Precoder Design based on Correlation Matrices for MIMO Systems

Hamid Reza Bahrami†, Tho Le-Ngoc‡, Amir Masoud Nasri Nasrabadi†, S. Hamidreza Jamali‡
†Department of ECE, McGill University, 3480 University, Montréal, Québec, Canada H3A 2A7
‡Department of ECE, University of Tehran, Tehran, Iran

Abstract – This paper presents a precoder design based only on the correlation matrices to avoid the need of perfect channel knowledge at the transmitter and receiver in practical MIMO systems. The structures for optimal linear precoders are derived for three criteria: minimum pairwise error probability, minimum mean squared-error, and maximum ergodic (mean) channel capacity. The minimum pairwise error probability and minimum mean squared-error criteria lead to the same optimum precoder structure. It is shown that, in general, the optimal linear transformations for the three criteria are the eigen-beamformers that transmit the signal along the eigenvectors of the transmit correlation matrix. Power loading across the eigen-beams is determined by considering the eigenvalues of both transmit and receive correlation matrices and can be viewed as a water-pouring policy. Simulation results show noticeable performance improvement over conventional systems, particularly when their transmit correlation matrix has low rank.

Keywords: MIMO Channel, Transmit Eigen-Beamforming, Precoder, Correlation, Water-pouring

I. INTRODUCTION

Design of linear transformations (precoders or beamformers), usually assumes full channel knowledge at the transmitter and aims to improve performance (i.e., higher information rates or lower bit error rates) by optimal allocation of resources such as power and bits over multiple antennas, based on the channel properties. However, perfect CSI is rarely available at the transmitter in practice [1]. Hence, it is more reasonable to assume that transmitter knows only the transmit and receive correlation matrices (i.e., partial channel knowledge at the transmitter). Knowing the transmit correlation matrix and assuming the receive correlation matrix to be identity, in [1], [7] and [12], an optimal transformation has been designed based on the pair-wise error probability (PEP) criterion. The receive correlation matrix is assumed to be unity based on the rich scattering environment near mobile units in a wireless link. This assumption is not always valid, e.g., in wireless uplink channels, and therefore the receive correlation matrix cannot be always neglected. In order to take advantage of correlation knowledge and based on capacity criteria, [2] and [6] used the eigen-decomposition of the average MIMO channel and thereupon implemented a water-filling approach across the eigen-modes of the correlation matrix.

This paper considers precoder designs based on both transmit and receive correlation matrices and derives the structures for optimal linear precoders under three criteria: minimum pairwise error probability, minimum mean squared-error, and maximum ergodic channel capacity.

II. CHANNEL MODEL

We consider the transmitter and receiver equipped with \( M \) and \( N \) antennas, respectively. The MIMO channel is represented by the \( N \times M \) matrix \( \mathbf{H} \) and its entry, \( h_{mn} \) is the complex-valued gain from the \( m \)th transmit to \( n \)th receive antenna. Assuming that the transmit and receive scattering radii are large compared to the distance between the transmitter and the receiver, the channel matrix model \( \mathbf{H} \) can be decomposed as [4]

\[
\mathbf{H} = \mathbf{R}^{1/2}\mathbf{G}\mathbf{T}^{1/2}
\]

(1)

where \( \mathbf{G} \) is an \( N \times M \) matrix with i.i.d. zero-mean complex Gaussian entries and \( \frac{1}{2} \) variance per dimension. Furthermore, \( \mathbf{R} = \mathbf{R}^{1/2} \mathbf{R}^{1/2} = \mathbf{E}\{\mathbf{HH}^H\} \) and \( \mathbf{T} = \mathbf{T}^{1/2} \mathbf{T}^{1/2} = \mathbf{E}\{\mathbf{H}^H\mathbf{H}\} \) are the receive and transmit correlation matrices, respectively. The entries of \( \mathbf{R} \) and \( \mathbf{T} \) are determined from the receive and transmit antenna spacing and angular spread [4]. The superscript \( \dagger \) denotes the Hermitian transposition (i.e., the operation of transposition combined with complex conjugation).

![Fig. 1: The system model](image)

Fig. 1 shows the block diagram of the MIMO system under consideration. The transmitter includes a space-time block code (STBC) encoder followed by a linear transformation governed by an \( M \times M \) matrix \( \mathbf{W} \). Input data, \( \mathbf{S} \), is first mapped to a codeword \( \mathbf{c} \) by the STBC encoder, and subsequently undergoes the linear transformation \( \mathbf{W} \) to form the codeword \( \mathbf{Wc} \) to be transmitted over the MIMO channel \( \mathbf{H} \). At the receive side, the noisy codeword is recovered by ML (Maximum Likelihood) decoding.

In other words, the transmitter combines ST coding with appropriate beamforming \( \mathbf{W} \). The transmitter is assumed to know \( \mathbf{T} \) and \( \mathbf{R} \) matrices. Based on this information, the transmitter design chooses the optimum \( \mathbf{W} \) according to the design criteria.

III. OPTIMAL TRANSFORMATION

3.1. Optimal Transformation using PEP Criterion

Based on the model described in the previous section, the pair-wise error probability (PEP), i.e., the probability that a transmitted codeword \( \mathbf{c} \) is erroneously received as a different...
codeword \( e \) can be upper-bounded as

\[
PEP \leq \exp \left(-\frac{E_s}{4N_0} \right) \text{tr} \left[ WH (c-e) (c-e)^H W^H H^H \right]
\]  
(2)

where \( N_0/2 \) is the noise variance per dimension, \( E_s \) is the average constellation energy, \( c \) and \( e \) are \( M \times K \) codeword matrices. For power constraint, \( \text{tr}(WH^H) \leq M \). Using (1),

\[
\text{tr}(AB) = \text{tr}(BA), \quad T^{1/2} W (c-e)(c-e)^H W^H (T^{1/2})^H = U A U^H \quad \text{and} \quad R^{1/2} [R^{1/2}]^H = V D V^H \quad \text{where} \quad V^H = G U \quad \text{and} \quad U^H = \Lambda \quad \text{(11)}
\]

we obtain \( \Phi = G U \) or equivalently, to maximize 

\[
\text{det} \left( \frac{I + \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \left( I \otimes \Phi \right) }{ \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \left( I \otimes \Phi \right) } \right) \quad \text{subject to} \quad \text{tr}(P) \leq \alpha M \quad \text{(10)}
\]

By averaging (11) with respect to independent Rayleigh-distributed components \( \{ g_{ij} \} \), we obtain the average MSE

\[
\text{MSE} = \frac{1}{4} \left( \prod_{i=1}^M \prod_{j=1}^M \left[ 1 + \left( \frac{2E_s}{N_0} \right) d_{ij} \right] \right)^{-1} \left( \text{det} \left( I + \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \right) \right)^{-1} \quad \text{(11)}
\]

Our goal is to obtain the minimum MSE (MMSE) in (11) or equivalently, to maximize \( \text{det} \left( I + \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \right) \), which is very similar to (6). In other words, (8) and (9) are also applicable to the problem using MMSE criterion.

### 3.3. Power Loading Policy under PEP and MMSE criteria

In the following we consider the special cases that lead to a simpler form for the power loading policy.

#### No Receive Correlation: For \( R = I \), we have \( D = I \) and (9) becomes

\[
\max_p \left\{ \text{det} \left( I + \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \right) \right\}^{1/2} \quad \text{subject to} \quad \text{tr}(P) \leq \alpha M \quad \text{(12a)}
\]

It can be shown that (12a) has the following solution for \( P 
\]

\[
p = \left( \text{v} \left( \left( \frac{E_s}{N_0} \right) \delta \right)^{-1} \right)^{1/2}
\]  
(12b)

where \( \delta \)'s are the diagonal entries of \( \Delta \) and \( v \) is a constant determined by the power constraint. This clearly has the form of the well-known water-pouring policy and coincides with the solution in [1]. From an intuitive point of view, when \( T \) has full rank (i.e., all \( \delta \)'s are positive), as SNR increases, the second term on right hand side of (12b) decreases and allows the power to be divided evenly among all eigen-values. Nevertheless, when some of the \( \delta \)'s are zero, the total power is divided among all other eigen-values of \( T \), according to the water-pouring policy.

#### No Transmit Correlation: For \( T = I \), \( \Delta = I \) and (9) becomes

\[
\max_p \left\{ \text{det} \left( I + \left[ \frac{2E_s}{N_0} \right] \left( D \otimes \Delta \right) \right) \right\}^{1/2} \quad \text{subject to} \quad \text{tr}(P) \leq \alpha M \quad \text{(13)}
\]

Due to perfect symmetry with respect to power \( p_i \), for all variations of \( D \), the expression adopts its maximum when \( P = \alpha I \), i.e., the final solution is independent of \( D \).

**General Case:** From Karush-Kuhn-Tucker optimization conditions [5], one may find the answer for a general MIMO system with arbitrary transmit and receive correlation matrices. However, the answer cannot be expressed in an explicit form (in terms of a water-pouring policy). We propose the following approximation of the solution

\[
p = \left( \text{v} \left( \left( \frac{E_s}{8N_0} \right) \delta \right)^{-1} \right)^{1/2}
\]  
(14)
where δ and ν are the same parameters defined in the special cases. This solution is in complete agreement with the solutions of two above special cases.

3.4. Optimal Transformation using Ergodic Capacity Criterion:

The ergodic capacity of the MIMO system in Fig. 1 is defined as

\[
C = E[\log_2 \det(1 + N_0^{-1}W^HHW)]
\]  

(15)

Again, the power constraint \(\|W\|^2 = M\) must be imposed on \(W\) to limit the total transmit power. Getting expectation from the log function in (15) is very hard (if not impossible). By applying the Jensen’s inequality [11] to \(E[\log_2 |det(A)|] \leq \log_2 |E[|det(A)|]|\), and using matrix expansions [8] we can derive a tight upper-bound on the ergodic capacity as

\[
C \leq C_{UB} = \log_2 E \left[ \det \left( 1 + \frac{1}{N_0} W^T \left( 1^2 \right) \Phi \Psi \left( 1^2 \right) \Psi^H \Phi^H W \right) \right]
\]  

(16)

where \(\det(A)\) denotes the determinant of a sub-matrix of \(A\) obtained by selecting the row and column subset from the matrix \(A\) indexed by \(\alpha_k^i = \{\alpha_k^i, \alpha_j^j, \text{...} \alpha_k^j\}\) and \(\alpha_k^j = \{\alpha_j^i, \alpha_j^j, \text{...} \alpha_j^i\}\), respectively. The cardinalities of the subsets \(\alpha_k^i\) and \(\alpha_k^j\) are \(k\). Furthermore, for a diagonal matrix, \(\det(A)\) is zero, \(\alpha_i \neq \alpha_j\).

Consider a matrix \(X = [x_1, x_2, \text{...} x_n]\) where \(x_i\) is an \(m \times 1\) vector with zero mean i.i.d. complex Gaussian entries with \(\frac{1}{2}\) variance per dimension. It follows that the product \(Y = XX^H\) has a Wishart distribution and \(E[\det(Y)] = n!/(m-n)!\) [9]. The entries of submatrices \((G_i^i)\) and \((\tilde{G}_i^i)\) in (16) are zero mean i.i.d. complex Gaussian random variables with \(\frac{1}{2}\) variance per dimension. Therefore, \(E[\det(G_i^i)\det(\tilde{G}^i_i)] = k!\) and

\[
C_{UB} = \log_2 \left[ \sum_{i=0}^{K} \sum_{\alpha_i^i} \frac{k!}{[N_0]^{k_i}} \det(\Delta \Phi \Psi^H \Phi \Psi \Phi^H \Psi^H \Phi^H) \right]
\]  

(17)

The above derivation uses the relations between \(W, \Psi, \Phi, T, \Phi\) and \(A\) as discussed in Section III. Since \(\Phi\) and \(\Psi\) are unitary matrices, the maximum value of \(C_{UB}\) in (17) is achieved when the product \(\Delta \Phi \Psi^H \Phi \Psi \Phi^H = \Phi^H\) is diagonal. For this purpose, we can set \(\Psi = \Phi^H\) and consider the following structure of precoder matrix \(W = \Phi \Psi \Gamma \Phi^H\) where \(\Gamma\) is an arbitrary unitary matrix. In this case, the power loading policy across the eigen-beams can be obtained by setting \(\Phi \Psi \Phi^H = I\) in (16), i.e.,

\[
C_{UB} = \log_2 \left[ E \left( \sum_{k=0}^{K} \sum_{\alpha_i^i} \left( \frac{1}{N_0} \right)^k \det(A) \det(\Phi^H) \right) \right]
\]  

(18)

3.5. Power Loading Policy using Ergodic Capacity Criterion:

**Capacity Bound:** For a MIMO system without any precoding, if there is no correlation in the channel, i.e., the correlation matrices are \(I\), we can find an upper-bound for capacity as

\[
C_{UB} = \log_2 \left[ \sum_{k=0}^{K} \frac{k!}{[N_0]^{k_i}} \left( \frac{M}{N} \right) \right]
\]  

(19)

The above result has also been derived in [2].

**No Receive Correlation:** For simplicity, consider the case of \(M = N = 2\). Let \(R = I\). Then \(D = I\) and the upper-bound in (19) is reduced to \(C_{UB} = \log_2 \left[ 1 + \text{tr}(\Delta P) + 2 \det(\Delta P) \right]\). For the sake of brevity, the factor \([N_0]^{-k}\) in (19) can be omitted by simply multiplying each diagonal elements of matrix \(\Delta\) by \([N_0]^{-k}\). After some simple manipulation, we obtain

\[
C_{UB} = \log_2 \left[ 1 + \text{tr}(I + \Delta P) + \det(\Delta P) \right]
\]  

(20)

Now the structure of the maximization problem changes to a simple water-pouring problem. Its solution is

\[
p_i = \left( v - [N_0/\delta^i] \right)^+ \]

(21)

where the constant \(v\) has to be found such that the power constraint on the summation of \(p_i\)'s is satisfied and \(\delta^i\) is the \(i^{th}\) diagonal entry of \(\Delta\). Note that the above result is similar to that obtained for the PEP criterion with no receive correlation shown by (12b).

**No Transmit Correlation:** In this case, we have \(T = I\) and consequently, \(\Lambda = I\). Therefore, (18) becomes

\[
C_{UB} = \log_2 \left[ 1 + \text{tr}(P) + \text{tr}(D) + 2 \det(P) \det(D) \right]
\]  

(22)

The right-hand side of (22) has perfect symmetry with respect to diagonal entries of \(D\). Furthermore, \(\text{tr}(P)\) is constant due to the power constraint. This implies that \(P = \alpha I\) is the solution to the optimization problem, similar to the same case for PEP criterion discussed in the previous section. This indicates that the receive correlation matrix has weaker effect in the power allocation problem as compared to the transmit correlation matrix.

**General Case:** Using the Karush-Kuhn-Tucker conditions [5], for a MIMO system, the solution can be found to have a water-pouring form rather than that proposed in [2] and can be expressed as

\[
p_i = \left( v - N_0 \text{tr}(D)/\delta^i \right)^+
\]  

(23)

where \(\delta^i\) is the \(i^{th}\) diagonal entry of \(\Delta\) and the constant \(v\) can be found via a water-pouring process such that the power constraint on the summation of \(p_i\)'s is satisfied. This answer is in agreement with the above special cases.

IV. SIMULATION RESULTS

4.1. PEP Precoder

In this section, the performance of the PEP (or MMSE) precoder, is evaluated by Monte-Carlo simulations on BPSK modulated symbols. A system with different number of transmit \((M)\) and receive \((N)\) antennas is considered. Alamouti full rate [10] schemes with different numbers of transmit and receive antennas are considered as the primary STBCs.
Both partial ($\rho_{ij}=0.5$) and full correlation ($\rho_{ij}=1$) transmit and receive matrices are considered where $\rho_{ij}$ is the correlation between two antennas $i \neq j$. Figs. 2-5 show the BER versus SNR curves. A precoding gain of approximately 3dB and 2.5dB is obtained in high correlation (Fig. 2-3) and low correlation cases (Fig. 4-5), respectively.

Simulations show that the precoding gain increases with the number of transmit/receive antennas. Comparing Figs. 2 and 3 (or equivalently Figs. 4 and 5) shows that the performance
gain achieved by using this precoder is mostly due to the transmit correlation.

We also consider the ergodic capacity of a $2 \times 2$ system with full and partial correlation, in transmitter and/or receiver. The results shown in Figs. 7 and 8 indicate that the precoder can provide an improvement of more than 10% in bit/channel/use. Again, the transmit correlation matrix has a major effect on the transmission rate. A channel transmit correlation matrix with low rank can degrade the transmission rate considerably while the effect of receive correlation matrix is negligible. The same conclusions are valid for the case of partially correlated transmit and/or receive side in Figs. 9-10 with smaller capacity improvement.

V. CONCLUSIONS

Using PEP, MMSE and ergodic capacity as the performance criteria, precoder designs based on the knowledge of MIMO channel transmit and receive correlation matrices are developed. The optimal transformations are eigen-beamformers, which transmit the signal along the eigenvectors of the transmit correlation matrix and power loading across the eigenbeams is determined based on the eigenvalues of the transmit (and receive) correlation matrix. Performance evaluation by simulation and analysis show that the receive correlation matrix has a negligible effect. PEP (and MMSE) precoders offer significant performance improvement, particularly when transmit and receive correlation matrices have low rank (full correlation).

REFERENCES