On the Expected Complexity Analysis of a Generalized Sphere Decoding Algorithm for Underdetermined Linear Communication Systems

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Abstract- This paper presents an analytical approach to evaluate the expected complexity of a generalized sphere decoding (GSD) algorithm, \( \lambda - GSD \), for underdetermined integer least-squares (ILS) problems. The analytical approach is used to derive the closed-form formulas to approximate the expected complexity of the \( \lambda - GSD \) algorithm, by utilizing special statistical properties in the transformed channel matrix. Analytical and simulation results in various scenarios are in good agreement, indicating that the proposed complexity analysis can be used as reliable complexity estimation for practical implementation of \( \lambda - GSD \) and can serve as reference for other GSD algorithms.

Keywords: Sphere Decoding (SD), Generalized SD (GSD), Maximum-likelihood (ML), Integer Least-Square (ILS), MIMO, \( \lambda \)-GSD.

I. INTRODUCTION

Sphere decoding (SD) [1], [2], [3] is an efficient searching method to obtain maximum-likelihood (ML) solution for NP-hard integer least-squares problems (ILS). SD can achieve polynomial complexity in number of unknowns for many practical communication problems with medium-to-high SNR ranges [4], [5], [6] and offers large complexity reduction over the exhaustive search method. However, when the ILS problem is underdetermined, zero elements appear in the diagonal terms of the upper-triangular matrix generated from QR or Cholesky decomposition before searching. As a result, the original SD cannot be used. Such underdetermined ILS problem arises in many applications, including MIMO detection with the number of transmit antennas larger than that of receiver antennas, MIMO detection with strongly correlated channel gains [7] or multi-user detection for overloaded CDMA-related systems (e.g., DS/MC-CDMA system when the number of active users is larger than the spreading length). To find optimal solution efficiently in such cases, generalized SD (GSD) algorithms, fully or partly based on SD, have been developed [7]-[11] for underdetermined or rank-deficient MIMO systems.

Acquiring the complexity order of an algorithm is significant for practical purpose: it can provide important indication on the feasibility in various practical scenarios. In underdetermined systems, the complexity has been shown only by simulation results for the GSD algorithms in [7]-[11]. Yet by simulation, their complexity is only measurable for small problem and/or constellation size or not-so-underdetermined systems. It is very time-consuming, if not impossible, to obtain complexity for large problems and/or constellations or heavily underdetermined systems, via simulation. To compensate for these issues, analytical approach has to be taken and only theoretical one can provide accuracy with confidence for complexity estimation in various scenarios. Thus, in this paper, we attempt to study analytically the expected complexity of the \( \lambda - GSD \) algorithm we previously proposed in [8] for underdetermined ILS problem. For the standard SD algorithm in full-column-rank systems, theoretical expected complexity studies have been carried out in [4], [5], [6] by assuming the distribution of the channel matrix follows i.i.d. Gaussian. Yet, even if i.i.d. Gaussian distribution is satisfied in the original underdetermined matrix, such analytical complexity results are not applicable to \( \lambda - GSD \), because the underdetermined ILS problem is transformed into a full-column-rank one before utilizing SD searching in \( \lambda - GSD \) algorithm, and the transformed channel matrix has some special structure and thus i.i.d. Gaussian assumption does not hold any more.

Therefore, in this paper, based on extension of the analysis framework in [4], [5], analytical closed-form formulas are derived to evaluate the expected complexity of \( \lambda - GSD \) [8] by taking into account the special statistical properties of the equivalent transformed channel matrix. Analytical and simulation comparison results with various system parameters confirm that the proposed analysis results can be used as reliable complexity estimation for practical implementation of \( \lambda - GSD \) and also can serve as references for other GSDs.

II. EXPECTED COMPLEXITY ANALYSIS FOR \( \lambda - GSD \)

The objective function of an underdetermined ILS problem in communications is,

\[
\min_{\lambda \in \mathbb{C}^n} \|y - \mathbf{H}x\|_2^2, \quad \text{subject to } y = \mathbf{H}x + \mathbf{v}, \quad (1)
\]

where \( y \in \mathbb{R}^m \) is the received vector, \( \mathbf{H} \in \mathbb{R}^{m \times n} \) is the underdetermined channel matrix with rank=\( m<n \), which is usually random and independent, and \( \mathbf{v} \) is a zero-mean AWGN vector. \( Z^n \) denotes the \( n \)-dim vector space with integer elements, of which \( A \) is a signal space with all possible \( n \times 1 \) transmitted vectors. GSD algorithms are developed to solve (1) efficiently. (1) is based on real systems but complex systems can be treated as real ones with doubled dimension (e.g.,[2], [3]). Thus, real systems are considered here for simplicity.

We previously proposed a GSD algorithm, \( \lambda \)-GSD, in [8] as an efficient detection scheme to solve(1). The basic idea is as follows. Partition \( \mathbf{H} = [\mathbf{H}_1 \mathbf{H}_2] \) where \( \mathbf{H}_1, \mathbf{H}_2 \) are \( m \times m \) and \( m \times (n-m) \) matrices respectively. Then partition \( x = [\mathbf{x}_1^T \mathbf{x}_2^T]^T \) so that the original linear model becomes \( y = [\mathbf{H}_1 \mathbf{H}_2][\mathbf{x}_1^T \mathbf{x}_2^T]^T + \mathbf{v} \). By introducing a trivial equation \( \mathbf{0} = \lambda \mathbf{x}_2 - \lambda \mathbf{x}_2 \) where \( \mathbf{0} \) is a zero-vector with length \( (n-m) \) and \( \lambda \) is a non-zero weighting factor, the original model \( y = \mathbf{H}x + \mathbf{v} \) can be further transformed to

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will be evaluated. The equivalent signal vector \( \mathbf{x} \) multiplied by the matrix \( \mathbf{H} \) is actually an i.i.d. Gaussian vector \( \mathbf{x} \), assuming the entries of \( \mathbf{H} \) are i.i.d. Gaussian distributed, and \( \mathbf{H} \) satisfies the condition in (2) is full-column-ranked. 

### S1. Selection of searching radius \( r \):

In the \( \lambda - \text{GSD} \) algorithm, for the transmitted vector \( \mathbf{x} \), we have the equivalent linear model, \( \bar{\mathbf{y}} = \mathbf{Hx} + [\mathbf{v}^T - \lambda \mathbf{x}_2^T]^T \), which leads to

\[
\| \bar{\mathbf{y}} - \mathbf{Hx} \|_2^2 = \| \mathbf{v} \|_2^2 + \lambda^2 \| \mathbf{x}_2 \|_2^2.
\]

By setting \( p_{fp} \) as a value slightly smaller than 1, it is desirable to set search radius \( r \) satisfying

\[
P(\| \bar{\mathbf{y}} - \mathbf{Hx} \|_2^2 = \| \mathbf{v} \|_2^2 + \lambda^2 \| \mathbf{x}_2 \|_2^2 \leq r^2) = p_{fp}.
\]

Since \( \mathbf{v} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}) \), \( \| \mathbf{v} \|_2^2 / 2\sigma^2 \) is a \( \chi^2 \) variable with \( m \) degrees of freedom. Based on the statistical property of \( \chi^2 \) distribution, the radius can be chosen as

\[
r^2 = \alpha m \sigma^2 + \lambda^2 \mathbb{E}[\| \mathbf{x}_2 \|_2^2].
\]

where the expected number of points in \( k \)-dim sphere of radius \( r \), \( E_p(k, r^2, \lambda) \), depends on \( \lambda \) and will be derived next.

### S2. Probability that a \( k \)-dim point \( x_{a,k^*} \) lies in a \( k \)-dim sphere of radius \( r \):

Now we need to know if an arbitrary \( x_a \) from the transmitted vector set, satisfies \( \| \bar{\mathbf{y}} - \mathbf{Hx} \|_2^2 \leq r^2 \) or

\[
\| \mathbf{H}(\mathbf{x} - x_a) + \mathbf{v} \|_2^2 + \lambda^2 \| \mathbf{x}_2 \|_2^2 \leq r^2 = \alpha m \sigma^2 + \lambda^2 \mathbb{E}[\| \mathbf{x}_2 \|_2^2].
\]

#### Lemma 1:

Given transmitted vector \( \mathbf{x} \), the probability that a \( k \)-dim point \( x_{a,k^*} \) lies in a \( k \)-dim sphere of radius \( r \) is

\[
P(\| x_{a,k^*} \|_2^2 \leq r^2 / \lambda^2) \text{, if } 1 \leq k \leq n - m
\]

\[
\gamma \left( \frac{r^2 - \lambda^2 \| x_{2a} \|_2^2}{2(\sigma^2 + \| x - x_a \|_2^2)} \right)^{\frac{k + m - n}{2}} \text{, if } n - m < k \leq n
\]

where \( k^* \) denotes the \( k \)-dim indices from \( (n-k+1) \) to \( n \).

#### Proof:

We start with \( k=n \). By defining \( \mathbf{w} = \mathbf{H}(\mathbf{x} - x_a) + \mathbf{v} \), (5) can be represented as \( \| \mathbf{w} \|_2^2 \leq r^2 - \lambda^2 \| x_{2a} \|_2^2 \). Elements of \( \mathbf{H} \) are i.i.d. and \( \sim \mathcal{N}(0, \mathbf{I}) \), and \( \mathbf{w} \sim \mathcal{N}((\mathbf{Hx} + \mathbf{v})^T, \sigma^2 \mathbf{I}) \), i.e., \( \mathbf{w} \) is actually an i.i.d. \( m \)-dim Gaussian vector. Therefore, \( \| \mathbf{w} \|_2^2 / (2\sigma^2 + \| x - x_a \|_2^2) \) is a \( \chi^2 \) random variable with \( m \) degrees of freedom. Thus we obtain

\[
P(\| x_{a,k^*} \|_2^2 \leq r^2 / \lambda^2) = \gamma \left( \frac{r^2 - \lambda^2 \| x_{2a} \|_2^2}{2(\sigma^2 + \| x - x_a \|_2^2)} \right)^{\frac{k + m - n}{2}} \text{.}
\]

Now consider \( k<n \). By QR decomposition, \( \mathbf{H} = \mathbf{QR} \) where

\[
\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{m \times m} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(n-m) \times (n-m)} \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} \mathbf{R}_{m \times m} & \mathbf{R}_{(n-m) \times m} \\ \mathbf{0} & \mathbf{I}_{(n-m) \times (n-m)} \end{bmatrix}
\]
(5) is equivalent to \( \| Q^H \hat{v} + R(x - x_n) \|_2^2 \leq r^2 \). Partitioning \( R = \left[ \begin{array}{c} \tilde{R}_{(n-k)(n-k)} \tilde{R}_{(n-k)k} \\ \tilde{R}_{nk} \end{array} \right] \), \( Q^H \hat{v} \triangleq u = \left[ \begin{array}{c} \tilde{u}_{n-k} \\ \tilde{u}_k \end{array} \right] \), we obtain
\[
\| Q^H \hat{v} + R(x - x_n) \|_2^2 = \begin{cases}
\| \tilde{u}_{n-k} \|_2^2 + \| \tilde{R}_{nk}(x - x_n) \|_2^2 + \| \tilde{R}_{nk} \|_2^2 & \text{if} \ \| \tilde{u}_{a,k} \|_2 \leq r^2,
\| \tilde{R}_{nk}(x - x_n) \|_2 & \text{if} \ \| \tilde{u}_{a,k} \|_2 > r^2.
\end{cases}
\]
\[
(8)
\]
For a \( k \)-dim sphere, we need to check if \( \| \tilde{u}_{k+1} \|_2 \leq r^2 \). It is equivalent to
\[
\| \tilde{u}_k + R_{(n-k)(n-k)}(x - x_n) \|_2^2 \leq r^2,
\]
where the LHS of (9) is the 2nd term in (8).

For \( 1 \leq k \leq n - m \), we have \( R_{nk} = \lambda \tilde{R}_{nk} \), \( \tilde{u}_k = -\lambda x_{a,k} \) due to \( Q = \left[ \begin{array}{c} Q_{nm \times n} 0 \\ 0 \end{array} \right] \). Hence, (9) becomes
\[
\| -\lambda x_{a,k} \|_2^2 + \| x - x_n \|_2^2 \leq r^2 \Rightarrow \lambda^2 \| x_{a,k} \|_2^2 \leq r^2 \Rightarrow P(x_{a,k} \text{ lies in the } \| x \|_2 \leq r^2)
\]
\[
= P \left( \| x_{a,k} \|_2^2 \leq r^2 \right) = P \left( \| x_{a,k} \|_2 \leq r \right)
\]
\[
(10)
\]
For \( (n - m) < k < n \), (9) can be equivalently written as
\[
\| \tilde{Q}^H \left( Q^T - \lambda \tilde{x}_a^T \right)^T \|_2^2 + \| R_{nk}(x - x_n) \|_2^2 \leq r^2 \Rightarrow \lambda^2 \| \tilde{x}_a \|_2^2 \leq r^2
\]
\[
(11)
\]
Since \( \tilde{H} \) is a i.i.d. random matrix and the distribution of each column is rotationally invariant from the left, the distribution of \( R_{nk} \) is the same as that of upper-triangular matrix obtained from the QR factorization of the submatrix
\[
\tilde{H}_{nk} \quad \tilde{H} = \left[ \begin{array}{c} \tilde{H}_{(n-k)(n-k)} \tilde{H}_{(n-k)k} \\ \tilde{H}_{nk} \end{array} \right] \quad \text{and} \quad \tilde{H}_{nk} = \tilde{H}_{(n-k)(n-k)} \tilde{H}_{(n-k)k} \tilde{H}_{nk}
\]
\[
(12)
\]
\[
43
\]
\[
(13)
\]
\[
\]
4-PAM (16QAM) constellation: For the 4-PAM signal set \{±3/2, ±1/2\} we name an element as a corner point if it is ±3/2 and as a centre point when it is ±1/2, as in [4].

When \(1 \leq k \leq n - m\), (6) in Lemma 1 is actually

\[
P(\|x_{n,k}\|^2 \leq r^2 / \lambda^2 = a \sigma^2 / \lambda^2 + \|x_2\|^2).
\]

(17)

While when \(n - m < k \leq n\), (6) becomes

\[
\gamma(\frac{a \sigma^2 + \lambda^2 (\|x_2\|^2 - \|x_{2n}\|^2)}{2a^2 + \|(x - x_{n})_k\|^2}, \frac{k + m - n}{2}).
\]

(18)

It’s more complicated than 2-PAM: First, let’s consider \(1 \leq k \leq n - m\) and calculate the expected number of points in each \(k\)-dim sphere. Assume \(k_2\) out of \(k\) elements of \(x_{a,k}\) are corner points and \(k_1\) out of \((n - m)\) elements of \(x_2\) are corner points, where \(0 \leq k_2 \leq k\) and \(0 \leq k_1 \leq (n - m)\). Then (17) is

\[
P\left(\{k_2/4 + (k - k_2)/4 \leq r^2 / \lambda^2 + 9k_1/4 + (n - m - k_1)/4\}\right).
\]

(19)

Define \(\delta(k_1, k_2) = \begin{cases} 1, & \text{if } 2(k_2 - k_1) \leq \frac{r^2}{\lambda^2} + \frac{(n - m - k_1)}{4} \leq 0 \\ 0, & \text{else} \end{cases}\)

and then the number of points fall into the \(k\)-dim sphere

\[
E_p(k, r^2, \lambda)_{1 \leq k \leq n} = \frac{1}{2^m} \sum_{k_1=0}^{n-m} \sum_{k_2=0}^{k-k_1} \sum_{k_2=0}^{k-k_1} 2^k \delta(k_1, k_2).
\]

(20)

When \(n - m < k \leq n\), we adopt the modified Euler’s generating function technique \(^3\) introduced in [4] to enumerate the number of points with \(\|x - x_{a}\|^2 = q \) in (18). For an arbitrary corner point \(x_1\), the Euler’s generating polynomial is \(\theta_0(x) = 1 + x + x^4 + x^9\) and if \(x_1\) is a centre point, the polynomial is \(\theta_1(x) = 1 + 2x + x^4\). The power of \(\theta_0(x)\) and \(\theta_1(x)\) represents information on possible values of \(\|x - x_{a}\|^2\) when \(x_1\) is corner and centre point respectively. Assume that \(k_2\) out of \((n - m)\) elements of \(x_{a}\) are corner points. Based on the modified Euler’s generating polynomial, we have

\[
\|x_{2n}\|^2 = 9k_2/4 + (n - m - k_2)/4 = 2k_2 + (n - m)/4.
\]

For \(k_1\) corner points out of \((n - m)\) elements of \(x_2\) (i.e., \(x_{r} = \{m+1\}, \ldots, n\))

Similarly, \(\|x_2\|^2 = 2k_1 + (n - m)/4\). Now \(\|x_{2n}\|^2 - \|x_2\|^2\) in (18) is equal to \(2(k_1 - k_2)\). For \(k_2\) corner points out of the remaining \((k - (n - m))\) elements of \(x_{a}\), the total number of corner points \(x_{a}\) is \(k_1 + k_2\). As a result, the number of combinations with \(\|x - x_{a}\||^2 = q\) is given by the coefficient of \(x^q\) in the polynomial

\[
\left(\frac{k_1}{k_2}\right)\left(1 + x + x^4 + x^9\right)^{k_1 + k_2}\left(1 + 2x + x^4\right)^{k - (k_1 + k_2)}.
\]

Hence for \(n - m < k \leq n\), we have

\[
E_p(k, r^2, \lambda)_{n-m < k \leq n} = \sum_{q} \sum_{k_1=0}^{n-m} \sum_{k_2=0}^{k-k_1} \sum_{k_2=0}^{k-k_1} 2^k \delta(k_1, k_2)
\]

(21)

Thus, the expected complexity of \(\lambda - GSD\) for 4PAM is

\[
E_p(k, r^2, \lambda)_{1 \leq k \leq n} = \sum_{k=1}^{n} f_p(k) \cdot E_p(k, r^2, \lambda)_{1 \leq k \leq n} + \sum_{k=n+1}^{n} f_p(k) \cdot E_p(k, r^2, \lambda)_{n-m < k \leq n}
\]

III. ILLUSTRATIVE RESULTS

In this section, the average complexity under underdetermined flat-fading MIMO scenarios will be studied with \(n\) transmit and \(m\) receive antennas \((m<n)\). We use the “accelerate” SD algorithm in [12] as the sub-algorithm for \(\lambda - GSD\) to obtain simulation results. The received signal is represented as \(y = Hx + v\), where \(x\) is the transmitted vector from M-QAM and \(v\) is the \(m \times 1\) AWGN noise with variance \(E(v^H v) = 2\sigma^2 I_m\). \(H_{[hi]}\) is an undetermined \(m \times n\) matrix where \(h_{ij}\) follows Rayleigh fading, i.e., \(h_{ij} \sim CN(0,1)\).

We define \(SNR = 10 \log_{10}(nE_s/(2q\sigma^2))\) dB, where \(E_s\) is the average M-ary QAM energy. The complex model is transferred
into a real one with $2n$ unknowns (from $L = \sqrt{M}$-PAM) before using $\lambda - \text{GSD}$. The average FLOPS for SD searching process are counted as simulation complexity measurement. Three metrics are to be compared to verify the proposed analysis: The simulated number of searching FLOPS for $\lambda - \text{GSD}$ without and with ordering algorithm [13] and the analytical expected number of FLOPS.

The analytical expected number of FLOPS, $C(m, n, \sigma^2, \lambda)$, is calculated for QPSK and 16QAM. We will compare the ratio of the Log value of the complexity $C$ (analytical or simulation) with the base of $L$ (L-PAM or $L^2$-QAM), for all the analytical and simulation results. The comparison results with various problem size $(m,n)$, SNR and $\lambda$ will be presented for QPSK and 16QAM. To calculate their analytical expected complexity, we roughly consider an $(m,n)$ QPSK system as a $(2m,2n)$ 2-PAM and an $(m,n)$ 16QAM as a $(2m,2n)$ 4-PAM system. The number of FLOPS/point in (16) and (21), is set to $f_p(k) = 2k + 9 + 2L$, for the standard SD in [4]. In the algorithm considered in [4], the searching starts from left to right, according to natural ordering. The SD algorithm [12] used in our simulation is slightly different, e.g., the searching in each layer starts from the ZF point and follows zig-zag order in the simulation. Hence, the actual $f_p(k)$ in our simulation platform could be slightly different from the one we used in analysis, but the difference is small compared to the overall FLOPS and we expect a negligible effect on the comparisons.

The expected complexity for constant-modulus QPSK is independent of $\lambda$ which is confirmed by simulations. Figure 3 also has simulation result of the GSD alg in [7], which can be treated with similar transformation to $\lambda$-GSD for QPSK. We can see that for QPSK, GSD in [7] may be more efficient than $\lambda$-GSD for some $\alpha$ (e.g., $\alpha = 1$ in [7]), but its complexity is more sensitive to $\alpha$.

Figure 3 is the complexity for QPSK vs. $\lambda$. As indicated by (16), the expected complexity for constant-modulus QPSK is independent of $\lambda$ which is confirmed by simulations. Figure 3 also has simulation result of the GSD alg in [7], which can be treated with similar transformation to $\lambda$-GSD for QPSK. We can see that for QPSK, GSD in [7] may be more efficient than $\lambda$-GSD for some $\alpha$ (e.g., $\alpha = 1$ in [7]), but its complexity is more sensitive to $\alpha$.

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4 The alg. in [7] transforms M-QAM’s into multiple QPSK’s before using SD while in $\lambda$-GSD, SD is directly used on M-QAMs without problem expansion. Thus, for QPSK they have similar performance and complexity. Yet, as M increases, the complexity [7] increases a lot due to the problem expansion.
Figure 4 16QAM: (log(C)/log(L)) Vs. n @SNR=28dB, m=2, λ=λ∗

The analytical and simulation results are in a good agreement for a 16QAM system with m=2, SNR=28dB, λ=λ∗=[0.0205, 0.0167, 0.0152, 0.0144] and n=3, 4, 5 and 6. λ∗ represents the choice of λ based on the criteria proposed in [8] to keep the performance degradation less than 1% compared to the ML performance. The analytical result is close to the simulation one for various n. Again, the simulation results show lower complexity than analysis with larger (n-m).

IV. CONCLUSIONS

This paper presents an analytical approach to derive closed-form formulas for the expected complexity of λ-GSD algorithm proposed in [8] for underdetermined ILS problem. Simulation was also used to obtain measured complexity results on some GSD algorithms in different scenarios. The simulation and analytical results are in good agreement, indicating that the proposed analysis is a versatile and effective tool to estimate the complexity of λ−GSD, especially for large systems where simulation becomes impractically long.

REFERENCES