MIMO Precoders Using Spatial and Path Correlations for Multipath Fading Channels

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Abstract—This paper presents MIMO precoder designs for frequency-selective channels based on the transmit, receive and channel path correlation matrices. Optimal linear precoder structures are derived based on ergodic capacity criterion for three different scenarios: uncorrelated channel paths but equal spatial correlation matrices, uncorrelated channel paths but unequal spatial correlation matrices, and correlated channel paths. For channels with uncorrelated paths, the precoding structure is composed of a number of parallel precoders designed for frequency-flat fading channels. Simulation results show that the proposed precoders outperform others based on spatial correlation in various propagation scenarios. Their achievable capacity is better in highly correlated (either spatial or path correlation) environments.

Keywords— MIMO channel, frequency-selective, spatial correlation, path correlation, precoder

I. INTRODUCTION

MIMO precoder design based on full channel knowledge at transmitter for performance enhancement has been investigated for fading channels (e.g., [1],[2],[3]). However, the assumption of full channel knowledge at the transmitter is not realistic in a time-varying fading environment since it is not possible to estimate and feedback accurate and instantaneous channel information to the transmitter. Hence, it is more reasonable to assume that transmitter knows only partial channel knowledge such as channel stochastic properties. In [4], precoding design based on the pairwise error probability (PEP) criterion and the knowledge of only the transmit correlation matrix for frequency-flat fading channels was presented. In [5], precoding designs based on the knowledge of transmit and receive correlation matrices at the transmit side were developed for three different criteria: PEP, MMSE and ergodic capacity. While frequency-flat fading channels have been considered for various precoding designs based on partial channel knowledge, there are very few papers addressing the frequency-selective fading environment. In [3], a precoder has been designed for an OFDM based MIMO system exploiting the transmit- and path-correlation properties. However, in the final design, the effect of path-correlation matrix was neglected. There are several reasons for the existence of path correlation in a multi-path channel. For example, the scatterers that are far from the transmit (or receive) antenna, can introduce a temporal correlation between their multi-path signals. In a keyhole channel [6], a transmitted signal can experience fading and delay spread before and after the pinhole. Due to the transmit and receive filtering structures, the fading coefficients from various paths with different delays can be statistically correlated and their correlation can be significant sometimes in some part of the channel (e.g., near the keyhole).

In [16], we have studied the structure of precoder in the frequency-selective channels when both transmit and path correlation knowledge is available at the transmit side. This paper is an extension to the work presented in [16] and considers MIMO precoding designs for general frequency-selective fading channels based on both spatial (transmit and receive) and path correlation matrices and derives the structures for optimal linear precoders under maximum ergodic (mean) channel capacity. Three separate cases are considered: 1) uncorrelated channel paths with similar spatial correlation, 2) uncorrelated channel paths with different spatial correlation, and 3) correlated channel paths. For uncorrelated channel paths, it is shown that the precoder is composed of a number of parallel precoders for frequency-flat fading channels. The power allocation to each precoder is based on the power of channel paths and the eigenvalues of transmit correlation matrix and can be calculated based on a water-pouring policy. Furthermore, in the case of similar spatial correlation, these parallel precoders have the same structure. For correlated paths, we show that the optimum precoder is an eigenbeamformer and the power allocation on each eigenmode follows a water-pouring strategy based on the product of the eigenvalues of transmit and path correlation matrices. Capacity improvement of the proposed precoders based on partial channel knowledge is investigated in different propagation scenarios.

The rest of the paper is organized as follows. Based on the representation of a frequency-selective fading channel by a model with $L$ effective paths, Section II develops a comprehensive model suitable to analyze a general MIMO system in a frequency-selective fading environment with emphasis on the spatial and path correlations. We consider different propagation scenarios and develop the channel model that fits to the three cases mentioned above. Using this model, Section III formulates the optimization problem to develop precoder/decoder designs that maximize the ergodic capacity criterion under the transmitted power constraints, based on the knowledge of spatial and path correlation matrices at the transmitter. The optimum precoding structures are derived for both uncorrelated and correlated channel paths. In Section IV, performance in terms of achievable ergodic capacity versus SNR of the proposed precoders is evaluated and compared with that of systems using either no precoding or spatial precoding in various scenarios by simulation. Finally, in Section V conclusion is presented.
II. SYSTEM MODEL

We consider a general MIMO communication system with \( M \) transmit and \( N \) receive antennas in a wireless frequency-selective fading environment. The equivalent complex baseband channel, including the RF transceivers and broadband wireless frequency-selective fading MIMO environment, is represented as a multi-path model with \( L \) effective paths where each path can be modeled as an \( N \times M \) matrix denoted by \( H_i \) with \( i = 0,1, \ldots, L-1 \). The \( N \times M \) matrix \( H_i(k) \) represents the spatial response corresponding to path \( i, i = 0,1, \ldots, L-1 \), at the instant \( k \), i.e., its entry, \( h_{nm}(k) \) is the complex-valued random gain from the \( n^\text{th} \) transmit to \( m^\text{th} \) receive antenna over the effective path \( i \) at the instant \( k \), assumed to be unchanged during a frame transmission. The system model can be written as:

\[
\bar{y}(k) = \bar{H} \bar{x}(k) + \bar{\eta}(k) \tag{1}
\]

where \( \bar{H} \) is the \( NP \times (P+L) \) block-Toeplitz channel matrix in (2):

\[
\bar{H} = \begin{bmatrix}
H_{0} & 0 & \cdots & 0 \\
H_{1} & H_{0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & H_{L-2} & H_{L-1}
\end{bmatrix}
\]  

In (1), we have assumed a transmitted block of \( P+L \) vectors of size \( M \times 1 \) where \( P \) is an arbitrary integer. We stack them in an \( M(P+L) \times 1 \) vector, \( \bar{x}(k) = [x_i(k+P+L), \ldots, x_i(k+P+L-1)]^T \). We stack \( P \) received snapshots in a \( PN \times 1 \) vector, \( \bar{y}(k) = [y_i(k+P+L), \ldots, y_i(k+P+L-1)]^T \), in which we eliminated the first \( L \) vectors to cancel the inter-block interference (IBI). Also, \( \bar{\eta}(k) = [\eta_i(k+P+L), \ldots, \eta_i(k+P+L-1)]^T \) is the \( PN \times 1 \) noise vector which is assumed to be circularly symmetric additive white zero-mean Gaussian noise.

Spatial correlation has been modeled by using a Kronecker product model, in which the channel correlation is the product of the transmit and receive correlation matrices. It is well known that the fading correlation is governed by the angle spread, the antenna spacing, and the wavelength [7]. Based on the Kronecker model, for each of the paths we can write:

\[
H_{i}(k) = R_{R,i}^{1/2} G_{i}(k) R_{T,i}^{1/2} \tag{3}
\]

where \( G_{i}(k) \) is an \( N \times M \) matrix with i.d. zero-mean complex Gaussian entries. Note that their variances are not the same across the channels. More specifically, \( \text{tr}(G_{i} G_{i}^H) = P_i \), where \( P_i \) is the power of the \( i^\text{th} \) path. Furthermore, \( R_R \) and \( R_T \) are \( N \times N \) and \( M \times M \) receive and transmit spatial correlation matrices, respectively. Entries of \( R_R \) and \( R_T \) can be determined from the receive and transmit antenna spacings, angular spread and angular power spectrum of the channel [7] and are the same for the \( L \) paths. Note that, the structure in (3) results from an assumption that transmit and receive scattering radii are large enough and only immediate surroundings of the antennas on one side (transmitter or receiver) have an impact on its antenna correlation without affecting that of the other side.

Based on the general model in (2), we can develop channel models suitable for modeling different sources of channel correlation, i.e. spatial and path correlation, in different propagation scenarios. In other words, we are mainly interested in three different scenarios: 1) uncorrelated paths with the same spatial correlation, 2) correlated paths with spatial correlation, and 3) uncorrelated paths with different spatial correlations. We consider each case separately and develop suitable model structures that ease the mathematical analysis of precoder design in the following sub-sections.

1) Uncorrelated paths with the same spatial correlation:
When the radius of all the local scatterer clusters are close to each other, i.e., when the angles of departure (AoD) and arrival (AoA) of all channel paths are almost the same, the spatial correlation matrices in (3) can be assumed the same for all channel paths, i.e. \( R_{R,i} = R_R \) and \( R_{T,i} = R_T \) \( \forall i : 0..L-1 \).

In this case, for all paths, the channel matrix for each path \( H_{i}(k) \) can be decomposed as:

\[
H_{i}(k) = R_{R}^{1/2} G_{i}(k) R_{T}^{1/2} \tag{4}
\]

Now, consider the first row of \( \bar{H} \) in (2), \( \bar{H} = [H_{i,1}, H_{i,2}, \ldots, H_{i,N}] \), \( \forall i : 0..L-1 \), can be decomposed as:

\[
\bar{H}_{i} = R_{R}^{1/2} G_{i}^{1/2} \tag{5}
\]

where \( R_{M(P+L)-1} \) and \( I_M \) are the \( M \times M \) identity matrices, and \( 0_{M(P+L)-1} \) are the \( M \times M \) zero matrices. Similarly, the \( j+1 \) row of \( \bar{H} \) in (2) is the right-shifted version of the first row by one \( (N \times M) \)-matrix and can be written as \( HE \), where \( E \) is an \( M \times (P+L) \)-matrix:

\[
E = \begin{bmatrix}
0_{M(P+L)-1} & I_M \\
I_M & 0_{M(P+L)-1}
\end{bmatrix}
\]

where \( I_{M(P+L)} \) and \( I_M \) are the \( M \times M \) identity matrices and \( 0_{M(P+L)-1} \) are the \( M \times (P+L) \)-matrix and \( 0_{M(P+L)-1} \) are the \( M \times M \) zero matrices, respectively. Similarly, the \( j+1 \) row of \( \bar{H} \) in (2) is the right-shifted version of the \( j^\text{th} \) row by one \( (N \times M) \)-matrix and can be written as the product of the \( j^\text{th} \) row by \( E \) for \( j=2,3,\ldots,(P-1) \). In other words, the channel model can be written as:

\[
\bar{H} = \begin{bmatrix}
H \times E \\
H \times E^2 \\
\vdots \\
H \times E^{P-1}
\end{bmatrix} = (I_p \otimes H) \times E = (I_p \otimes H) \times (E^{P-1}) \tag{5}
\]

where \( I_p \) is the \( P \times P \) identity matrix and \( \otimes \) stands for Kronecker product. \( E \) has the following properties:

- \( \forall i, \ 0 < i \leq P-1 : \det(E^i) = \pm 1 \), \( E(E^i)^T = I \) and hence \( E(E^i)^T \) has the same eigenvalues and eigenvectors as the identity matrix. The eigenvalues of \( E \) have the same absolute values as those of the identity matrix.
- Given an \( M(P+L) \times M(P+L) \)-matrix \( A \) and the eigen-decompositions of \( A = U A U^H \) and \( A = U_i A_i U_i^H \), \( A_i = A \) and \( U_i = U E^{i-1} \).

From the above properties, (4) and (5), the channel model can be written as:

\[
\bar{H} = (I_p \otimes R_R^{1/2} G(1_{P+L} \otimes R_T^{1/2})) \times E \tag{6}
\]

in which \( R_R \) and \( R_T \) are the \( N \times N \) and \( M \times M \) receive and transmit correlation matrices defined in (4), and \( G = [G_{L-1}, G_{L-2}, \ldots, G_0, 0, \ldots, 0] \) is an \( N \times M(P+L) \)-matrix composed of \( P+L \) matrices of size \( N \times M \).
2) Correlated paths in presence of spatial correlation:

We consider now the case of $L$ correlated channel paths with the same spatial transmit and receive correlation matrices for all $L$ paths. From (5), it is easy to check that in this case, the channel model can be written as:

$$\mathbf{H} = (\mathbf{I}_P \otimes \mathbf{R}_P) \mathbf{G} (\mathbf{R}_P \otimes \mathbf{R}_T^2) \mathbf{E} \label{eq:correlated}
$$

where $\mathbf{G}$ is a block diagonal matrix with diagonal matrices $\mathbf{G}$ is an $N \times (P+L)$ block diagonal matrix whose elements, $\mathbf{G}_i$'s are $N \times (P+L)$ diagonal matrices with i.i.d. zero-mean, complex Gaussian entries and $\frac{1}{2}$ variance per dimension. The remaining entries of this matrix are zero.

3) Uncorrelated paths with different spatial correlation:

Unlike the previous case, the spatial correlation matrices in this case are not the same for all channel paths, i.e., (4) is not valid and we have to use general equation in (3). A possible physical interpretation of this case is when the channel is composed of a number of local scattering clusters located in the vicinity of transmitter and receiver, however, the angles of arrival (AoA) and departure (AoD) of each scattering cluster is not the same. Assuming that every channel path has different transmit and receive correlation matrices, one can develop the following model:

$$\mathbf{H} = (\mathbf{I}_P \otimes \mathbf{H}) \mathbf{E} = (\mathbf{I}_P \otimes \mathbf{R}_P^{1/2}) \mathbf{G} (\mathbf{R}_P \otimes \mathbf{R}_T^2) \mathbf{E} \label{eq:uncorrelated}
$$

where $\mathbf{H}$ is a block diagonal matrix with diagonal matrices $\mathbf{H}$ is an $N \times (P+L)$ block diagonal matrix whose elements, $\mathbf{H}_i$'s are $N \times (P+L)$ diagonal matrices with i.i.d. zero-mean, complex Gaussian entries and $\frac{1}{2}$ variance per dimension. The remaining entries are zero, i.e., $\mathbf{0}_{N \times M}$ denotes an $N \times M$ zero matrix. Furthermore, $\mathbf{R}_P$ and $\mathbf{R}_T$ are $N \times (P+L)$ and $(P+L) \times (P+L)$ transmit and receive correlation matrices with the structures:

$$\mathbf{R}_P = \begin{bmatrix}
\mathbf{R}_{P,1} & \mathbf{R}_{P,2} & \cdots & \mathbf{R}_{P,L}
\end{bmatrix}
$$

and

$$\mathbf{R}_T = \begin{bmatrix}
\mathbf{R}_{T,1} & \mathbf{R}_{T,2} & \cdots & \mathbf{R}_{T,L-1}
\end{bmatrix}
$$

where $\mathbf{R}_{P,i}$ and $\mathbf{R}_{T,j}$ are, respectively, the $N \times N$ receive and $M \times M$ transmit correlation matrices associated with the $i$th channel path as defined in (3).

Now the system model consists of a channel with $M$ and $N$ transmit and receive antennas and $L$ channel paths and precoding/decoding matrices at the transmitter and receiver can be written as:

$$\mathbf{y}(k) = \mathbf{W}_R \mathbf{H}(k) \mathbf{W}_T \mathbf{x}(k) + \mathbf{W}_R \mathbf{a}(k) \label{eq:sys_model}
$$

where $\mathbf{y}(k)$ and $\mathbf{a}(k)$ is the $NP \times 1$ output vector by stacking $P$ subsequent $N \times 1$ received and noise vectors $\mathbf{y}$ and $\mathbf{a}$ in long vectors (for suitable elimination of the IBF effects as previously discussed), $\mathbf{x}(k)$ is the $M(P+L) \times 1$ transmitted vector that can be constructed by stacking $P+L$ subsequent $M \times 1$ input vectors $\mathbf{x}$ in (1) in a vector, $\mathbf{H}$ is $NP \times (P+L)$ composite channel matrix defined in (1) and $\mathbf{W}_T$ and $\mathbf{W}_R$ are the $M(P+L) \times (P+L)$ and $NP \times NP$ precoder and decoder matrices, respectively. Our objective is to find $\mathbf{W}_T$ and $\mathbf{W}_R$ based on a performance meter while we assume that we have full channel knowledge at the receiver but just spatial and path correlation matrices at the transmitter.

III. OPTIMAL PRECODER DESIGN

We assume that the receiver has the perfect channel information but transmitter knows only spatial and path correlation matrices. Our objective is to design the precoder $(\mathbf{W}_T)$ and decoder $(\mathbf{W}_R)$ matrices to maximize the ergodic capacity for a given total transmit power. Using the channel matrix descriptions in the previous section, the capacity proof is a simple extension of the well-known capacity result for single-input single-output (SISO) channels with memory, similar to the scenario discussed in [8]. Subsequently, for sufficiently large values of $P$, the ergodic capacity per dimension can be derived as:

$$C \equiv \frac{1}{P} E \left[ \log_2 \det(\mathbf{I}_{NP} + \mathbf{\gamma} \mathbf{W}_R \mathbf{H} \mathbf{W}_T^H \mathbf{H}^H \mathbf{W}_R^H) \right] \label{eq:capacity}
$$

where $\mathbf{W}_R$ is the $(P+L) \times (P+L)$ precoding matrix, $\mathbf{W}_R$ is the $NP \times NP$ decoding matrix to be optimized and $\mathbf{\gamma}$ is the channel signal to noise ratio (SNR), $E[\cdot]$ denotes expectation over different realizations of channel matrix $\mathbf{H}$ and the superscript $H$ denotes the Hermitian transposition. Assuming that the spatial and path correlation matrices are available at the transmitter, we want to design the precoding and decoding matrices $\mathbf{W}_T$ and $\mathbf{W}_R$ to maximize (10) under the power constraint criterion on $\mathbf{W}_T$ based on the limited power at the transmitter.

First, consider the optimization of decoder matrix $\mathbf{W}_R$. With the assumed knowledge of the instantaneous channel information at the receiver, a reasonable criterion to design a linear receiver $\mathbf{W}_R$, for given $\mathbf{H}$ and $\mathbf{W}_T$, is to maximize instantaneous mutual information (rather than its average):

$$I(\mathbf{y}, \mathbf{x}) = \log_2 \det(\mathbf{I}_{NP} + \mathbf{\gamma} \mathbf{W}_R \mathbf{H} \mathbf{W}_T^H \mathbf{H}^H \mathbf{W}_R^H) \label{eq:inst_mutual}
$$

The following Lemma specifies the structure of the optimum decoder when full channel knowledge is available at the receiver.

**Lemma 1:** Any decoding matrix of form $\mathbf{W}_R = \mathbf{\Gamma} \mathbf{W}_R^H \mathbf{H}^H \mathbf{W}_R^H$ maximizes the instantaneous mutual information in (11) with an arbitrary $MP \times (P+L)$ matrix $\mathbf{\Gamma}$ which $\mathbf{\Gamma} \mathbf{\Gamma}^H = \mathbf{I}$.

**Proof:** Using instantaneous mutual information formula the proof is straightforward. \hfill \square

1 In fact, the ergodic capacity of a frequency-selective fading channel based on (1) can be found by limitation over the average mutual information when the dimension of the channel matrix tends to infinity [9]-[11]:

$$C = \lim_{P \to \infty} \frac{1}{P} E \left[ \log \det(\mathbf{I} + \mathbf{\gamma} \mathbf{H} \mathbf{H}^H) \right]$$
Plugging the decoding matrix $W_R = \Gamma W_T^H \tilde{H}^H$ into (10) will result in:

$$C = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (12)$$

Now, our objective is to find the precoding matrix $W_T$ that maximize the right hand side of (13) while we assume that only spatial and path correlation matrices are available at the transmitter. Generally, getting the expectation of (13) is very difficult. Therefore, by applying Jensen’s in equality [12] we get an upper-bound on ergodic capacity and we maximize that upper-bound by finding $W_T$. Note that $\log \det$ function in (12) is a concave function. Therefore, Jensen’s inequality can be applied. Applying Jensen’s in equality to (12) will result in:

$$C \leq C_{UB} = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (13)$$

We will first start by deriving the optimum precoding matrix for uncorrelated paths (case 1 in Section II) and then extend the solutions to the correlated channel paths. For that, we substitute $\tilde{H}$ from (6) into (14) to get:

$$C_{UB} = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (14)$$

The following Lemma gives us the structure of the precoder.

**Lemma 2:** The $M(P+L) \times M(P+L)$ precoding matrix that optimizes the right hand side of (14) can be written as: $W_T = \text{diag}(W_j)$, $i = 0, 1, \ldots, (L+P-1)$ where $W_j$ is an $M \times M$ matrix. The problem of finding the optimal precoder can be reduced to a symbol-wise precoding problem in which the optimum precoder is an $M \times M$ precoder applies to each of the $P+L$ vectors, separately. Furthermore, optimal $W_j$’s can be found using the eigen-decomposition of the transmit correlation matrix and have the form $W_j = \Phi \Sigma_i^{1/2} \Gamma_j$, where $\Gamma_j$’s are $M \times M$ arbitrary unitary matrices, $\Sigma_i$’s are $P+L$ diagonal matrices, and $\Phi$ is the $M \times M$ transmit eigenmatrix matrix resulting from eigen decomposition of $R_P$.

**Proof:** Using the properties of Kronecker product, one can rewrite (13) as:

$$C_{UB} = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (15)$$

Using matrix identity det $(1 + AB) = \det(I + BA)$ and the eigen-decompositions of the nonnegative definite matrix $R_T^H R_T^2 \Sigma_i^{1/2} \Gamma_j$, we have $V$ and $\Sigma_i^H$ are unitary matrices and $D$ is a diagonal matrix that contains non-negative eigenvalues, (14) can be written as:

$$C_{UB} = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (16)$$

Since $V$ and $\Sigma_i^H$ are unitary matrices, $\hat{G} = V^H G$ and $\hat{G}^H = G^H V$ have the same distribution as $G$ and $G^H$, i.e., they are zero-mean, circularly symmetric complex Gaussian random matrices. By taking the expectation from (15) and some simple manipulations, we obtain:

$$C_{UB} = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma W_T^H \tilde{H}^H \bar{H}^H W_T)$$  \hspace{1cm} (17)$$

Consider now the eigen-decompositions $W_T W_T^H = \Psi \Sigma_i^H \Psi^H$ and $R_T = \Phi \Delta \Phi^H$. Using the properties of $E$, it is interesting to see that the eigen-structure of $(I_{P+L} \otimes R_T)$ does not change under $E$ transformation since the diagonal structure of $(I_{P+L} \otimes R_T)$ is a block diagonal matrix with similar diagonal blocks. On the other hand, we can combine $(P \otimes I_M)$ and $(E^i)^T$ and sum up the resulted matrices to get $(P_T \otimes I_M)$ where $P_T = \sum_{i=0}^{P-1} (P \otimes I_M)$. $(E^i)^T$ is a diagonal matrix that is constructed by summation of entries of $P$, i.e., its $j$-th diagonal element is $P_j = \sum_{l=0}^{L} P_{jl}$ for $j=1,2,\ldots,L$ and $P_{jl} = \sum_{l=1}^{P} P_{2l-1}$ for $j=1,2,\ldots,L$. For an assumed $P=L$, (17) can be rewritten as:

$$C = \frac{1}{P} \log_2 \det(I_{M(P+L)} + \gamma \text{tr}(D)(P_T \otimes I_M))$$  \hspace{1cm} (18)$$

Our aim is to maximize $C_{UB}$ in (18) subject to the constraint on the total transmit power, i.e., $\text{tr}(W_T W_T^H) \leq \text{constant}$. Using Hadamard’s inequality [13], it can be shown that we need $\Psi = (I_{P+L} \otimes \Phi)$. This identity ensures that the argument of the determinant function in (18) is a diagonal matrix. Therefore, the precoding matrix can be written as:

$$W_T = (I_{P+L} \otimes \Phi) \Sigma_i^{1/2} \Gamma_j$$  \hspace{1cm} (19)$$

where $\Gamma_j$ is an arbitrary unitary matrix that has no effect on the system performance and can be set to identity for simplicity. Therefore, $W_T$ will be a block diagonal matrix with diagonal block $W_j = \Phi \Sigma_i^{1/2} \Gamma_j$ and it proves the Lemma 2. □

This proof is similar to what we presented in [16], except that we also consider receive correlation matrix as well. In other words, the optimal linear precoding matrix $W_T$ consists of $P+L$ eigen-beamformers whose orthogonal beams point to the eigenvectors of the transmit correlation matrix, $R_T$. This also can be interpreted as $P+L$ parallel precoder for frequency
flat channels. Furthermore, the diagonal matrices $\Sigma_i$’s are in fact the power loadings on each of the eigen-beamformers.

The power loading policies across the eigen-beams can be obtained by substituting $\Psi = (I_{P+L} \otimes \Phi)$ in (19) as:

$$\max \log_2 \det(I_{M(P+L)} + k\gamma \text{tr}(D)(P_T \otimes I_M))$$

$$= \Sigma \begin{pmatrix} \frac{1}{2} \end{pmatrix}_{i=1}^{P+L} \Sigma_i = -i \Sigma_i$$

where $i=1,2,\ldots,M(P+L)$. (21) can be considered as a form of water pouring [15]. In (21), $[x]^+ = \max(0,x)$ for a scalar $x$, $\delta_i$’s are the eigenvalues of transmit correlation matrix $R_T$, $\mu$ is the constant determined by the power constraint, and $\sigma_i$ and $p_i$’s are the diagonal entries of $\Sigma$ and $P_T$, respectively.

We now consider the case of uncorrelated channel paths with the unequal spatial correlation matrices (case 3 in Section II). The following Lemma specifies the structure of the precoder in this case.

**Lemma 3:** The $M(P+L) \times M(P+L)$ linear precoding matrix, $W_T$, that maximizes the ergodic capacity in (14) for the channel model in (8) is a block diagonal matrix $W_T = \text{diag}(W_i)$, with $(P+L)$ optimal $M \times M$-matrices $W_i = \Phi_i \Sigma_i^{1/2} \Delta_i$, where $\Delta_i$’s are random matrices, $\Phi_i$’s are diagonal matrices, and $\Sigma_i$’s are diagonal matrices resulting from eigen-decomposition of transmit correlation matrices $R_T$’s, $l=0,1,\ldots,(L-1)$.

**Proof:** By substituting (8) into (13), proof follows the same steps as that of Lemma 2 and we ignore it for brevity. \qed

In this case, the diagonal entries of $\Sigma$ can be obtained from the following power loading equations:

$$\sigma_i = [\mu - (\kappa \gamma \text{tr}(D)p_i^{(\text{mod}L+P)}\delta_i^{(\text{mod}M)})^{-1}]^+$$

where $p_i$’s are the diagonal entries of $P_T = \sum_{l=0}^{P+L-1}(P \cdot \text{diag}(\text{tr}(D))) \otimes I_M)(E_i)^T$ and $\delta_i$’s are the diagonal entries of $\Delta_i$’s which are the result of eigen-decomposition of $R_T = \sum_{l=1}^{P+L-1} \begin{pmatrix} R_T \cdot E_i \end{pmatrix}^T = \text{diag}(\tilde{\Delta}_i) \Phi_i$, $i = 0,1,\ldots,(P+L-1)$.

In other words, when the spatial correlation matrices are not the same for different channel paths, the structure of the precoder is also block-diagonal and therefore it can be decoupled into $(P+L)$ frequency-flat $M \times M$ precoders. However, the construction of the $(P+L)$ precoders requires to solve the eigen-decomposition of an $M(P+L) \times M(P+L)$ matrix $R_T$, (or equivalently, $L$ different transmit correlation matrices of size $M \times M$), and hence, it is more complicated than the small $(M \times M)$ eigen-decomposition for transmit correlation matrix $R_T$ in the previous case of similar spatial correlation matrices.

Now we consider the last case, i.e. the case of correlated channel paths. For that we plug (7) into (12) to get the upper-bound similar to (13) for the case of uncorrelated channel paths. Like the previous cases, the following Lemma indicates the structure of the precoder in this case.

**Lemma 4:** The linear precoding $M(P+L) \times M(P+L)$ matrix, $W_T$, that maximizes the upper bound on ergodic capacity in the case of correlated channel paths has the form $W_T = \Phi \Sigma^{1/2} \Gamma$, where $\Gamma$ is an arbitrary unitary matrix, $\Sigma$ is a diagonal matrix and $\Phi$ is a unitary matrix that can be calculated from Kronecker product of the eigenvector matrices of transmit and path correlation matrices ($R_T$ and $R_F$).

**Proof:** Proof follows the same steps as that of Lemma 2 and we ignore it for briefness. \qed

In this case, the optimal linear precoding matrix $W_T$ is an eigen-beamformer with orthogonal beams pointing to a matrix that is a function of $R_F \otimes R_T$ and can be written as:

$$\Phi = \sum_{l=0}^{P+L-1}(E_i^T)^T \Phi_0 E_i^T$$

where $\Phi_0$ is the eigen-vector matrix of $R_F \otimes R_T$, i.e. $R_F \otimes R_T = \Phi_0 \Phi_i^T$. The diagonal matrix $\Sigma$ can be found via a power loading on each of the eigen-beamformers:

$${\sigma_i} = \left[\mu - (\kappa \gamma \text{tr}(D) \delta_{i+1})^{-1}\right]^+ = \left[\mu - (\kappa \gamma \text{tr}(D) r_{i+1})^{-1}\right]^+$$

where $i=1,2,\ldots,M(L+P)$, $\sigma_{i+1}$ and $\delta_{i+1}$ are the $(M(i+1))$th diagonal elements of $\Sigma$ and $\Delta$, respectively ($\Delta$ is the eigenvalue matrix of $R_F \otimes R_T$, $r_{i+1}$’s and $\delta_{i+1}$’s are the eigenvalues of $R_T$ and $R_F$, and $\mu$ is a constant determined by the power constraint.

From Lemmas 2-4, the general precoding structure for frequency-selective fading channels can be expressed as $W_T = \Phi \Sigma^{1/2} \Gamma$ where $\Gamma$ is an arbitrary unitary matrix that can be set to identity for simplicity, and the unitary matrix $\Phi$ and the diagonal matrix $\Sigma$ have the following structures:

- For uncorrelated channel paths with equal spatial correlation matrices, $\Phi = I_{P+L} \otimes \Phi_0$ where $\Phi_0$ is the eigen-vector matrix of $R_T$, $\Sigma = I_{P+L} \otimes \Sigma_0$ and $\Sigma_0$ can be found from (21).
- For uncorrelated channel paths with unequal spatial correlation matrices, $\Phi = \text{diag}(\Phi_i)$ where $\Phi_i$’s are the eigen-vector matrices of $R_T$, $i = 0,1,\ldots,P+L-1$, $\Sigma_0 = \text{diag}(\Sigma_0)$ and $\Sigma_0$ can be found from (22).
- For correlated channel paths: $\Phi = \sum_{l=0}^{P+L-1} (E_i^T)^T \Phi_0 E_i^T$ where $\Phi_0$ is the eigen-vector matrix of $R_F \otimes R_T$, and $\Sigma$ can be found from (24).

**IV. Numerical Results**

We consider the capacity improvement of our precoding scheme over a MIMO frequency-selective channel with equal-power paths, and different number of transmit antennas and paths: $N=2$, $M=2$, 4 and $L=2$, 4 and $P=6$.

For a frequency-selective fading channel that is highly correlated in space and frequency, Figure 1 shows the capacity-
versus-SNR curves for three systems: without precoding, with spatial precoder [5], and with the proposed precoder. The simulation results indicate that the proposed precoder outperforms the spatial precoder and achieves better capacity with increased numbers of channel paths and transmit antennas. Furthermore, the gain in achievable capacity of the proposed precoder is increased with larger values of $L$ and $M$ due to the mitigation of path correlation by the proposed precoder, e.g., as compared to the spatial precoder, at SNR=10dB, the proposed precoder provides a capacity gain of about 10% for $M=L=2$, and 25% for $M=L=4$.

Figure 1. Ergodic capacity with highly correlated transmit and receive antennas and channel paths

Although the objective function is ergodic capacity, Figure 2 also shows a considerable advantage in the system using the proposed precoding scheme in term of outage capacity as compared to systems using no precoding or spatial precoding.

Figure 2. Outage capacity with highly correlated transmit and receive antennas and channel paths

V. CONCLUSION

We proposed a precoder structure for MIMO systems in a frequency-selective fading environment, based on the knowledge of transmit and receive antenna and channel path correlations. Optimum precoding designs to maximize the ergodic capacity under constraint on transmitted power were developed for three different cases: uncorrelated channel paths with similar spatial correlation, uncorrelated channel paths with different spatial correlation, and correlated channel paths. The precoder structures in the cases of uncorrelated channel paths are composed of $P+L$ parallel precoders for frequency-flat fading channels. The power assignment to each precoder and the power allocation over the eigenmodes of each precoder was calculated based on the power of channel paths and eigenvalues of transmit correlation matrix. In the case of correlated channel paths, the precoder structure is in fact an eigen-beamformer with the beams refer to the eigenvectors of the Kronecker product of path and transmit correlation matrices. Furthermore, the power allocated to each eigenmode can be obtained from a water-pouring policy which is specified by the product of eigenvalues of transmit antenna and path correlation matrices.

Simulation results for MIMO systems in a frequency-selective fading environment with different scenarios indicate that the proposed precoder can increase the system ergodic capacity in presence of spatial and path correlations and its offered capacity gain is increased with the level of correlation and numbers of antennas and channel paths. The effectiveness of the proposed precoder is more pronounced in the environment with severe spatial and path correlation.

REFERENCES


