mid P \downarrow \mid !B \mid N \otimes N \mid N \oplus N \\
P & := \bot \mid N \downarrow \mid ?B \mid P \boxdot P \mid P \& P
\end{align*}
\]

Definition 121. We define pre-sequents \( \Delta \vdash \Gamma ; \Theta \) where \( \Delta, \Theta \) contain negative (perennial) formulas, \( \Theta \) containing at most one formula, and \( \Gamma \) contains only behaviors.

Proposition 45 supposes that we are working with behaviors, and cannot be used to interpret contraction in full generality. It is however possible to show in a similar way that contraction can be interpreted when the sequent contains at least one behavior (this is the next proposition). This restriction of the context is necessary: without behaviors in the sequent one cannot interpret the contraction since the inflation property is essential for showing that \( (1/2)\phi(\lnot a) \otimes \psi(\lnot a) + (1/2) \phi \) is an element of \( \phi(\lnot A) \otimes \psi(\lnot A) \).

Proposition 122. Let \( A \) be a conduct and \( \phi, \psi \) be disjoint delocations of \( !V A \). Let \( C \) be a behavior and \( \theta \) a delocation disjoint from \( \phi \) and \( \psi \). Then the project \( C^w \phi,\psi,\theta \) is an element of the behavior:

\[
(\lnot A \otimes C) \rightarrow (\phi(\lnot A) \otimes \psi(\lnot A) \otimes \theta(C))
\]

Proof. The proof follows the proof of Proposition 45. We show in a similar manner that the project \( C^w \phi,\psi,\theta ::(a \otimes c) \) is universally equivalent to:

\[
\frac{1}{2}\phi(\lnot a) \otimes \psi(\lnot a) \otimes \theta(C) + \frac{1}{2} \phi
\]

Since \( !A \) is a perennial conduct and \( C \) is a behavior, \( (\phi(\lnot A) \otimes \psi(\lnot A) \otimes \theta(C)) \) is a behavior. Thus \( C^w \phi,\psi,\theta ::(a \otimes c) \) is an element in \( (\phi(\lnot A) \otimes \psi(\lnot A) \otimes \theta(C)) \). Finally we showed that the project \( C^w \phi,\psi,\theta \) is an element of \( (\lnot A \otimes C) \rightarrow (\phi(\lnot A) \otimes \psi(\lnot A) \otimes \theta(C)) \), and that the latter is a behavior.

In a similar way, the proof of distributivity relies on the property that \( A + B \subset A \& B \) which is satisfied for behaviors but not in general. It is therefore necessary to restrict to pre-sequents that contain at least one behavior in order to interpret the \( \& \) rule. Indeed, we can think of a pre-sequent \( \Delta \vdash \Gamma ; \Theta \) as the conduct\(^\text{11}\)

\[
\left( \bigvee_{N \in \Delta} N \downarrow \right) \left( \bigvee_{B \in \Gamma} B \right) \left( \bigvee_{N \in \Theta} N \right)
\]

\(^{11}\)This will actually be the exact definition of its interpretation.
<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \vdash \Gamma, C; \quad \Delta \vdash \Gamma几_{1}, C； \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \text{cut}$</td>
<td>(a) Identity Group</td>
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<th>Rule</th>
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<tbody>
<tr>
<td>$\Delta \vdash \Gamma几_{1}, C几_{1}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \Delta \vdash \Gamma几_{1}, C几_{2}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \varphi$</td>
<td>(b) Multiplicative Group</td>
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<td>$\Delta \vdash \Gamma几_{1}, C几_{1}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \Delta \vdash \Gamma几_{1}, C几_{2}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \psi$</td>
<td>(c) Additive Group</td>
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<td>$\Delta \vdash \Gamma几_{1}, C几_{1}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \Delta \vdash \Gamma几_{1}, C几_{2}; \quad \Delta \vdash \Gamma几_{2}, C几_{2}; \quad \varphi^{\text{mix}}$</td>
<td>(d) Exponential Group</td>
</tr>
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Figure 25: Rules for the sequent calculus $\text{ELL}_{\text{comp}}$

Such a conduct is a behavior when the set $\Gamma$ is non-empty and the set $\Theta$ is empty, but it is neither a perennial conduct nor a co-perennial conduct when $\Gamma = \emptyset$. We will therefore restrict to pre-sequents such that $\Gamma \neq \emptyset$ and $\Theta = \emptyset$.

**Definition 123 (Sequents).** A sequent $\Delta \vdash \Gamma; \Theta$ is a pre-sequent $\Delta \vdash \Gamma; \Theta$ such that $\Gamma$ is non-empty and $\Theta$ is empty.

**Definition 124 (The Sequent Calculus $\text{ELL}_{\text{comp}}$).** A proof in the sequent calculus $\text{ELL}_{\text{comp}}$ is a derivation tree constructed from the derivation rules shown in Figure 25 page 68.

7.2. Truth

The notion of success is the natural generalization of the corresponding notion on graphs [Sei12b, Sei12a]. The graphing of a successful project will therefore be a disjoint union of “transpositions”. Such a graphing can be represented as a
graph with a set of vertices that could be infinite, but since we are working with equivalence classes of graphings one can always find a simpler representation: a graphing with exactly two edges.

**Definition 125.** A project \( a = (a, A) \) is *successful* when it is balanced, \( a = 0 \) and \( A \) is a disjoint union of transpositions:

- for all \( e \in E^A \), \( \omega^A_e = 1 \);
- for all \( e \in E^A \), \( \exists e^* \in E^A \) such that \( \phi^A_e = (\phi^A_e)^{-1} \) — in particular \( S^A_e = T^A_e \) and \( T^A_e = S^A_e \);
- for all \( e, f \in E^A \) with \( f \not\in \{e, e^*\} \), \( S^A_e \cap S^A_f \) and \( T^A_e \cap T^A_f \) are of null measure;

A conduct \( A \) is *true* when it contains a successful project.

**Lemma 126.** If \( a = (0, A) \) is successful, one can find a representative \( \hat{A} \) of the equivalence class modulo \( \sim \) of \( A \) such that \( E^\hat{A} \) has its cardinality equal to 2.

**Proof.** One can define a partition \( E_1, E_2 \) of \( E^A \) such that for all couple \( e, e^* \) in \( E^A \),

\[
\hat{A} = (1, \bigcup_{e \in E_1} \phi^A_e : \bigcup_{e \in E_1} S^A_e \to \bigcup_{e \in E_2} T^A_e), (1, \bigcup_{e \in E_2} \phi^A_e : \bigcup_{e \in E_2} S^A_e \to \bigcup_{e \in E_2} T^A_e)\]

It is clear that \( A \) is a refinement of \( \hat{A} \) which concludes the proof.

**Proposition 127** (Consistency). The conducts \( A \) and \( A^\perp \) cannot be simultaneously true.

**Proof.** We suppose that \( a = (0, A) \) and \( b = (0, B) \) are successful projects in the conducts \( A \) and \( A^\perp \) respectively. Then:

\[
\ll a, b \gg_m = \ll A, B \gg_m
\]

If there exists a cycle whose support is of strictly positive measure between \( A \) and \( B \), then \( \ll A, B \gg_m = \infty \). Otherwise, \( \ll A, B \gg_m = 0 \). In both cases we obtained a contradiction since \( a \) and \( b \) cannot be orthogonal.

**Proposition 128** (Compositionnality). If \( A \) and \( A \rightarrow B \) are true, then \( B \) is true.

**Proof.** Let \( a \in A \) and \( f \in A \rightarrow B \) be successful projects. Then:

- If \( \ll a, f \gg_m = \infty \), the conduct \( B \) is equal to \( T_{VS} \), which is a true conduct since it contains \((0, \emptyset)\);
- Otherwise \( \ll a, f \gg_m = 0 \) (this is shown in the same manner as in the preceding proof) and it is sufficient to show that \( F : A \) is a disjoint union of transpositions. But this is straightforward: to each path there corresponds an opposite path and the weights of the paths are all equal to 1, the conditions on the source and target sets \( S_x \) and \( T_x \) are then easily checked.

Finally, if \( A \) and \( A \rightarrow B \) are true, then \( B \) is true.
7.3. Interpretation of proofs

To prove soundness, we will follow the proof technique used in our previous papers [Sei12b, Sei12a]. We will first define a localized sequent calculus and show a result of full soundness for it. The soundness result for the non-localized calculus is then obtained by noticing that one can always localize a derivation. We will consider here that the variables are defined with the carrier equal to an interval in \( \mathbb{R} \) of the form \([i, i+1]\).

Definition 129. We fix a set \( \mathcal{V} = \{X_i(j)\}_{i,j \in \mathbb{N} \times \mathbb{Z}} \) of localized variables. For \( i \in \mathbb{N} \), the set \( X_i = \{X_i(j)\}_{j \in \mathbb{Z}} \) will be called the variable name \( X_i \), and an element of \( X_i \) will be called a variable of name \( X_i \).

For \( i,j \in \mathbb{N} \times \mathbb{Z} \) we define the location \( \sharp X_i(j) \) of the variable \( X_i(j) \) as the set
\[
\{ x \in \mathbb{R} | 2^i(2j + 1) \leq m < 2^i(2j + 1) + 1 \}
\]

Definition 130 (Formulas of \( \text{locELL}_{\text{comp}} \)). We inductively define the formulas of \( \text{localized polarized elementary linear logic} \) \( \text{locELL}_{\text{comp}} \) as well as their locations as follows:

- **Behaviors:**
  - A variable \( X_i(j) \) of name \( X_i \) is a behavior whose location is defined as \( \sharp X_i(j) \).
  - If \( X_i(j) \) is a variable of name \( X_i \), then \( (X_i(j))^\perp \) is a behavior whose location is \( \sharp X_i(j) \).
  - The constants \( T_{\sharp \Gamma} \) are behaviors whose location is defined as \( \sharp \Gamma \).
  - The constants \( 0_{\sharp \Gamma} \) are behaviors whose location is defined as \( \sharp \Gamma \).
  - If \( A, B \) are behaviors with respective locations \( X, Y \) such that \( X \cap Y = \emptyset \), then \( A \& B \) (resp. \( A \& B \), resp. \( A \& B \), resp. \( A \oplus B \)) is a behavior whose location is \( X \cup Y \);
  - If \( A \) is a perennial conduct with location \( X \) and \( B \) is a behavior whose location is \( Y \) such that \( X \cap Y = \emptyset \), then \( A \otimes B \) is a behavior with location \( X \cup Y \);
  - If \( A \) is a co-perennial conduct whose location is \( X \) and \( B \) is a behavior with location \( Y \) such that \( X \cap Y = \emptyset \), then \( A \& B \) is a behavior and its location is \( X \cup Y \);

- **Perennial conducts:**
  - The constant \( 1 \) is a perennial conduct and its location is \( \emptyset \);
  - If \( A \) is a behavior or a perennial conduct and its location is \( X \), then \( !A \) is a perennial conduct and its location is \( \Omega(X \times [0,1]) \);
  - If \( A, B \) are perennial conducts with respective locations \( X, Y \) such that \( X \cap Y = \emptyset \), then \( A \otimes B \) (resp. \( A \& B \)) is a perennial conduct whose location is \( X \cup Y \).
• **Co-perennial conducts:**
  
  - The constant $\perp$ is a co-perennial conduct;
  
  - If $A$ is a behavior or a co-perennial conduct and its location is $X$, then $\text{?}A$ is a co-perennial conduct whose location is $\Omega(X \times [0,1])$;
  
  - If $A,B$ are co-perennial conducts with respective locations $X,Y$ such that $X \cap Y = \emptyset$, then $A \not\not B$ (resp. $A \& B$) is a co-perennial conduct whose location is $X \cup Y$;

If $A$ is a formula, we will denote by $\sharp A$ the location of $A$. A sequent $\Delta \vdash \Gamma$; of locELL$_{comp}$ must satisfy the following conditions:

- the formulas of $\Gamma \cup \Delta$ have pairwise disjoint locations;
- the formulas of $\Delta$ are all perennial conducts;
- $\Delta$ is non-empty and contains only behaviors.

**Definition 131** (Interpretations). An **interpretation basis** is a function $\Phi$ which associates to each variable name $X_i$ a behavior of carrier $[0,1]$.

**Definition 132** (Interpretation of locELL$_{comp}$ formulas). Let $\Phi$ be an interpretation basis. We define the interpretation $I_\Phi(F)$ along $\Phi$ of a formula $F$ inductively:

- If $F = X_i(j)$, then $I_\Phi(F)$ is the delocation (i.e. a behavior) of $\Phi(X_i)$ defined by the function $x \mapsto 2^j(2j + 1) + x$;
- If $F = (X_i(j))^\perp$, we define the behavior $I_\Phi(F) = (I_\Phi(X_i(j))^\perp$;
- If $F = T_{\Phi^i}$ (resp. $F = 0_{\Phi^i}$), we define $I_\Phi(F)$ as the behavior $T_{\Phi^i}$ (resp. $0_{\Phi^i}$);
- If $F = 1$ (resp. $F = \perp$), we define $I_\Phi(F)$ as the behavior $1$ (resp. $\perp$);
- If $F = A \oplus B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \oplus I_\Phi(B)$;
- If $F = A \not\not B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \not\not I_\Phi(B)$;
- If $F = A \& B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \& I_\Phi(B)$;
- If $F = !A$ (resp. $?A$), we define the conduct $I_\Phi(F) = !I_\Phi(A)$ (resp. $?I_\Phi(A)$).

Moreover, a sequent $\Delta \vdash \Gamma$; will be interpreted as the $\not\not$ of formulas in $\Gamma$ and negations of formulas in $\Delta$, which will be written $\not\not \Delta \vdash \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not \not\not
• if $\pi$ is composed of a single rule $T_\Gamma$, we define $I_\Phi(\pi) = o_\Gamma$;

• if $\pi$ is obtained from $\pi'$ by using a $\mathcal{R}$ rule, a $\mathcal{R}^{mix}$ rule, a $\otimes_{p}$ rule, or a $I$ rule, then $I_\Phi(\pi) = I_\Phi(\pi')$;

• if $\pi$ is obtained from $\pi_1$ and $\pi_2$ by performing a $\otimes$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \otimes I_\Phi(\pi_2)$;

• if $\pi$ is obtained from $\pi'$ using a $\mathcal{A}$ rule or a $\oplus_i$ rule introducing a formula of location $V$, we define $I_\Phi(\pi) = I_\Phi(\pi') \otimes 0^V$;

• if $\pi$ of conclusion $\vdash \Gamma, A_0 \& A_1$ is obtained from $\pi_0$ and $\pi_1$ using a $&$ rule, we define the interpretation of $\pi$ in the same way it was defined in our previous paper [Sei12a];

• if $\pi$ is obtained from $\pi_1$ and $\pi_2$ through the use of a promotion rule $!$, we think of this rule as the following "derivation of pre-sequents":

\[
\frac{\pi_1}{\Delta_1 \vdash \Gamma_1, C_1; \Delta_2 \vdash \Gamma_2, C_2; \ !} \quad \frac{\Delta_2 \vdash \Gamma_2, C_2; \ !}{\Delta_1, \Delta_2, ! \Gamma_2 \vdash \Gamma_1 \otimes ! C_2; \ \otimes_{mix}}
\]

As a consequence, we first define a delocation of $! I_\Phi(\pi)$ to which we apply the implementation of the functorial promotion. Indeed, the interpretation of

\[
\mathcal{R} \mathcal{A}^\perp \mathcal{R} \mathcal{R} \mathcal{A} \Gamma
\]

can be written as a sequence of implications. The exponential of a well-chosen delocation is then represented as:

\[
!(\phi_1(A_1) \to (\phi_2(A_2) \to \ldots (\phi_n(A_n) \to \phi_{n+1}(A_{n+1})) \ldots))
\]

Applying $n$ instances of the project implementing the functorial promotion to the interpretation of $\pi$, we obtain a project $p$ in:

\[
!(\phi_1(A_1)) \to !(\phi_2(A_2)) \to \ldots !(\phi_n(A_n)) \to !(\phi_{n+1}(A_{n+1}))
\]

which is the same conduct as the one interpreting the conclusion of the promotion "rule" in the "derivation of pre-sequents" we have shown. Now we are left with taking the tensor product of the interpretation of $\pi_2$ with the project $p$ to obtain the interpretation of the $!$ rule;

• if $\pi$ is obtained from $\pi$ using a contraction rule $ctr$, we write the conduct interpreting the premise of the rule as $(!A \otimes !A) \to D$. We then define a delocation of the latter in order to obtain $(\phi(!A) \otimes \psi(!A)) \to D$, and take its execution with $ctr$ in $!A \to (!A \otimes !A)$;
if \( \pi \) is obtained from \( \pi_1 \) and \( \pi_2 \) by applying a cut rule or a cut\(^{pol} \) rule, we define \( I_\varphi(\pi) = I_\varphi(\pi_1) \subseteq I_\varphi(\pi_2) \).

**Theorem 134** (locELL\( \text{comp soundness} \)). *Let \( \Phi \) be an interpretation basis. Let \( \pi \) be a derivation in locELL\( \text{comp} \) of conclusion \( \Delta \vdash \Gamma \). Then \( I_\varphi(\pi) \) is a successful project in \( I_\varphi(\Delta \vdash \Gamma) \).*

**Proof.** The proof is a simple consequence of of the proposition and theorems proved before hand. Indeed, the case of the rules of multiplicative additive linear logic was already treated in our previous papers [Sei12b, Sei12a]. The only rules we are left with are the rules dealing with exponential connectives and the rules about the multiplicative units. But the implementation of the functorial promotion (Proposition 118) uses a successful project do not put any restriction on the type of conducts we are working with, and the contraction project (Definition 45 and Proposition 122) is successful. Concerning the multiplicative units, the rules that introduce them do not change the interpretations.

As it was remarked in our previous papers, one can chose an enumeration of the occurrences of variables in order to “localize” any formula \( A \) and any proof \( \pi \) of ELL\( \text{comp} \); we then obtain formulas \( A^e \) and proofs \( \pi^e \) of locELL\( \text{comp} \). The following theorem is therefore a direct consequence of the preceding one.

**Theorem 135** (Full ELL\( \text{comp Soundness} \)). *Let \( \Phi \) be an interpretation basis, \( \pi \) an ELL\( \text{comp} \) proof of conclusion \( \Delta \vdash \Gamma \); and \( e \) an enumeration of the occurrences of variables in the axioms in \( \pi \). Then \( I_\varphi(\pi^e) \) is a successful project in \( I_\varphi(\Delta^e \vdash \Gamma^e) \).*

### 8. Contraction and Soundness for Polarized Conducts

#### 8.1. Definitions and Properties

In this section, we consider a variation on the definition of additive connectives, which is constructed from the definition of the formal sum \( a + b \) of projects. Let us first try to explain the difference between the usual additives \& and \( \oplus \) considered until now and the new additives \( \tilde{\&} \) and \( \tilde{\oplus} \) defined in this section. The conduct \( A \& B \) contains all the test that are necessary for the set \( \{ a' \otimes o \mid a' \in A \} \cup \{ b' \otimes o \mid b' \in B \} \) to generate the conduct \( A \oplus B \), something for which the set \( a + b \) is not sufficient. For the variant of additives considered in this section, it is the contrary that happens: the conduct \( A \tilde{\&} B \) is generated by the projects of the form \( a + b \), but it is therefore necessary to add to the conduct \( A \tilde{\oplus} B \) all the needed tests.

**Definition 136.** Let \( A, B \) be conducts of disjoint carriers. We define \( A \& B = (A + B)^\perp \). Dually, we define \( A \tilde{\oplus} B = (A \perp \& B \perp)^\perp \).

These connectives will be useful for showing that the inclusion \( !A \& B \subset !A \tilde{\oplus} B \) holds when \( A, B \) are behaviors. We will first dwell on some properties of these connectives before showing this inclusion. Notice that if one of the two conducts \( A, B \) is empty, then \( A \& B \) is empty. Therefore, the behavior \( \emptyset \tilde{\oplus} \) is a kind
of absorbing element for \&. But the latter connective also has a neutral element: 1! Notice that the fact that \& and \otimes share the same unit appeared in Girard’s construction\(^\text{12}\) of geometry of interaction in the hyperfinite factor [Gir11].

Notice that at the level of denotational semantics, this connective is almost the same as the usual \& (apart from units). The difference between them is lost during the quotient operation.

**Proposition 137.** Distributivity for \& and \oplus is satisfied for behaviors.

**Proof.** Using the same project than in the proof of Proposition 24, the proof consists in a simple computation. 

**Proposition 138.** Let \(A, B\) be behaviors. Then \(\{a \otimes \circ \mid a \in A\} \cup \{b \otimes \circ \mid b \in B\} \subseteq A \oplus B\).

**Proof.** We will show only one of the inclusions, the other one can be obtained by symmetry. Chose \(f + g \in A \perp + B \perp\) and \(a \in A\). Then:

\[
\langle f + g, a \otimes \circ \rangle_m = \langle f, a \otimes \circ \rangle_m + \langle g, a \otimes \circ \rangle_m = \langle f, a \rangle_m
\]

Using the fact that \(g\) and \(a\) have null wagers. 

**Proposition 139.** Let \(A, B\) be proper behaviors. Then every element in \(A \oplus B\) is observationally equivalent to an element in \(\{a \otimes \circ \mid a \in A\} \cup \{b \otimes \circ \mid b \in B\} \subseteq A \oplus B\).

**Proof.** Let \(c \in A \oplus B\). Since \((A \perp + B \perp) \perp = A \oplus B\), we know that \(c \perp a + b\) for all \(a + b \in A^+ + B^+\). By the homothety lemma (Lemma 25), we obtain, for all \(\lambda, \mu\) non-zero real numbers 0:

\[
\langle c, \lambda a + \mu b \rangle_m = \lambda \langle c, a \rangle_m + \mu \langle c, b \rangle_m \neq 0, \infty
\]

We deduce that one expression among \(\langle c, a \rangle_m\) and \(\langle c, b \rangle_m\) is equal to 0. Suppose, without loss of generality, that it is \(\langle c, a \rangle_m\). Then \(\langle c, a' \rangle_m = 0\) for all \(a' \in A \perp\). Thus \(\langle b, c \rangle_m \neq 0, \infty\) for all \(b \in B \perp\). But \(\langle b \otimes \circ, c \rangle_m = \langle b, c \circ \rangle_m\). We finally have that \(\circ \in B \perp\) and \(\circ \in A \oplus B\).

**Proposition 140.** Let \(A, B\) be proper behaviors. Then \(A \& B\) is a proper behavior.

**Proof.** By definition, \(A \& B = (A + B)^\perp\). But \(A, B\) are non empty contain only one sliced wager-free projects. Thus \(A + B\) is non empty and contains only one sliced wager-free projects. Thus \((A + B)^\perp\) satisfies the inflation property. Moreover, if \(a + b \in A + B\), we have that \(a + b + \lambda c = (a + \lambda c) + b\). Since \(A\) has the inflation property, \(A + B\) has the inflation property. Thus \((A + B)^\perp\) contains only wager-free projects. Moreover, \((A + B)^\perp = A^\perp \oplus B^\perp\) and it is therefore non-empty by the preceding proposition (because \(A^\perp, B^\perp\) are non empty). Then \((A + B)^\perp\) is a proper behavior, which allows us to conclude.

\(^{12}\)Our construction differs slightly from Girard, which is why our additives don’t share the same unit as the multiplicatives.
**Proposition 141.** Let $A, B$ be behaviors. Then $!(A \& B) \subset !A \otimes !B$.

**Proof.** If one of the behaviors among $A, B$ is empty, $!(A \& B) = 0 = !A \otimes !B$. We will now suppose that $A, B$ are both non-empty.

Chose $\bar{f} = (0, F)$ a one-sliced wager-free project. We have that $\bar{f}' = n_F/(n_F + n_G)\bar{f} \in A$ if and only if $\bar{f} \in A$ from the homothety lemma (Lemma 25). Moreover, since $A$ is a behavior, $\bar{f}' \in A$ is equivalent to $\bar{f}'' = \bar{f}' + \sum_{i<n_G} (1/(n_F + n_G))o_i \in A$.

Since the weighted thick and sliced graphing $\frac{n_F}{n_F + n_G} F + \sum_{i=1}^{n_G} \frac{1}{n_F + n_G} \emptyset$ is universally equivalent to a one-sliced weighted thick and sliced graphing $F'$, we obtain finally that the project $(0, F')$ is an element of $A$ if and only if $\bar{f} \in A$. We define in a similar way, being given a project $g$, a weighted graphing with a single slice $G'$ such that $(0, G') \in B$ if and only if $g \in B$.

We are now left to show that $!((0, F') \otimes (0, G')) = !(\bar{f} + g)$. By definition, the graphing of $!((0, F') \otimes (0, G'))$ is equal to $!\iota F' \cup !\iota G'$. By definition again, the graphing of $!(\bar{f} + g)$ is equal to $!\iota (F \cup G) = !\iota F^{11} \cup !\iota G^{12}$, where $t_1$ (resp. $t_2$) denotes the injection of $\iota F$ (resp. $\iota G$) into $\iota F' \cup \iota G'$. We are now left to notice that $!\iota F^{11} = !\iota F'$ since $F^{11}$ and $F'$ are variants one of the other. Similarly, $!\iota G^{12} = !\iota G'$. Finally, we have that $!A \otimes !B = !\iota A \otimes !\iota B$ which is enough to conclude.

**Lemma 142.** Let $A$ be a conduct, and $\phi, \psi$ disjoint delocations. There exists a successful project in the conduct

$$A \rightrightarrows \phi(A) \& \psi(A)$$

**Proof.** We define $c = \exists a \phi \otimes \psi(V \downarrow) + \exists a \phi \otimes \psi(V \uparrow)$. Then for all $a \in A$:

$$c :: a = \phi(a) \otimes \psi(V \downarrow) + \psi(a) \otimes \phi(V \uparrow)$$

Thus $c \in A \rightrightarrows \phi(A) \& \psi(A)$. Moreover, $c$ is obviously successful. \qed

**Proposition 143.** Let $A$ be a behavior, and $\phi, \psi$ be disjoint delocations. There exists a successful project in the conduct

$$?\phi(A) \mp \psi(A) \rightrightarrows ?A$$

**Proof.** If $\bar{f} \in ?\phi(A) \mp \psi(A)$, then we have $f \in ?(\phi(A) \mp \psi(A))$ by Proposition 141. Moreover, we have a successful project $c$ in $A \Downarrow \rightrightarrows \phi(A) \mp \psi(A)$ using the preceding lemma. Using the successful project implementing functorial promotion we obtain a successful project $c' \in !A \Downarrow \rightrightarrows !(\phi(A) \mp \psi(A))$. Thus $c'$ is a successful project in $?\phi(A) \mp \psi(A) \rightrightarrows ?A$. Finally, we obtain, by composition, that $\bar{f} :: c'$ is a successful project in $?A$. \qed

**Corollary 143.1.** Let $A, B$ be behaviors, and $\phi, \psi$ be respective delocations of $A$ and $B$. There exists a successful project in the conduct

$$!(A \& B) \rightrightarrows !(\phi(A) \& \psi(B))$$

---

13The implication $a \in A \Rightarrow a + \lambda o \in A$ comes from the definition of behaviors, its reciprocal is shown by noticing that $a + \lambda o - \lambda o$ is equivalent to $a$. 75
Proof. It is obtained as the interpretation of the following derivation (well formed in the sequent calculus we define later on):

\[
\begin{array}{l}
\frac{\vdash A, A^-; \text{ax}}{\vdash A \oplus B^-, A} \quad \frac{\vdash B, B^-; \text{ax}}{\vdash A \oplus B^-, B} \\
\frac{\vdash (A \& B) \vdash !A}{\vdash (A \& B) \vdash !B} \\
\frac{\vdash (A \& B), !((A \& B)) \vdash !A \otimes !B}{\vdash (A \& B) \vdash !A \otimes !B} \quad \text{ctr}
\end{array}
\]

The fact that it is successful is a consequence of the soundness theorem (Theorem 160).

8.2. Polarized conducts

8.2.1. Definitions

The notions of perennial and co-perennial conducts are not completely satisfactory. In particular, we are not able to show that an implication \(A \rightarrow B\) is either perennial or co-perennial when \(A\) is a perennial conduct (resp. co-perennial) and \(B\) is a co-perennial conduct (resp. perennial). This is an important issue when one considers the sequent calculus: the promotion rule has to be associated with a rule involving behaviors in order to in the setting of behaviors (using Proposition 102). Indeed, a sequent \(\vdash ?\Gamma, !A\) would be interpreted by a conduct which is neither perennial nor co-perennial in general. The sequents considered are for this reason restricted to pre-sequent containing behaviors.

We will define now the notions of negative and positive conducts. The idea is to relax the notion of perennial conduct in order to obtain a notion negative conduct. The main interest of this approach is that positive/negative conducts will share the important properties of perennial/co-perennial conducts while interacting in a better way with connectives. In particular, we will be able to interpret the usual functorial promotion (not associated to a \(\otimes\) rule), and we will be able to use the contraction rule without all the restrictions we had in the previous section.

Definition 144 (Polarized Conducts). A positive conduct \(P\) is a conduct satisfying the inflation property and containing all daemons:

- \(p \in P \Rightarrow p + \lambda 0 \in P\);
- \(\forall \lambda \in \mathbb{R} - \{0\}, \forall n \lambda = (\lambda, (V^P, \emptyset)) \in P\).

A conduct \(N\) is negative when its orthogonal \(N^-\) is a positive conduct.

Proposition 145. A perennial conduct is negative. A co-perennial conduct is positive.

Proof. We already showed that the perennial conducts satisfy the inflation property (Proposition 96) and contain daemons (Proposition 98).

Proposition 146. A conduct \(A\) is negative if and only if:
• \( A \) contains only wager-free projects;

• \( a \in A \Rightarrow 1_A \neq 0. \)

**Proof.** If \( A^\perp \) is a positive conduct, then it is non-empty and satisfies the inflation property, thus \( A \) contains only wager-free projects by Proposition 26. As a consequence, if \( a \in A \), we have that \( \langle a, \Delta a \rangle_m = \lambda 1_A \) thus the condition \( \langle a, \Delta a \rangle_m \neq 0 \) implies that \( 1_A \neq 0. \)

Conversely, if \( A \) satisfies that stated properties, we distinguish two cases. If \( A \) is empty, then it is clear that \( A^\perp \) is a positive conduct. Otherwise, \( A \) is a non-empty conduct containing only wager-free projects, thus \( A^\perp \) satisfies the inflation property (Proposition 27). Moreover, \( \langle a, \Delta a \rangle_m = 1_A \neq 0 \) as a consequence of the second condition and therefore \( \Delta a \in A^\perp \). Finally, \( A^\perp \) is a positive conduct, which implies that \( A \) is a negative conduct. \( \square \)

The polarized conducts do not interact very well with the connectives \& and \( \circ \). Indeed, if \( A, B \) are negative conducts, the conduct \( A \& B \) is generated by a set of wager-free projects, but it does not satisfy the second property needed to be a negative conduct. Similarly, if \( A, B \) are positive conducts, then \( A \& B \) will obviously have the inflation property, but it will contain the project \( \Delta a \) (which implies that any element \( c \) in its orthogonal is such that \( 1_c = 0. \)) We are also not able to characterize in any way the conduct \( A \& B \) when \( A \) is a positive conduct and \( B \) is a negative conduct, except that it is has the inflation property. However, the notions of positive and negative conducts interacts in a nice way with the connectives \( \circ, \& \). Indeed, if \( A, B \) are negative conducts, the conduct \( A \& B \) is generated by a set of wager-free projects, but it does not satisfy the second property needed to be a negative conduct. Similarly, if \( A, B \) are positive conducts, then \( A \& B \) will obviously have the inflation property, but it will contain the project \( \Delta a \) (which implies that any element \( c \) in its orthogonal is such that \( 1_c = 0. \)) We are also not able to characterize in any way the conduct \( A \& B \) when \( A \) is a positive conduct and \( B \) is a negative conduct, except that it is has the inflation property. However, the notions of positive and negative conducts interacts in a nice way with the connectives \( \circ, \& \).

### 8.2.2. Polarized Conducts and Connectives

**Proposition 147.** The tensor product of negative conducts is a negative conduct. The \& of negative conducts is a negative conduct.

**Proof.** We know that \( A \circ B = \varnothing \) if one of the two conducts \( A \) and \( B \) is empty, which leaves us to treat the non-empty case. In this case, \( A \circ B = (A \& B)^\perp \) is the bi-orthogonal of a non-empty set of wager-free projects. Thus \( (A \circ B)^\perp \) satisfies the inflation property. Moreover \( \langle a \circ b, \Delta a \rangle_m = 1_B 1_A \lambda \) which is different from zero since \( 1_A, 1_B \) both are different from zero. Thus \( \Delta a \in (A \circ B)^\perp \), which shows that \( A \circ B \) is a negative conduct since \( (A \circ B)^\perp \) is a positive conduct.

The set \( A^\perp \mid_B \) content les démons car \( \Delta a \circ o = \Delta a_1 \), and \( \Delta a \in A^\perp \). It has the inflation property since \( (b + \lambda o) \circ o = b \circ o + \lambda o \). Thus \( (A^\perp \mid_B)^\perp \) is a negative conduct. Similarly, \( (B^\perp \mid_B)^\perp \) is a negative conduct, and their intersection is a negative conduct since the properties defining negative conducts are are preserved by intersection. As a consequence, \( A \& B \) is a negative conduct.

In the case of \( \circ \), we will use the fact that \( A \oplus B = (A \mid_B \cup B \mid_A)^\perp \). If \( a \in A \), \( a \circ o = b \) has a null wager and \( 1_B = 1_A \neq 0. \) If \( A \) is empty, \( (A \mid_B)^\perp \) is a positive conduct. If \( A \) is non-empty, then Proposition 27 allows us to state that \( (A \mid_B)^\perp \) has the inflation property. Moreover, the fact that all elements in \( a \circ o = b \) satisfy \( 1_B \neq 0 \) implies that \( \Delta a \in (A \mid_B)^\perp \) for all \( \lambda \neq 0. \) Therefore, \( (A \mid_B)^\perp \) is a positive conduct.
As a consequence, \( A \uparrow B \) is a negative conduct. We show in a similar way that \( B \uparrow A \) is a negative conduct. We can deduce from this that \( A \uparrow B \cup B \uparrow A \) contains only projects \( c \) with zero wager and such that \( 1_c \neq 0 \). Finally, we showed that \( A \oplus B \) is a negative conduct.

**Corollary 147.1.** The \( \mathcal{R} \) of positive conducts is a positive conduct, the \& of positive conducts is a positive conduct, and the \( \oplus \) of positive conducts is a positive conduct.

**Proposition 148.** Let \( A \) be a positive conduct and \( B \) be a negative conduct. Then \( A \otimes B \) is a positive conduct.

*Proof.* Pick \( \varepsilon \in (A \otimes B)^\perp = A \rightarrow A^\perp \). Then for all \( b \in B \), \( \varepsilon : b = (1_B f + 1_F b, F :: B) \) is an element of \( A^\perp \). Since \( A^\perp \) is a negative conduct, we have that \( 1_F 1_B \neq 0 \) and \( 1_B f + 1_F b = 0 \). Thus \( 1_F \neq 0 \). Moreover, \( B \) is a negative conduct, therefore \( 1_B \neq 0 \) and \( b = 0 \). The condition \( 1_B f + 1_F b = 0 \) then becomes \( 1_B f = 0 \), i.e. \( f = 0 \).

Thus \( (A \otimes B)^\perp \) is a negative conduct, which implies that \( A \otimes B \) is a positive conduct. \( \square \)

**Corollary 148.1.** If \( A \) is a positive conduct and \( B \) is a positive conduct, \( A \rightarrow B = (A \otimes B^\perp)^\perp \) is a positive conduct.

**Corollary 148.2.** If \( A, B \) are negative conducts, then \( A \rightarrow B \) is a negative conduct.

*Proof.* We know that \( A \rightarrow B = (A \otimes B^\perp)^\perp \). We also just showed that \( A \otimes B^\perp \) is a positive conduct, thus \( A \rightarrow B \) is a negative conduct. \( \square \)

**Proposition 149.** The tensor product of a negative conduct and a behavior is a behavior.

*Proof.* Let \( A \) be a negative conduct and \( B \) be a behavior. If either \( A \) or \( B \) is empty (or both), \( (A \otimes B)^\perp \) equals \( T_{V_A \cup V_B} \) and we are done. We now suppose that \( A \) and \( B \) are both non empty.

Since \( A, B \) contain only wager-free projects, the set \( \{ a \otimes b \mid a \in A, b \in B \} \) contains only wager-free projects. Thus \( (A \otimes B)^\perp \) has the inflation property: this is a consequence of Proposition 27. Suppose now that there exists \( \varepsilon \in (A \otimes B)^\perp \) such that \( f \neq 0 \). Soit \( a \in A \) and \( b \in B \). Then \( \ll \varepsilon, a \otimes b > \gg_m = f 1_A 1_B + [F, A :: B]_m \).

Since \( 1_A \neq 0 \), we can define \( \mu = -[F, A \cup B]/(1_A f) \), and \( b + \mu a \in B \) since \( B \) has the inflation property. We then have:

\[
\ll \varepsilon, a \otimes (b + \mu a) > \gg_m = f 1_A (1_B + \mu) + [F, A :: (B + \mu a)]_m
\]

\[
= f 1_A \frac{-[F, A \cup B]_m}{1_A f} + [F, A :: B]_m
\]

\[
= 0
\]

This is a contradiction, since \( \varepsilon \in (A \otimes B)^\perp \). Thus \( f = 0 \).

Finally, we have shown that \( (A \otimes B)^\perp \) has the inflation property and contains only wager-free projects. \( \square \)

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Corollary 149.1. If A is a negative conduct and B is a behavior, A \rightarrow B is a behavior.

Proposition 150. The weakening (on the left) of negative conducts holds.

Proof. Let A, B be conducts, N be a negative conduct, and pick f ∈ A \rightarrow B. We will show that \langle f \otimes \circ_N \rangle is an element of A \otimes N \rightarrow B. For this, we pick a ∈ A and n ∈ N. Then for all b’ ∈ B^\perp,

\langle f \otimes \circ_N \rangle \colon (a \otimes n), b' \rangle = 
\langle f \otimes \circ_N \rangle \colon (a \otimes n) \otimes b' \rangle = 
\langle f \otimes \circ_N \rangle \colon (a \otimes b') \otimes n \rangle = 
1_F(1_A b' + 1_N 1_B a) + 1_N 1_B f + \llbracket F \cup 0, A \cup B' \cup N \rrbracket
= 1_F(1_N 1_B b' + 1_N 1_B a) + 1_N 1_B f + \llbracket F \cup 0, A \cup B' \cup N \rrbracket
= 1_N(1_F(1_B b' + 1_B a) + 1_B f) + 1_N F, A \cup B' \rangle
= 1_N \langle f, a \otimes b' \rangle

Since 1_N \neq 0, \langle f \otimes \circ_N \rangle \colon (a \otimes n), b' \rangle \neq 0, \infty if and only if \langle f, a \otimes b' \rangle \neq 0, \infty. Therefore, for all a \otimes n ∈ A \otimes N, \langle f \otimes \circ_N \rangle \colon (a \otimes n) ∈ B. This shows that f \otimes \circ_N is an element of A \otimes N \rightarrow B by Proposition 47.

8.2.3. Sequent Calculus

We now describe a sequent calculus which is much closer to the usual sequent calculus for Elementary Linear Logic. We introduce once again three types of formulas — (B)ehaviors, (P)ositive, (N)egative. The sequents we will be working with will be the equivalent to the notion of pre-sequent introduced earlier.

Definition 151. We once again define three types of formulas — (B)ehavior, (P)ositive, (N)egative — by the following grammar:

\begin{align*}
B & := X | X^\bot | 0 | T | B \otimes B | B \boxdot B | B \boxprod B | B \\& B | B \otimes N | B \boxprod P \\
N & := 1 | !B | \downarrow N | N \otimes N | N & N | N \boxdot N | N \boxprod P \\
P & := \downarrow | ?B | \uparrow P | P \boxdot P | P \& P | P \boxprod P | N \otimes P
\end{align*}

Definition 152. A sequent A \Rightarrow \vdash \Gamma; \Theta is such that A, \Theta contain only negative formulas, \Theta containing at most one formula and \Gamma containing only behaviors.
Figure 27: Sequent Calculus ELLpol
Definition 153 (The System ELLpol). A proof in the system ELLpol is a derivation tree constructed from the derivation rules shown in Figure 27.

Remark. Even though one can consider the conduct $A \& B$ when $A, B$ are negative conducts, no rule of the sequent calculus ELLpol allows one to construct such a formula. The reason for that is simple: since in this case the set $A + B$ is not necessarily included in the conduct $A \& B$, one cannot interpret the rule in general (since distributivity does not necessarily holds). The latter can be interpreted when the context contains at least one behavior, but imposing such a condition on the rule could lead to difficulties when considering the cut-elimination procedure (in case of commutations). We therefore whose to work with a system in which one introduces additive connectives only between behaviors. Notice however that a formula built with an additive connective between negative sub-formulas can still be introduced by a weakening rule.

Proposition 154. The system ELLpol possesses a cut-elimination procedure.

Proof. We will not do the straightforward proof of this proposition. □

Definition 155. We fix $\mathcal{V} = \{X_i(j)\}_{i,j \in \mathbb{N} \times \mathbb{Z}}$ a set of localized variables. For $i \in \mathbb{N}$, the set $X_i = \{X_i(j)\}_{j \in \mathbb{Z}}$ will be referred to as the name of the variable $X_i$, and an element of $X_i$ will be referred to as a variable of name $X_i$.

For $i, j \in \mathbb{N} \times \mathbb{Z}$ we define the location $\sharp X_i(j)$ of the variable $X_i(j)$ as the set
\[
\{ x \in \mathbb{R} | 2^i(2j + 1) \leq m < 2^i(2j + 1) + 1 \}
\]

Definition 156 (Formulas of locELLpol). We inductively define the formulas of locELLpol together with their locations as follows:

- Behaviors:
  - A variable $X_i(j)$ of name $X_i$ is a behavior whose location is defined as $\sharp X_i(j)$;
  - If $X_i(j)$ is a variable of name $X_i$, then $(X_i(j))^+$ is a behavior of location $\sharp X_i(j)$.
  - The constants $T_{\sharp \Gamma}$ are behaviors of location $\sharp \Gamma$;
  - The constants $0_{\sharp \Gamma}$ are behaviors of location $\sharp \Gamma$.
  - If $A, B$ are behaviors of respective locations $X, Y$ such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \mathcal{V} B$, resp. $A \& B$, resp. $A \oplus B$) is a behavior of location $X \cup Y$;
  - If $A$ is a negative conduct of location $X$ and $B$ is a behavior of location $Y$ such that $X \cap Y = \emptyset$, then $A \otimes B$ is a behavior of location $X \cup Y$;
  - If $A$ is a positive conduct of location $X$ and $B$ is a behavior of location $Y$ such that $X \cap Y = \emptyset$, then $A \mathcal{V} B$ is a behavior of location $X \cup Y$;

- Negative Conducts:
- The constant 1 is a negative conduct;
- If A is a behavior or a negative conduct of location X, then !A is a negative conduct of location \( \Omega(X \times [0,1]) \);
- If A, B are negative conducts of locations X, Y such that X \( \cap \) Y = \( \emptyset \), then A \( \otimes \) B (resp. A \( \oplus \) B, resp. A \& B) is a negative conduct of location X \( \cup \) Y;
- If A is a negative conduct of location X and B is a positive conduct of location Y, A \( \nabla \) B is a negative conduct of location X \( \cup \) Y.

• Positive Conducts:
- The constant \( \bot \) is a positive conduct;
- If A is a behavior or a positive conduct of location X, then ?A is a positive conduct of location \( \Omega(X \times [0,1]) \);
- If A, B are positive conducts of locations X, Y such that X \( \cap \) Y = \( \emptyset \), then A \( \& \) B (resp. A \& B, resp. A \( \oplus \) B) is a positive conduct of location X \( \cup \) Y;
- If A is a negative conduct of location X and B is a positive conduct of location Y, A \( \otimes \) B is a positive conduct of location X \( \cup \) Y.

If A is a formula, we will denote by \( \sharp \) A its location. We also define sequents \( \Delta \vdash \Gamma \Theta \) of locELL\(_{pol} \) when:
- formulas in \( \Gamma \cup \Delta \cup \Theta \) have pairwise disjoint locations;
- formulas in \( \Delta \) and \( \Theta \) are negative conducts;
- there is at most one formula in \( \Theta \);
- \( \Gamma \) contains only behaviors.

**Definition 157** (Interpretations). We define an interpretation basis as a function \( \Phi \) which maps every variable name \( X_i \) to a behavior of carrier \( [0,1] \).

**Definition 158** (Interpretation of locELL\(_{pol} \) formulas). Let \( \Phi \) be an interpretation basis. We define the interpretation \( I_\Phi(F) \) along \( \Phi \) of a formula \( F \) inductively:
- If \( F = X_i(j) \), then \( I_\Phi(F) \) is the delocation (i.e. a behavior) of \( \Phi(X_i) \) along the function \( x \mapsto 2^{i(2j + 1)}x \);
- If \( F = (X_i(j))^\perp \), we define the behavior \( I_\Phi(F) = (I_\Phi(X_i(j)))^\perp \);
- If \( F = T_\sharp \) (resp. \( F = 0_\sharp \)), we define \( I_\Phi(F) \) as the behavior \( T_\sharp \) (resp. \( 0_\sharp \));
- If \( F = 1 \) (resp. \( F = \bot \)), we define \( I_\Phi(F) \) as the behavior \( 1 \) (resp. \( \bot \));
- If \( F = A \oplus B \), we define the conduct \( I_\Phi(F) = I_\Phi(A) \oplus I_\Phi(B) \);
- If \( F = A \nabla B \), we define the conduct \( I_\Phi(F) = I_\Phi(A) \nabla I_\Phi(B) \);
Moreover a sequent $\Delta \vdash \Theta$ will also represent this formula by the equivalent formula $\otimes$ basis. We define the interpretation $I$ then implies the following result.

**Definition 159** (Interpretation of $\text{locELL}_{\text{pol}}$ proofs). Let $\Phi$ be an interpretation basis. We define the interpretation $I_\Phi(\pi)$ — a project — of a proof $\pi$ inductively:

- if $\pi$ consists in an axiom rule introducing $\vdash (X, (j))$, we define $I_\Phi(\pi)$ as the project $\forall \exists$ defined by the translation $x \leadsto 2^{(2j' - 2j)} + x$;
- if $\pi$ consists solely in a $\exists$ rule, we define $I_\Phi(\pi) = o_\exists$;
- if $\pi$ consists solely in a $\forall$ rule, we define $I_\Phi(\pi) = o_\forall$;
- if $\pi$ is obtained from $\pi'$ by a $\forall$ rule, a $\otimes_{sl}$ rule, a $\otimes_{d}$ rule, a $\exists^\text{mix}$ rule, or a $\forall_1$ rule, then $I_\Phi(\pi) = I_\Phi(\pi')$;
- if $\pi$ is obtained from $\pi_1$ and $\pi_2$ by applying a $\forall$ rule, a $\otimes_{sl}$ rule, a $\otimes_{d}$ rule, or a $\exists^\text{mix}$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \otimes I_\Phi(\pi_2)$;
- if $\pi$ is obtained from $\pi'$ by a $\exists$ rule or a $\exists_1$ rule introducing a formula of location $V$, we define $I_\Phi(\pi) = I_\Phi(\pi') \oplus V$;
- if $\pi$ of conclusion $\vdash \Gamma, A_0 \& A_1$ is obtained from $\pi_0$ and $\pi_1$ by applying a $\&$ rule, we define the interpretation of $\pi$ as it was done in our earlier paper [Sei12a];
- if $\pi$ is obtained from $\pi'$ by applying a promotion rule $!$ or $!_{\text{pol}}$, we apply the implementation of the functorial promotion rule to the project $I_\Phi(\pi')$ $n - 1$ times, where $n$ is the number of formulas in the sequent;
- if $\pi$ is obtained from $\pi$ by applying a contraction rule $\text{ctr}$, we define the interpretation of $\pi$ as the execution between the interpretation of $\pi'$ and the project implementing contraction described in Proposition[122];
- if $\pi$ is obtained from $\pi_1$ and $\pi_2$ by applying a $\exists$ rule or a $\exists_{\text{pol}}$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \cap I_\Phi(\pi_2)$.

We then obtain a soundness result for the localized calculus $\text{locELL}_{\text{pol}}$ which then implies the following result.

**Theorem 160.** Let $\Phi$ be an interpretation basis, $\pi$ a proof of $\text{ELL}_{\text{pol}}$ of conclusion $\Delta \vdash \Gamma; \Theta$, and $e$ an enumeration of the occurrences of variables in the axioms of $\pi$. Then $I_\Phi(\pi^e)$ is a successful project in $I_\Phi(\Delta^e \vdash \Gamma^e; \Theta^e)$.
9. Conclusion and Perspectives

In this paper, we extended the setting of Interaction Graphs in order to deal with all connectives of linear logic. We showed how one can obtain a soundness result for two versions of Elementary Linear Logic. The first system, which is conceived so that the interpretation of sequents are behaviors, seems to lack expressivity and it may appear that elementary functions cannot be typed in this system. The second system, however, is very close to usual ELL sequent calculus, and, even though one should prove it, the proofs from the type of natural numbers \( \text{nat} \) to itself seem to correspond to elementary functions from natural numbers to natural numbers, as it is the case with traditional Elementary Linear Logic \cite{DJ03}.

Though the generalization from graphs to graphings may seem a big effort, we believe the resulting framework to be extremely interesting. We should stress that with little work, one should be able to show that interpretations of proofs can be described by finite means. Indeed, the only operation that may turn an interval into an infinite number of intervals is the promotion rule, and one should be able to show that, up to a suitable delocation, the promotion of a project defined on a finite number of rational intervals is defined on a finite number of rational intervals.

This finite description of projects would however be lost if one were to consider continuous dialects in addition to discrete ones. All the definitions and properties of thick and sliced graphings obviously hold in this setting and one can obtain all the results described in this paper. The question of whether we would gain some expressivity by extending the framework in this way is still open. We believe that it may be a way to obtain more expressive exponentials — such as the usual exponentials of linear logic.

More generally, now that this framework has been defined and that we have shown its interest by providing a construction for elementary exponentials, we believe the definition and study of other exponential connectives may be a work of great interest. First, these new exponentials would co-exist with each other, making it possible to study their interactions. Secondly, even if the definition of exponentials for full linear logic may be a complicated task, the definition of low-complexity exponentials may be of great interest.

The last path for expansion of this work is about second order quantification. As we already explained, we believe that it is possible to extend the soundness result to second order quantification by considering a generalized notion of graphing in which edges can be chosen to be transports of measure instead of measure-preserving maps. The main issue about localized second order quantification is that at some point one needs to instantiate the \( \forall \) quantification, and this may turn out to be impossible if the carrier of the quantifier does not match the carrier of the formula one wants to instantiate it with. To deal with this issue, we propose to use transports of measure, that will resize the proofs in order to solve this issue. There is a bit of work concerning the adaptation of the results of this paper (especially the existence of circuit-quantifying maps) to this generalized notion of graphing, but we believe this can be done.
References


