Estimating Crossing Fibers: A Tensor Decomposition Approach

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Abstract—Diffusion weighted magnetic resonance imaging is a unique tool for non-invasive investigation of major nerve fiber tracts. Since the popular diffusion tensor (DT-MRI) model is limited to voxels with a single fiber direction, a number of high angular resolution techniques have been proposed to provide information about more diverse fiber distributions. Two such approaches are Q-Ball imaging and spherical deconvolution, which produce orientation distribution functions (ODFs) on the sphere. For analysis and visualization, the maxima of these functions have been used as principal directions, even though the results are known to be biased in case of crossing fiber tracts. In this paper, we present a more reliable technique for extracting discrete orientations from continuous ODFs, which is based on decomposing their higher-order tensor representation into an isotropic component, several rank-1 terms, and a small residual. Comparing to ground truth in synthetic data shows that the novel method reduces bias and reliably reconstructs crossing fibers which are not resolved as individual maxima in the ODF. We present results on both Q-Ball and spherical deconvolution data and demonstrate that the estimated directions allow for plausible fiber tracking in a real data set.

Index Terms—DW-MRI, Q-Ball, spherical deconvolution, fiber tracking, higher-order tensor, tensor decomposition.

1 INTRODUCTION

Diffusion-weighted magnetic resonance imaging (DW-MRI) [28] is a common tool for in vivo investigation of the human central nervous system. In particular, taking several diffusion-weighted images in different directions allows for conclusions about the orientation of coherently organized nerve fibers. Under the assumption of a single coherent tract per voxel, the diffusion tensor (DT-MRI) model [5] has proven sufficient to infer the dominant fiber direction. Unfortunately, in cases of partial voluming as well as for crossing, touching, fanning or bending fiber configurations, the diffusion tensor may become degenerate (i.e., have two larger eigenvalues of similar magnitude, such that the principal direction is ill-defined) or have an apparently well-defined principal direction which is no longer aligned with any real fiber direction (cf. Figures 1 (a) and (b)).

To gain more information about such voxels, techniques with a higher angular resolution have been proposed. Two of them are Q-Ball imaging [37] and spherical deconvolution [36], which use a more flexible orientation distribution function (ODF) to indicate multimodal fiber distributions. When interpreting such data, it is desirable to estimate the principal fibers within a voxel that led to a given ODF. Unfortunately, this is an under-determined inverse problem. As a simple ad-hoc solution, it has become common to treat the maxima of the ODF as approximate principal directions, both for assessing the accuracy of reconstruction schemes [36, 19, 42, 16] and for visualization through glyphs [21, 16], orientation plots [16], color maps [33], and fiber tracking [18, 22, 33].

However, there are many situations in which the number and orientation of ODF maxima deviate from the real tracts: In the case of crossing fibers, one important reason for this is that the fact that each single fiber bundle leads to an ODF peak of finite width. In the joint ODF, these peaks interfere, and the maxima of the resulting distribution no longer coincide with the modes of the original peaks. This fact is demonstrated in Figure 1 (a), which shows a Q-Ball of a synthetic 65° fiber crossing for which inferring principal directions from ODF maxima underestimates the included angle by approximately 10°. Reducing the angle to 55° (Figure 1 (b)) even leads to a single maximum, which is not aligned with any of the fiber directions. In this case, the direction estimated from the Q-Ball profile is not better than the one from traditional DT-MRI. Such effects are known [36] and have been demonstrated in high detail [42].

Extracting individual fibers from an ODF requires assumptions about the shape of single fiber peaks. For example, from the ODF in Figure 1 (b) alone, it is not clear whether the voxel contains two (or more) crossing tracts or only a single one which causes a broad peak, e.g., because of fanning structures. On the other hand, it has been demonstrated that even a multimodal ODF does not necessarily indicate multiple tracts, but may be due to strong bending [37]. In this paper, we explicitly address the case of crossing fibers, which are assumed to each cause a narrow peak in the ODF. Then, estimating fibers amounts to finding a set of such peaks whose sum approximates the observed ODF. We do this by converting the spherical harmonic series which is commonly used to describe ODFs into a higher-order tensor representation and decomposing it into rank-1 tensors, which model the individual peaks. In synthetic test cases, this yields more accurate results than a simple maximum extraction (cf. right-hand side of Figures 1 (a) and (b)). In real data, it provides plausible and reproducible results. For example, Figure 1 (c) presents a Q-Ball from a voxel in which transcallosal fibers are known to intersect the corona radiata [18]. Unlike a maximum search, our method estimates two fibers, whose orientations agree with the expected directions.

The remainder of this paper is organized as follows: Section 2 provides the necessary background on high angular resolution diffusion imaging and higher-order tensors, and Section 3 discusses related work. Section 4 motivates our choice of rank-k approximations for addressing the fiber extraction problem and presents the resulting algorithm. Finally, Section 5 demonstrates the advantages of our approach both on synthetic and real data, before Section 6 concludes the paper.

2 BACKGROUND

This section provides a brief review of the background on high angular resolution diffusion imaging and higher-order tensors which is necessary to understand the remainder of the paper.

2.1 High Angular Resolution Diffusion Imaging

Classical diffusion tensor imaging is based on the Stejskal-Tanner equation [34]. It describes measured signal intensity \( S(\mathbf{g}) \) in the presence of a diffusion sensitizing gradient in direction \( \mathbf{g} \) as a function of unweighted intensity \( S_0 \), the apparent diffusion coefficient (ADC) \( D(\mathbf{g}) \), and an acquisition constant \( b \):

\[
S(\mathbf{g}) = S_0 e^{-bD(\mathbf{g})}
\]  

(1)

Diffusion tensor imaging [5] assumes that \( D(\mathbf{g}) \) is adequately described by a quadratic form \( D(\mathbf{g}) = \mathbf{g}^T D \mathbf{g} \), where the \( 3 \times 3 \) diffusion tensor

\[
D = \begin{bmatrix}
D_{xx} & D_{xy} & D_{xz} \\
D_{yx} & D_{yy} & D_{yz} \\
D_{zx} & D_{zy} & D_{zz}
\end{bmatrix}
\]
tensor $D$ is estimated from at least six measurements in non-collinear directions $g_i$, plus an unweighted measurement of $S_0$. Validation studies (e.g., [29, 13]) have indicated that in voxels with a single predominant fiber orientation, the direction of highest ADC (i.e., the principal eigenvector of $D$), is well-aligned with this fiber direction.

To deal with voxels in which the single fiber assumption does not hold, it has been proposed to model the apparent diffusivities $D(g_i)$ in a more flexible way, using spherical harmonics [17, 2] or, equivalently, higher-order tensors [30]. However, such approaches do not easily reveal fiber directions, because maxima in the ADC profile no longer align with those in the case of fiber crossings [38]. In fact, apparent diffusivity maxima do not even provide a usable approximation of fiber directions: In the case of a 90° crossing, maxima are offset by 45°.

Q-space methods [10] allow more direct conclusions about possible fiber directions. They describe the spin displacement within the experimental diffusion time $δ$ by a displacement vector $r$ and reconstruct the voxel-averaged probability density $P(r)$ of these vectors. Q-space imaging exploits the fact that $P(r)$ is related to the attenuated signal $S(q)$ via the Fourier transform $ℱ$:

$$P(r) = ℱ[S(q)]$$

(2)

$S(q)$ is now given as a function of the diffusion wavevector $q = (2π)^{-1}γδg$, where $γ$ is a nucleus-specific constant, $δ$ and $g$ are diffusion time and gradient direction as above. While a Cartesian sampling of q-space, followed by a discrete Fourier transform has been performed to obtain $P(r)$ (e.g., [41]), this procedure is infeasible for routine investigation. Moreover, for practical analysis, $P$ is integrated in radial direction to obtain an orientation distribution function (ODF) $ψ(u)$ of directions $u$ on the unit sphere ($∥u∥ = 1$), which discards much of the acquired information.

Q-Ball imaging [37] remedies both problems: It uses the Funk-Radon transform, a mapping from the sphere to the sphere, which treats each point as a pole and assigns the integral over the associated equator to it. Q-Ball imaging exploits the fact that the Funk-Radon transform of the measurements $S(g_i)$, which are taken with fixed diffusion time $δ$ (i.e., on a ball in q-space), provides a good approximation to $ψ(u)$. Q-Ball reconstruction is greatly simplified by working in a spherical harmonics basis, which allows for an analytic solution of the Funk-Radon transform [3, 19, 16].

A third way of estimating fiber orientations from high angular resolution measurements is spherical deconvolution [36]: It is based on the assumption that the signal attenuation from a single, well-organized fiber population can be described by an axially symmetric response function $R(θ)$ which is assumed constant over the whole brain and for all types of fibers. Then, the measured signal $S(θ, φ)$ is expressed as the convolution of a fiber orientation density function $F(θ, φ)$ with $R(θ)$, where $θ$ and $φ$ are polar and azimuth angles, respectively:

$$S(θ, φ) = F(θ, φ) * R(θ)$$

(3)

After $S$ has been measured, the response function $R$ is estimated from voxels which are thought to contain a single fiber tract. The fiber distribution $F$ is then obtained by spherical deconvolution.

Both Q-Ball imaging and spherical deconvolution result in functions on the sphere, yet with different semantics: While a Q-Ball orientation distribution functions defines the probability of a spin displacement, an orientation density function from spherical deconvolution yields a fiber fraction. The proposed method is applicable to both types of functions, since they share the problem of separating the contributions of individual peaks in the case of crossing fibers. For simplicity, we use the term “orientation distribution function” (ODF) for both methods throughout this paper.

2.2 Spherical Harmonics and Higher-Order Tensors

Both spherical deconvolution and modern Q-Ball reconstruction techniques use spherical harmonics, which form an orthonormal basis for complex functions on the unit sphere, much like the Fourier series offers an orthonormal basis over an interval in Cartesian space. The spherical harmonic $Y^m_l$ for order $l$ and phase factor $m ≤ l$ is given as

$$Y^m_l(θ, φ) = \sqrt{\frac{2l+1}{4π}\frac{(l-m)!}{l!}} P^m_l(\cos θ) e^{imφ}$$

(4)

where $P^m_l$ is an associated Legendre polynomial and $i$ is the imaginary unit. Since ODFs are real-valued and exhibit antipodal symmetry, we employ the restricted basis used in [16]. For $h = 0, 2, 4, . . . , l$ and $m = −h, . . . , 0, . . . , h$, $j := (h^2 + h + 2)/2 + m$ and

$$Y_j := \begin{cases} \sqrt{2} \cdot \text{Re}(Y^h_l) & \text{if } -h \leq m < 0 \\ Y^h_l & \text{if } m = 0 \\ \sqrt{2} \cdot \text{Im}(Y^h_l) & \text{if } 0 < m \leq h \end{cases}$$

(5)

where Re and Img denote real and imaginary parts, respectively. For a spherical harmonics series up to even order $l$, this results in $(l + 1)(l + 2)/2$ terms.

Higher-order tensors can be used to provide an alternative representation of functions on the sphere. In this work, we treat order-$l$ tensors $D$ in a given orthonormal Cartesian coordinate system as quantities whose elements are addressed by $l$ indices. The employed tensors are supersymmetric, i.e., their elements $D_{i_1 i_2 . . . i_l}$ are invariant under arbitrary permutations of indices $i_1 . . . i_l$. A function $D(g)$ on the unit sphere is given by their induced homogeneous form. For an order-$l$ tensor $D$ in dimension $n = 3$, it reads:

$$D(g) = \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \cdots \sum_{i_l=1}^{3} D_{i_1 i_2 \cdots i_l} g_{i_1} g_{i_2} \cdots g_{i_l}$$

(6)

To ensure antipodal symmetry, we use only even orders $l$. Tensorial representations are equivalent to spherical harmonics in the sense that any function $D$ in the form of Equation (6) can be re-written as a linear combination of spherical harmonics, $D = \sum_j c_j Y_j$, with $Y_j$ from Equation (5), and vice versa [30]. A matrix that relates $D_{i_1 i_2 . . . i_l}$ to $c_j$ is found by writing $g$ in spherical coordinates and solving

$$c_j = \int_0^{2π} \int_0^π D(g(θ, φ)) Y_j(θ, φ) \sin θ dθ dφ$$

(7)

symbolically for each $j$, which is simplified by software like Maple or Mathematica. Working with tensor representations of orientation distribution functions will allow us to relate the problem of decomposing them into individual peaks to recent work on low-rank tensor approximations in multilinear algebra.

In the following section, we will use the standard tensor scalar product $⟨A, B⟩ = \sum_i \sum_{i'} \sum_{i''} A_{i i' i''} B_{i' i''}$ and the corresponding norm $∥A∥ = \sqrt{⟨A, A⟩}$.

3 RELATED WORK

Several previous works [1, 38, 27, 9, 32, 8] have used $k$-tensor models to resolve crossing fiber configurations. Similarly, a recent work used Bayesian inference to estimate the parameters of a mixture of
Bingham distributions [24]. In both cases, the measured signal is modeled directly, using several second-order tensors. Our work differs from this conceptually in that we extract discrete directions from pre-computed Q-Ball or spherical deconvolution models. Moreover, we build on completely different mathematical methods, since we use rank-k higher-order tensors to approximate existing models. In practice, our approach proved to deal reliably even with three-fiber crossings (cf. Sections 5.3 and 5.5), while no k-tensor models have been presented so far which can estimate more than k = 2 crossing tracts in a stable manner.

Our method is based on a higher-order tensor decomposition. In the context of diffusion tensor processing, higher-order tensors have previously been analyzed by Basser and Pajevic [6] using spectral decomposition and by Kindlmann et al. [25] based on invariance gradients and rotation tangents. However, both works are concerned with fourth-order covariance tensors that arise from the DT-MRI model and do not address high angular resolution techniques.

4 Tensor Decomposition for Inferring Fiber Directions
Section 4.1 will introduce the fundamental idea of our approach, and the formal model used to separate individual fiber contributions. Then, Section 4.2 describes how to find rank-1 tensor approximations, which is an important task within our method. Finally, Section 4.3 provides the complete algorithm and a listing in pseudocode.

4.1 Rank-1 Tensors as Fiber Terms and Rank-k Approximations for Estimating Crossings
An order-l rank-1 tensor \( D \) is one that can be written as the outer product of l vectors \( v^1 \otimes \cdots \otimes v^l \), i.e.,

\[
D_{i_1 \cdots i_l} = v^1_{i_1} v^2_{i_2} \cdots v^l_{i_l}
\]

for all indices. From this, the rank of a tensor is defined as the smallest number \( R \) such that the tensor can be written as a sum of \( R \) rank-1 terms [20]. Unlike in the matrix case \( (l=2) \), where column rank, row rank, and outer product rank all coincide, there exist other notions of general tensor rank [14], but they are not relevant to our problem.

The homogeneous form \( D(g) \) of a symmetric rank-1 tensor

\[
\mathcal{D}(s,v) = s \cdot v^1 \otimes \cdots \otimes v^l
\]

defined from a scalar \( s \) and a real unit vector \( v \) provides a sharp, non-oscillating, non-negative, axially symmetric peak of height \( s \) at \( v \). Its sharpness grows with order \( l \), reflecting the higher angular resolution provided by tensors of higher order. We choose symmetric rank-1 tensors as suitable and computationally convenient models of the narrow peaks into which we would like to decompose the ODF. It is safe to make this choice, since other analytic, heuristic, or empirical models of a single fiber ODF can be converted to the assumed rank-1 tensor model in a simple preprocessing step. Examples of this will be given in Sections 5.1 and 5.2.

The best rank-1 approximation \( \mathcal{D}(s,v) \) of a tensor \( \mathcal{D} \) in the sense of minimizing the residual norm \( \| \mathcal{D} - \mathcal{D}(s,v) \| \) is given by the vector \( v \) which maximizes the absolute value of its homogeneous form \( D \) and the scalar \( s = D(v) \) [14]. When modeling fiber peaks as rank-1 tensors, the current practice of selecting ODF maxima can thus be interpreted as finding an optimal fit for each tract independently from the others, and the bias caused by this in case of crossing fibers can be considered a consequence of the fact that the sum of k non-orthogonal, locally optimal rank-1 approximations generally does not yield an optimal rank-k approximation [26].

The idea behind the proposed method is to improve a fiber estimate by refining the corresponding rank-k tensor approximation of the ODF. This takes into account the interference between non-orthogonal peaks, which is ignored by simple maxima extraction. Unfortunately, there are no methods which find the best rank-k approximation for general \( k \) and tensors of order \( l \geq 2 \). In fact, an optimal approximation may not exist, since the set of tensors \( \{ \mathcal{D} \mid \text{rank}(\mathcal{D}) \leq k \} \) is generally not closed. In other words, for rank \( k > 1 \) and order \( l > 2 \), there exist sequences \( \mathcal{D}^{(n)} \) of rank-k tensors whose limit \( \mathcal{D} \) has some rank \( R > k \), making it impossible to define a best rank-k approximation of \( \mathcal{D} \) [15].

Even though a decomposition into a minimal number \( R \) of rank-1 terms is known to exist and is referred to alternatively as “canonical decomposition” or “parallel factor analysis”, efficient algorithms for computing it are currently only available for tensors of dimension \( n = 2 \) or order \( l \leq 3 \) and thus are not applicable to our problem [12]. Despite these discouraging insights, we propose a stable and efficient algorithm that employs rank-1 approximations to successfully decompose tensor representations of ODFs.

4.2 Finding Rank-1 Approximations
Our algorithm involves computing the best rank-1 approximation to a higher-order tensor, which is equivalent to maximizing the absolute value of its homogeneous form on the unit sphere. There exists a higher-order power method for this task, which generalizes the power method for finding the largest eigenvector of a matrix [14]. Unfortunately, its supersymmetric variant is only guaranteed to converge for tensors whose induced homogeneous form is either non-negative or non-positive [26]. While Q-Ball ODFs are non-negative by definition, neither the residuals which occur in our algorithms nor results from spherical deconvolution have this property. Therefore, we employ a gradient descent technique with Armijo stepsize, as described in [40]. Even though a similar fixed stepsize algorithm has previously been used for fiber tracking in Q-Ball data [22], the adaptive stepsize proved critical to guarantee convergence in our experiments.

4.3 A Practical Algorithm to Resolve Crossings
The goal of our algorithm will be to decompose the higher-order tensor representation of the ODF into an optional isotropic part, several rank-1 terms which represent individual fiber peaks, as well as a small residual, which covers noise and factors outside the model. The isotropic part is needed in case of Q-Ball data to capture the “ambient” part of the diffusion, which cannot be assigned to any specific fiber compartment. For spherical deconvolution data, it may be omitted.

The proposed algorithm works iteratively: Initially, the full input ODF is assigned to the residual \( \mathcal{R} \). Any subsequent step reduces the residual norm \( \| \mathcal{R} \| \), until convergence is reached. First, the isotropic part of \( \mathcal{R} \) is found by computing the mean of its homogeneous form, \( \mu = \text{mean}(\mathcal{R}) \), and multiplying it with the isotropic tensor \( \mu \mathcal{R} \) whose homogeneous form is identically one. Since the spherical harmonic expansion \( \mathcal{R}_0^0 \) is a constant function, closed formulas for both \( \mu \) and \( \mathcal{R} \) are found from the matrix that relates spherical harmonic coefficients to tensor components [31].

The key part of the problem is to find a rank-F approximation of \( \mathcal{R} \), where \( F \) is the desired number of fiber tracts \( \mathcal{F} \). Since no algorithms for this are known and a global optimum may not even exist, our strategy is to do an iterative local optimization, which repeatedly optimizes each rank-1 term, while keeping all others fixed. When adding a new term, this involves a rank-1 approximation approx(\( \mathcal{R} \)) of \( \mathcal{R} \), as described in Section 4.2. To improve an existing rank-1 term \( \mathcal{F}_i \), an approximation of \( \mathcal{R}_i + \mathcal{F}_i \) is sought. In this case, we start a gradient descent at the previous optimum \( v \), denoted refine(\( v, \mathcal{R} + \mathcal{F} \)). This procedure is repeated until the residual norm no longer changes significantly. The final result may not be an unconstrained optimum, but is locally optimal in the sense that it cannot be improved by changing any term individually. Experimental results indicate that this weak notion of optimality is sufficient to provide a remarkable enhancement over simple maximum extraction, which does not take into account interference between fiber terms at all.

A final, non-trivial question is how many fibers should look for in a given ODF. Previous authors (e.g., [2, 24, 8]) have approached this with computationally costly statistical methods. To allow for a faster tracking process, our algorithm employs a simple heuristic that was found effective in noisy synthetic data: A rank-(\( F + 1 \)) approximation is accepted if it reduces the residual norm to at least \( \theta_{\text{norm}} \in [0,1] \) times the residual norm of the rank-\( F \) approximation (i.e., its contribution to
INPUT AND PARAMETERS

- \( O_{in} \): orientation distribution function to be analyzed
- \( F_{max} \): maximum number of fibers to extract

OUTPUT

- \( \hat{F}_i \): final fiber term \( i \)

IMPORTANT VARIABLES

- \( \mathcal{F}(n) \): fiber term \( i \) at inner iteration \( n \)
- \( v^{(n)} \): vector part of \( \mathcal{F}_i^{(n)} \)
- \( F \): current number of fiber terms
- \( R, \mathcal{A}(n) \): current residual, residual before inner iteration \( n \)

CONSTANTS

- \( \mathcal{I} \): isotropic tensor of constant value one
- \( \epsilon \): small scalar value greater than zero

\[
\mathcal{R} := \mathcal{R}^{(1)} := \hat{O}_{in}; F := 1; n := 1;
\]

repeat // outer iteration

repeat // inner iteration

if \( \mathcal{O}_{in} \) is a Q-Ball ODF then

\[
\mathcal{R} := \mathcal{R} - \text{mean}(\mathcal{R}) \cdot \mathcal{I}; // \text{improve isotropic estimate}
\]

end

for \( i := 1 \ldots F \) do // improve all \( F \) fiber terms

if \( \mathcal{F}^{(n)} \) is defined then

\[
\mathcal{F}_i^{(n+1)} := \text{refine}(v^{(n)}, \mathcal{R} + \mathcal{F}_i^{(n)});
\]

else

\[
\mathcal{F}_i^{(n+1)} := \text{approx}(\mathcal{R});
\]

end

end

\( n := n + 1; \mathcal{A}^{(n)} := \mathcal{R}; \)

until \( \|\mathcal{A}^{(n)}\| > (1 - \epsilon) \cdot \|\mathcal{A}^{(n-1)}\| \) // test convergence

if accept(\( \mathcal{F}_i^{(n)} \)) then // accept fiber terms

\( F := F + 1; \)

end

until \( F > F_{max} \) or not accept(\( \mathcal{F}_i^{(n)} \))

Table 1. Pseudocode of the proposed algorithm for estimating fiber tracts. All used functions are explained in the text.

5 RESULTS

First, we will concentrate on experiments on synthetic data, which allow us to validate our method against ground truth. On synthetic data, we contrast Q-Ball reconstruction (Section 5.1) with spherical deconvolution (Section 5.2), and confirm that our algorithm reliably reconstructs three-fiber crossings (Section 5.3) and remains stable under varying volume fractions (Section 5.4). Then, we turn to a real dataset, and demonstrate that tensor decomposition gives clear advantages over maximum extraction for fiber tracking (Section 5.5).

5.1 Synthetic Q-Ball Data

We have generated synthetic diffusion-weighted MRI measurements \( S(g) \) according to Equation (1), using two Gaussian compartments with equal volume fractions and fractional anisotropy \( FA = 0.87 \) (\( FA = \sqrt{3/2} |D - \text{mean}(D)| / |D| \) [7]). This model is commonly used to simulate crossing fiber populations for validation purposes ([37, 19, 42]). As measurement parameters, we selected 60 gradient directions based on electrostatic repulsion [23] and a \( b \)-value of \( b = 3000 \) s/mm\(^2\) (as in [36, 19, 16], among others). Rician noise at two levels (SNR\(_0\) = 40 and SNR\(_0\) = 20) was added to the signals \( S(g) \).

Note that SNR\(_0\) refers to the signal-to-noise ratio in the unweighted data. The resulting SNR in diffusion-weighted images is lower and depends on the exact gradient direction and fiber setup. For example, in a simple one-fiber experiment, SNR\(_0\) = 40 leads to an effective SNR between 22 (perpendicular to the fiber) and 0.25 (along the fiber). Starting from 90\(^o\), we decreased the angle between the simulated tracts in steps of 5\(^o\), until our method failed to resolve them reliably. For each fiber configuration and each noise level, we took 1 000 samples.

In a first experiment, we estimated Q-Balls of order \( l \) = 4 from the synthetic data, using the analytic solution of the Funk-Radon transform and Laplace-Beltrami regularization with smoothing parameter \( \lambda = 0.004 \) [16]. For comparison with the decomposition results, we computed the discrete maxima of the resulting ODFs using a refined icosahedral tessellation of the sphere and improved their accuracy via gradient descent (cf. Section 4.2). In case of more than two maxima, we selected the largest ones. In the decomposition, we set \( F_{max} = 2 \). The included angle of the estimated fibers was computed and compared to the ground truth. Figure 2 presents mean and standard deviation of the reconstruction error for maximum finding (red, circles) and tensor decomposition (black, crosses). As previously reported [42], maxima exhibit a clear bias starting at around 80\(^o\), which is greatly reduced in the decomposition results. Moreover, the decomposition reliably reconstructs crossings far beyond the point at which the individual maxima have merged. This can be explained by the similarity of the decomposition to a deconvolution: Effectively, the decomposition finds a discrete set of delta distributions whose convolution with the kernel defined by the homogeneous form of a rank-1 tensor approximates the Q-Ball.

It has been pointed out [19] that the effective angular resolution of analytic Q-Ball imaging is affected both by the measurement process and the reconstruction: The angular resolution of diffusion measurements is limited by the employed \( b \)-value. Additionally, truncating the spherical harmonic expansion introduces a point spread function in the reconstruction, which becomes narrower for increasing order \( l \). If the latter effect dominates, it should be appropriate to decrease the width of the assumed fiber peak, as is done when modeling it as an order-\( l \) rank-1 tensor. Unfortunately, the results on Q-Balls with order \( l = 6 \) (SNR\(_0\) = 40) in Figure 3 (a) suggest that this is not the case: Compared to Figure 2 (a), the bias of the decomposition (black) is much higher, indicating that an order-6 rank-1 tensor is a less suitable model of a single fiber peak than the wider order-4 rank-1 tensor.

As proposed in Section 4.1, we can reduce this bias by mapping a more suitable model \( M^a \) to the model \( M^a \) assumed by our algorithm.
This is done by deconvolving the ODF with $M^c$, followed by a convolution with $M^p$. In the spherical harmonic basis, this is a simple operation: An axially symmetric model $M$ is described by a single scalar value $m_h$ per harmonic order $h$ and the result of convolving it with a spherical harmonic series with coefficients $c_j$ is given by $c_j' = m_h · c_j$, where $h$ is the harmonic order of $c_j$ (cf. Section 2.2).

As a proof of concept, we let $M^c$ be the order-4 rank-1 tensor, which gave usable results in the previous experiment. $M^p$ is the order-6 rank-1 tensor. Then, the values $m_h^c$ and $m_h^p$ are found as the spherical harmonic coefficients $c_j$ of the corresponding rank-1 tensors. The resulting conversion factors $m_h = m_h^c / m_h^p$ by which the $c_j$ are multiplied are given as $m_h = [5/7, 5/6, 105/77, 1]$ for $h = [0, 2, 4, 6]$, respectively. Note that since $m_h^p = 0$, we do not perform a deconvolution on harmonic order $h = 6$. Figure 3 (b) shows that this heuristic sharpening reduces the bias both for the decomposition and for maximum finding.

From this experiment, we conclude that reliably extracting fibers from Q-Ball ODFs generally requires explicit modeling of the single fiber response, e.g., via an additional deconvolution step. Since using an empirical estimate of the appropriate deconvolution kernel would be conceptually very similar to spherical deconvolution, we will concentrate on the latter technique in the remainder of this work.

5.2 Spherical Deconvolution

The similarity between tensor decomposition and spherical deconvolution which has been discussed above does not imply that one could replace the other. Existing methods for spherical deconvolution yield a continuous ODF, and we will show that decomposing it still has advantages over taking maxima when discrete directions are desired. On the other hand, it has been emphasized that the fiber distribution described by the ODF holds more information than just the principal directions (e.g., evidence on the amount and orientation of fiber spread) [36] and there have been initial attempts to exploit it [33].

For our experiments, we have implemented spherical deconvolution as described in [36]. When setting the response function $R$, one explicitly specifies the shape of the peak that will result from the deconvolution of a training sample. For this, Tournier et al. use a truncated spherical harmonic representation of a delta distribution (cf. appendix of [35]). As an alternative, we employed the peak described by a rank-1 tensor, whose spherical harmonic coefficients $c_j$ are found as above.

Compared to truncated delta peaks, the non-oscillating rank-1 peaks gave smoother ODFs and their non-negativity reduced the undesired non-physical negative lobes in the resulting ODFs. They also produced best results with our decomposition, which explicitly assumes the rank-1 model. When finding maxima at order $l = 6$, the ringing in the truncated delta peak leads to a bias which oscillates for varying crossing angles, but remains tolerable over a wider range. For order $l = 4$, the ringing is so strong that a rank-1 tensor also gives more accurate maxima. To be as fair as possible to both methods, presented results use rank-1 peaks for the decomposition and maximum extraction at $l = 4$, truncated delta peaks for maximum finding at $l = 6$. Figure 4 presents results from spherical deconvolution of the same data as above, with $l = 6$. Again, the decomposition (black) exhibits less bias and is applicable in a wider range than maximum extraction (red).

5.3 Three-Fiber Crossings

Resolving crossings of three fiber bundles has proven difficult for many previous approaches: Tuch [38] reported that $k$-tensor estimation becomes unstable for $k = 3$. Bergmann et al. [9] discuss general $k$, but only present results for $k = 2$. Both Kreher et al. [27] and Peled et al. [32] explicitly restrict themselves to the two-fiber case.

To investigate this more difficult case, we generated additional data which uses three Gaussian fiber compartments. The respective principal axes were chosen such that their endpoints form an equilateral triangle on the sphere and any pair of them includes angle $\alpha$. Again, $\alpha$ was decreased gradually, starting from $90^\circ$. All other measurement parameters were chosen as above. ODFs were reconstructed using spherical deconvolution with orders $l = 4$ and $l = 6$.

Figure 5 shows that the rank-$k$ approximation remains stable for $k = 3$ and allows one to resolve three-fiber crossings even at low orders and relatively small angles (SNR$_0 = 40$). At the higher noise level (SNR$_0 = 20$, $l = 6$), it remained possible to reliably reconstruct three-fiber crossings down to $\alpha = 40^\circ$ (data not shown).

In all our experiments, we have found that under low noise (SNR$_0 = 40$), models of order $l = 6$ could reliably represent angles which were impossible to reconstruct from noisier data (SNR$_0 = 20$). From this observation, we conclude that under realistic measurement conditions, noise rather than model complexity will be the limiting factor for angular resolution already at order $l = 6$.

5.4 Estimating Volume Fractions and Fiber Number

To test the stability of our method under varying volume fractions, we fixed a two-fiber crossing at $60^\circ$ and gradually decreased the volume fraction of the weaker tract, until reconstruction failed (spherical deconvolution, SNR$_0 = 20$, $l = 6$). Both relative magnitude of maxima and relative magnitude of the fiber terms, $||F_{\text{weak}}||/(||F_{\text{weak}}|| + ||F_{\text{strong}}||)$, gave usable estimates of relative volume fractions (cf. Figure 6 (a)). Even more importantly, maximum extraction found the weaker fiber in less than 50% of the cases already for ratio 0.3 : 0.7, while the decomposition still worked below 0.2 : 0.8. However, the standard deviation of the estimated angle became so high that we do not report it for smaller values (Figure 6 (b)). A three-fiber crossing at $50^\circ$ with one dominant tract at ratio 0.6 : 0.2 : 0.2 was still correctly reconstructed by the decomposition in 85% of the cases.

In a final experiment on synthetic data, we validated the heuristic...
Fig. 6. Under varying volume fractions (VF), ODF decomposition (black crosses) reconstructs the weaker tract over a wider range than maximum finding (red circles).

<table>
<thead>
<tr>
<th>SNR&lt;sub&gt;0&lt;/sub&gt; = 40</th>
<th>SNR&lt;sub&gt;0&lt;/sub&gt; = 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber count estimated by our heuristic</td>
<td></td>
</tr>
<tr>
<td>1 tract</td>
<td>2</td>
</tr>
<tr>
<td>2 tracts</td>
<td>0</td>
</tr>
<tr>
<td>3 tracts</td>
<td>0</td>
</tr>
<tr>
<td>Number of above-average ODF maxima</td>
<td></td>
</tr>
<tr>
<td>1 tract</td>
<td>1</td>
</tr>
<tr>
<td>2 tracts</td>
<td>163</td>
</tr>
<tr>
<td>3 tracts</td>
<td>88</td>
</tr>
</tbody>
</table>

Table 2. In a fiber detection experiment, our heuristic proved to be a more reliable indicator of fiber number than ODF maxima.

which determines the fiber number in our algorithm (cf. Section 4.3). Classification was tested on 1000 voxels each with a single fiber, two fibers, and three fibers, respectively. Directions were chosen uniformly at random such that all included angles were above 45°, all volume fractions were equal. Under these conditions, our classification proved reliable, even under noise (cf. Table 2). For comparison, we specify the number of ODF maxima whose magnitude is above the ODF mean.

We expect the estimation of both volume fractions and the number of fiber compartments to be more challenging in the presence of fiber spread in real data. However, it is reassuring to note that the decomposition improves upon maximum extraction also in this respect.

5.5 Improvement in Fiber Tracking

The real-world dataset used to test our algorithm in practice consisted of 60 diffusion-weighted images (three averages each) at isotropic voxel size 1.72 mm<sup>3</sup> and b = 1000 s/mm<sup>2</sup>. At such low b-values, only very few crossings are resolved as individual maxima in unsharpened Q-Ball data, so we used filtered spherical deconvolution for a fair comparison of the decomposition method against maximum extraction. As in [36], a DT-MRI model was first estimated and the average fractional anisotropy (FA) of the 300 voxels with highest FA was computed. Based on this, the deconvolution kernel R was set from the synthetic signal of a prolate tensor with the same FA, using noise attenuation vector β = [1 1 1 0.6] (cf. [36]).

In a first experiment, we tried to track the lateral transcallosal fibers (TF) that run through the corpus callosum (CC) (cf. labels in Figure 7 (c)). It is known that DT-MRI tractography (Figure 7 (a)) only captures the dominant U-shaped callosal radiation (CR) [39]. The ability to find the lateral fibers with high angular resolution imaging has previously been demonstrated using simple streamline tracking along the most collinear ODF maximum on high b-value diffusion spectrum data [18] and using probabilistic tractography on sharpened Q-Balls [4].

We have tried to reproduce the tract using a deterministic higher-order tensor tracking algorithm, similar to [21]. However, we additionally allowed for tract splitting in cases where a second maximum was found within 30° of the current tracking direction. Since Q-Ball estimation, spherical deconvolution, and the conversion between spherical harmonics and tenor coefficients are all linear operations, one may equivalently interpolate diffusion-weighted images, spherical harmonic coefficients, or tensor components. For efficiency, we used component-wise trilinear interpolation of the tensor field before extracting principal directions.

With seed points in the corpus callosum, the lateral fibers were not found when following ODF maxima (Figure 7 (b)). When using decomposition results instead, the tract could be reconstructed (c). A visual comparison of maxima (e) to decomposition results (f) reveals that the latter separates the diverging transcallosal fibers earlier and reconstructs their crossing with fibers from the internal capsule more reliably (cf. ellipses in (f)), which facilitates tracking. In the background of all figures, a slice of co-registered T<sub>1</sub> data is shown for context.

Figure 8 confirms that this result can be reproduced along a large part of the corpus callosum. Here, only fibers which leave a corridor of ±20 mm around the mid-sagittal plane have been colored to visually emphasize the transcallosal fibers. Note that the DT-MRI result in (a) includes a part of the inferior fronto-occipital fasciculus (in

![Image](a) Tracts from DT-MRI (b) Tracts from ODF maxima (c) Tracts from ODF decomposition)

Fig. 8. Tractography of the full corpus callosum, shown from above. Only the decomposition result (c) reliably includes transcallosal fibers.
Fig. 9. Tensor decomposition allows tracking through the triple crossing of cortico-spinal tract (blue), corpus callosum (red) and superior longitudinal fasciculus (green) even at \( b = 1000 \) and \( l = 4 \).

The low number of parameters in order-4 models makes them particularly attractive for experiments with a relatively low number of measurements. Due to the limited measurement time feasible in clinical practice, there has been some interest in resolving crossing tracts in such settings (e.g., [9, 32]). Figure 9 illustrates the advantages of tensor decomposition for order \( l = 4 \). It presents \( 4 \times 4 \) voxels from a coronal slice of the three-fiber crossing between cortico-spinal tract (blue), transcallosal fibers (red), and superior longitudinal fasciculus (green). Since many peaks have merged (cf. the ODF profiles in (a)), the extracted maxima in (b) miss one or two of the crossing tracts in most voxels. In comparison, the decomposition in (c) reconstructs them more reliably, allowing all three tracts to be tracked through the crossing region (e), while the red transcallosal fibers are mostly blocked at the crossing when using ODF maxima (d). To avoid visual clutter, tracts have been terminated when leaving a small region of interest.

In a final experiment on real data, we have seeded the tractography within the decussation of the superior cerebellar peduncle (dscp), a location in the brainstem where parts of the superior cerebellar peduncle (scp) cross to the opposite hemisphere. Due to partial voluming, this region also contains a part of the adjacent corticospinal / corticopontine tract (cst/cpt). Figure 10 compares results of maximum tracking and tensor decomposition in a view from posterior / superior. Again, the decomposition makes tracking through this complex configuration more reliable. In particular, a part of the cst/cpt is reconstructed.

6. Conclusion and Future Work

In many visualization methods for Q-Ball and spherical deconvolution data, finding the directions of crossing fibers from a continuous orientation distribution function (ODF) is a crucial step. In this work, we have shown that taking into account the interference between the signals from different tracts by approximating the ODF as a sum of individual fiber peaks provides estimates of much higher accuracy than the current practice of extracting maxima. Recently, it has been proposed to go to high harmonic orders (up to \( l = 12 \)) in spherical deconvolution, to reduce bias in fiber estimates and to represent close tracts by individual maxima in the ODF. Since this increases the number of model parameters above the number of measurements which are typically available, it requires a non-linear, constrained ODF reconstruction, which involves additional, heuristic parameters [35]. In contrast, we have demonstrated that the angular resolution of Q-Ball or spherical deconvolution models is not reached when crossing fibers are no longer separated by individual maxima. Rather, a subsequent decomposition step reliably reconstructs fibers over a considerable angular range, at greatly reduced bias. In our synthetic experiments, angular resolution was bounded by assumed measurement noise rather than model resolution already for order \( l = 6 \).

Addressing the problem of separating crossing fibers via tensor decomposition has been motivated by the great utility of the eigenvector decomposition in DT-MRI. While higher-order tensors have previously been considered as an alternative to spherical harmonics in the context of diffusion imaging [30, 31, 21], the proposed connection to rank-1 tensors and rank-\( k \) tensor approximations is new. It provides a useful way to formalize the problem and allowed us to find an efficient algorithm for crossing fiber estimation, which draws on existing techniques for rank-1 approximation. It is our hope that the link between the analysis of orientation distribution functions and recent efforts in multilinear algebra (e.g., [11, 15, 40]) will become even more useful as more results become available in this interesting area of research.

Evaluating the algorithm against ground truth in synthetic data has shown that it resolves crossing fibers more effectively than existing methods and experiments on real data have given plausible results. However, several aspects of the inverse problem in diffusion-weighted imaging remain challenging: In many cases, it is difficult to find the correct number of fibers in a voxel and to tell crossings from diverging or bending configurations. As future work, we plan to quantify the confidence of fiber estimates and to explore both coherence over spatial neighborhoods and prior knowledge to increase it.

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REFERENCES


