Joint Channel Estimation and Synchronization with Inter-carrier Interference Reduction for OFDM

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Abstract- This paper proposes a pilot-aided joint channel estimation and synchronization scheme for burst-mode orthogonal frequency division multiplexing (OFDM) systems. The scheme eliminates the need of an IFFT block while keeping the low number of parameters to be estimated for low complexity without sacrificing the performance and convergence speed. For fast convergence and high performance, we develop a linearized cost function of the carrier frequency offset (CFO), sampling clock frequency offset (SFO) and channel impulse response (CIR) coefficients based on received signal samples and pilot tones in frequency domain and the corresponding recursive least square (RLS) estimation and tracking algorithm. For channel responses, CIR coefficients are estimated to benefit their low number and then transformed to the channel transfer function in order to keep low complexity. The ICI introduced by rotation due to CFO and SFO is analyzed and modeled, and a simple maximum-likelihood (ML) scheme based on the preamble is developed for coarse estimation of initial CFO and SFO values to be used in suppression of dominant ICI effects and in fine RLS estimation and tracking. Simulation results demonstrate that, in large practical ranges of CFO and SFO values, the proposed pilot-aided joint channel estimation and synchronization scheme provides a receiver performance remarkably close to the ideal case of perfect channel estimation and synchronization.

Index terms- Orthogonal frequency division multiplexing (OFDM), joint channel estimation and synchronization, recursive least square (RLS) algorithm, inter-carrier interference (ICI), channel impulse response, carrier frequency offset, sampling frequency offset.

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) techniques have been employed intensively in various broadband communications systems for their robustness and high spectral efficiency in frequency-selective fading channels. However, along with the potential benefits of the multi-carrier based transmission, the inherent drawback is its vulnerability to synchronization errors such as the carrier frequency offset (CFO) and the sampling clock frequency offset (SFO). As CFO, SFO and channel distortion have mutual effects, joint channel estimation and synchronization could provide better accuracy at the cost of high complexity. Various joint channel estimation and synchronization techniques have been recently proposed. In [1], [2], SFO is assumed to be zero, while CFO was excluded in [3]. In [4], both CFO and SFO are considered in a joint synchronization and channel estimation scheme performed in time domain (TD) to reduce the number of channel coefficients to be estimated. The TD joint estimation of channel distortion, CFO and SFO parameters requires the TD version of the recovered signals for adaptive computation, and hence, needs an IFFT block, equivalent to an OFDM modulator [4]. To reduce the complexity, it is desired to avoid this IFFT by performing joint estimation of CFO, SFO and channel response in frequency domain. However, as shown in this paper, CFO and SFO introduce rotations in time domain, which create large inter-carrier interference (ICI) in frequency domain and hence greatly degrade the estimation performance. Furthermore, the number of coefficients of the channel impulse response (CIR) is much smaller than that of channel transfer function (or channel frequency response).

This paper proposes a pilot-aided joint channel estimation and synchronization scheme that eliminates the need of an IFFT block while keeping the low number of parameters to be estimated for low complexity without sacrificing the performance and convergence speed. For fast convergence and high performance, we develop a linearized cost function of the SFO, CFO and CIR coefficients based on received signal samples and pilot tones in frequency domain and the corresponding recursive least square (RLS) estimation and tracking algorithm. For channel responses, CIR coefficients are estimated to benefit their low number and then transformed to the channel transfer function in order to keep low complexity. The ICI introduced by rotation due to CFO and SFO is analyzed and modeled, and a simple maximum-likelihood (ML) scheme based on the preamble is developed for coarse estimation of initial CFO and SFO values to be used in suppression of dominant ICI effects and in fine RLS estimation and tracking. Simulation results demonstrate that, in large practical ranges of CFO and SFO values, the proposed pilot-aided joint channel estimation and synchronization scheme provides a receiver performance remarkably close to the ideal case of perfect channel estimation and synchronization.

The rest of the paper is organized as follows. Section II describes the system model and analyzes the effects of CFO, SFO and channel distortion. Based on these results, the ICI reduction technique is proposed in Section III along with the analysis of residual ICI to show the feasibility of joint channel estimation and tracking.
estimation and synchronization in frequency domain. Section IV presents the derivations and development of the RLS joint channel estimation and synchronization algorithm. Section V derives the ML scheme based on the preamble for coarse estimation of initial CFO and SFO values. Simulation results for various conditions and schemes along with Cramer-Rao lower bounds (CRLB) are presented and discussed in Section VI. Finally, Section VII provides concluding remarks.

II. SYSTEM MODEL

Consider an OFDM transmitter shown in Fig. 1a. Let $X_m(k)$ be the complex-valued data symbol conveyed by the sub-carrier $k$ of the $m$-th OFDM symbol. Each OFDM symbol consists of $K<N$ information bearing sub-carriers, where $N$ is FFT size. After CP insertion and D/A converter, the transmitted baseband signal can be represented as

$$s(t) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=-K/2}^{K/2-1} X_m(k)e^{\frac{j2\pi kn}{N}} U(t - mT),$$

where $T$ is the sampling period at the output of IFFT, $N_g$ denotes the number of CP samples, $T_e = N_g T$, $T_s = (N/N_g)T$ is the OFDM symbol length after CP insertion, $u(t)$ is the unit step function, and $U(t) = u(t) - u(t - T_e)$. At the receiver, after CP removal, the received samples at rate $1/T$ in time domain can be determined by

$$r_{m,n} = \frac{e^{\frac{j2\pi n}{N}}}{N} \sum_{k=-K/2}^{K/2-1} X_m(k)H(k)e^{\frac{j2\pi kn}{N}} + w_{m,n}$$

where $n = 0, 1, ..., N - 1$, $N_m = N_g + m(N/N_g)$. The complex-valued Gaussian noise sample, $w_{m,n}$, has zero mean and variance of $\sigma^2$. $H(k) = \sum_{i=0}^{L-1} h_i e^{\frac{j2\pi kn}{N}}$ is the channel frequency response where $\{h_0, h_1, ..., h_{L-1}\}$ represents the effective channel impulse response (CIR). To completely remove ISI, the CP length must be longer than the channel length, $L$. The SFO and CFO terms are expressed in terms of the transmit sampling interval $T$ as $\eta = \Delta T/T$, $\Delta T = T' - T$ and $\epsilon = \Delta NT$, respectively, and $\eta = (1 + \eta)\epsilon$. In practice, the sampling timing error, $\Delta T$, and carrier frequency error, $\Delta f_c$, are usually kept very smaller than the sampling interval, $T$, and RF carrier frequency, $f_c$ respectively. However, since the RF carrier frequency is normally much higher than the sampling frequency $1/T$, the SFO term $\eta << 1$ but the CFO term $\epsilon$ can be large and $\eta = \epsilon$.

Unlike the traditional FD channel estimation, in the hybrid FD-TD estimation, the CIR, $\{h_0, h_1, ..., h_{L-1}\}$, is estimated based on the observations of received sub-carriers in FD.

Fig. 1: Proposed architecture of joint CIR/CFO/SFO estimation for burst-mode OFDM transmission.

After FFT, the received FD sample is $Y_m(k) = \sum_{n=0}^{N-1} r_{m,n}e^{-\frac{j2\pi nk}{N}}$.

After some straightforward manipulations, we obtain

$Y_m(k) = \sum_{i=0}^{L-1} X_m(i)H(i)e^{\frac{j2\pi ik}{N}} + \delta_{\theta} + W_m(k)$

where

$W_m(k) = \sum_{n=0}^{N-1} w(n+N_m)e^{-\frac{j2\pi nk}{N}}$, $\epsilon = i\eta + \epsilon_\eta \cdot \sin(c(x)) = \frac{\sin(\pi x)}{\pi x}$

and $\delta_{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{j2\pi i\eta}{N}(x-k)} = \sin(c(x) + i - k) e^{j\epsilon(x) + i - k}$.

In the first summation in (3), the sub-carrier of interest corresponds to the term $i=k$, while other terms with $i\neq k$ represent ICI. It can be seen that, in an ideal case with zero SFO and CFO, $\epsilon_\eta = 0$, $\delta_{\theta} = 1$ for $i=k$ and $\delta_{\theta} = 0$ for $i\neq k$, and $Y_m(k) = X_m(k)H(k) + W_m(k)$. It is noted that the derived expression for the ICI in (3) is more accurate than that in [7].

III. ICI REDUCTION BY TD CFO-SFO COMPENSATION

As shown in (2) and (3), SFO and CFO introduce rotation in time domain and both attenuation and ICI in frequency domain. Attenuation can be compensated in a symbol-by-symbol manner. However, removing ICI requires all detected symbols. Hence, ideally, it is better to remove the rotation in time...
domain to prevent ICI in frequency domain. Based on the derivations of (3), it is noted that only the common factor \( e^{-\frac{j2\pi c_n}{N}} \) and individual coefficients \( e^{-\frac{j2\pi m_k}{N}(1+\eta)} \) embedded in the summation at (2) result in the ICI in (3). The common factor can be removed from the received time-domain sample. However, the correction of the individual coefficients requires the knowledge of detected symbols in frequency domain, which is not available. Fortunately, the common factor has the major influence in \( \delta_a \) due to the large CFO term \( \epsilon \) while the effect of the individual coefficient is minor in \( \delta_a \) since the SFO term \( \eta \ll 1 \). As a result, to suppress the common factor, the received time-domain sample in (2) is multiplied by 
\[
e^{-\frac{j2\pi \eta c_n}{N}}
\]
prior to FFT as shown in Fig. 1,
\[
r_m e^{(k)} = r_m e^{-\frac{j2\pi \eta c_n}{N}},
\]
where \( \eta \) is the forgetting factor of the embedded in the iterations used for estimation, \( \epsilon \) is the sub-carrier index of the pilot tone currently used at the \( i \)-th iteration in the \( m \)-th OFDM data symbol, \( \lambda \) is the forgetting factor of the RLS algorithm, and \( X_m(k) \) is the used pilot tone value. It is noted that all tones are employed as pilot ones in preamble of a burst.

To make use of the linear RLS approach \[5\] for estimating the unknown CIR, CFO, SFO, the non-linear estimation error \( \epsilon \) needs to be linearized about the estimates at the \((i-1)\)-th iteration by using the first-order Taylor’s series approximation \[4\] as follows:
\[
e_i = Y_m(k) - \left[ f(X_m(k), \hat{\omega}_{i-1}) + \nabla f(X_m(k), \hat{\omega}_{i-1}) (\hat{\omega}_{i-1} - \hat{\omega}_{i-1}) \right],
\]
where
\[
f(X_m(k), \hat{\omega}) = X_m(k) \hat{H}(k) e^{\frac{j2\pi \eta c_n}{N}} \tilde{\delta}_{i/k},
\]
the weight vector at the \( i \)-th iteration \( \hat{\omega}_i = [\text{Re}\{\tilde{\delta}_{i/k}^{(1)}\}, \ldots, \text{Im}\{\tilde{\delta}_{i/k}^{(L+1)}\}] \) contains the estimates of CIR, CFO and SFO. The gradient vector, \( \nabla f(X_m(k), \hat{\omega}) \), is derived in Appendix.

Based on RLS approach \[5\], the iterative computation of these estimated parameters can be done as follows:

1. Initialize \( \hat{\omega}_0 \) and \( P_0 = \delta^4 I_{2L+2} \), where \( \delta \) is the regularization parameter, \( I_{2L+2} \) is the \((2L+2) \times (2L+2)\) identity matrix.

2. Update the parameters at the \( i \)-th iteration
\[
K_i = \frac{P_{i-1} \nabla f^T(X_m(k), \hat{\omega}_{i-1})}{\lambda + \nabla f^T(X_m(k), \hat{\omega}_{i-1}) P_{i-1} \nabla f^T(X_m(k), \hat{\omega}_{i-1})},
\]
\( \lambda \) denotes the forgetting factor
\[
P_i = \frac{1}{\lambda} (P_{i-1} - K_i \nabla f^T(X_m(k), \hat{\omega}_{i-1}) P_{i-1}),
\]
\( e_i = Y_m(k) - f(X_m(k), \hat{\omega}_{i-1}) \),

3. Update estimates at the \( i \)-th iteration
\[
\hat{\omega}_j = \hat{\omega}_{i-1} + e_i K_j.
\]

It should be noted that the use of RLS-based algorithm gives the joint estimation technique the rapid acquisition and low steady-state error. In burst-mode OFDM transmission, rapid acquisition will enable the estimation technique to function properly with reduced/short preamble length while the required error estimation is still achieved.

In the demodulator in Fig. 1, the CIR, CFO, SFO estimates are updated on a symbol-by-symbol basis for the ML sub-carrier.

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\( ^1 \) Estimation of CFO and SFO will be discussed in Section IV

\( ^2 \) The initial values are discussed in Section V
detector, while the tracking block updates the CIR, CFO and SFO estimates on an iteration-by-iteration basis. Since the number of CIR coefficients is much smaller than FFT size, a simplified FFT therefore can be employed to generate channel transfer function for both the ML sub-carrier detector in demodulator and reconstruction of the transmitted signal in tracking block.

V. ML CFO-SFO ESTIMATOR

Due to the possibility of multiple local minima caused by the non-linearity of the cost function, the initial guesses of estimated parameters for adaptive estimation must fall in a specific vicinity of their actual values. Consequently, the large initial errors between the initial guesses and true values would cause the instability of the RLS-based iterative computation. To alleviate such deterioration, we propose a simple ML CFO-SFO estimator to obtain coarse estimates of the initial CFO and SFO values after the acquisition phase by using the two long training symbols in preamble (IEEE 802.11a Standard in [6]). Let \( m_i \) and \( m_i + 1 \) be the time indices of the first and second long training symbols in preamble, respectively. Based on the FD observations in preamble, we define the following term,

\[
y(k) = \frac{X_{m_i}(k)Y_{m_{i+1}}(k)}{X_{m_{i+1}}(k)Y_{m_i}(k)} + \mathcal{E}(k),
\]

where \( N_p = N + N_g \) and the FD error sample, \( \mathcal{E}(k) \), can be expressed by

\[
\mathcal{E}(k) = \frac{X_{m_i}(k)W_{m_{i+1}}(k) - X_{m_{i+1}}(k)W_{m_i}(k)e^{j\frac{2\pi N_p}{N}\delta_k}}{X_{m_{i+1}}(k)X_{m_i}(k)H(k)e^{j\frac{2\pi(N_m+N_g)}{N}\delta_k}} + \delta_k.
\]

ICI parts are herein absorbed in \( W_m(k) \).

The FD error sample \( \mathcal{E}(k) \) can be approximated to be Gaussian distributed. As a result, based on the use of pilot tones of the two long training symbols, we define a ML cost function as follows,

\[
f(\varepsilon, \eta) = \sum_{k \in I_p} \left| Y(k) - e^{j\frac{2\pi N_p}{N}[\eta \epsilon + (1+\eta)]} \right|^2,
\]

where \( Y(k) \) is determined by the FD observations in (15) and \( I_p \) is the set of sub-carrier indices of pilot tones in preamble.

As a result, in the absence of CIR knowledge, the coarse estimates of CFO and SFO can be simply obtained by

\[
\hat{\epsilon}, \hat{\eta} = \arg \min_{\varepsilon, \eta} \sum_{k \in I_p} \left| Y(k) - e^{j\frac{2\pi N_p}{N}[\eta \epsilon + (1+\eta)]} \right|^2. \tag{17}
\]

As can be seen in (17), the complexity of the coarse CFO/SFO estimation depends on the step sizes for two-dimensional search over practical ranges of CFO and SFO. The above coarse CFO-SFO estimates are then used as initial CFO and SFO values for the RLS-based joint CIR, CFO and SFO estimation & tracking (in Section IV) while the initial guesses of CIR are set to zeros. These coarse CFO-SFO estimates are also used for ICI suppression in Section III.

VI. SIMULATION RESULTS AND DISCUSSIONS

Computer simulation was conducted to evaluate the performance of the proposed joint channel estimation and synchronization scheme. We set the OFDM system parameters based on the IEEE 802.11a uncoded systems. QPSK signal constellation was employed for OFDM symbols of 52 data sub-carriers and 4 equally spaced pilot tones of the same power. A burst format of two long identical training symbols and 225 data OFDM symbols was used in simulation. In the joint estimation implementation, to ensure the convergence of acquisition phase for iterative computation of a coarse CIR estimate, the elements of gradient vector corresponding to the CFO and SFO parameters were set to zeros in the first long training symbol. We also assumed that residual CFO values are yielded by correlation-based acquisition phase during the short training symbols in preamble. The propagation channel was assumed to be exponentially decaying Rayleigh fading one with the length of 5 and the RMS delay spread of 25ns (IEEE 802.15-00). In TD CFO/SFO compensator, the terms, \( \mathcal{E}^c \) and \( \eta^c \), are updated on a symbol-by-symbol basis by using the existing CFO and SFO estimates, respectively.

In Figs 2 and 3, the measured mean square errors (MSE) of the CIR, CFO and SFO estimates are plotted together with their corresponding CRLB values during the estimation period of the pilot-aided joint estimation approach. It is observed that the value of forgetting factor smaller than 0.99 causes the instability of the estimation approach. In addition, the numerical results demonstrate that the proposed estimation approach achieves the best performance in term of MSE values with infinite memory (forgetting factor equals to one) and regularization parameter, \( \delta = 10 \).

As an ultimate performance metric, the bit error rate (BER) of the ML-based sub-carrier detector is used to assess the pilot-aided joint estimation approach in various scenarios. In the demodulator, the ML-based sub-carrier detector yields an estimate of the transmitted FD data symbol, \( X_{m_i}(k) \), by minimizing (individually for each sub-carrier) the following metric over all possible transmitted sub-carrier values, \( X_m(k) \):

\[
\hat{X}_m(k) = \arg \min_{X_m(k)} \left| Y_m(k) - X_m(k)\hat{H}(k)e^{j\frac{2\pi}{N} n_s \delta_k} \right|^2. \tag{18}
\]

Fig. 4 demonstrates the detrimental effect of the synchronization error, CFO, on the OFDM system performance with QPSK constellation. In order to understand the contribution of various parameters to the system performance, we consider various cases as follows. As can be observed from Group A of BER values in Fig. 4, even with the use of the perfect estimates of CIR and SFO, the BER performance of the ML sub-carrier detector still degrades dramatically if CFO effect is neglected at receiver. Group B of BER values implicitly indicate the need of the ML CFO-SFO estimator to further widen the allowable range of CFO in the joint channel estimation and synchronization. Specifically, Group D of BER values exhibits a remarkably enhanced robustness of the ML sub-carrier detector against the residual CFO by using the pilot-aided joint estimation approach and the ML CFO-SFO.
estimator. As a result, in the presence of large residual CFO (up to 0.2) and SFO (up to 1123 ppm in simulation while the practical values of SFO is about 40ppm in IEEE 802.11a Standard), the BER performance corresponding to these imperfect estimates of CIR, CFO and SFO is very close to the BER under the ideal case of perfect channel estimation and synchronization (CFO=0, SFO=0). In addition, ICI-caused BER degradation is depicted by Group C of BER values where ICI components are completely preserved. In the absence of the TD CFO-SFO compensation, the BER performance degrades considerably as CFO is greater than 0.01 due to the dominant effect of ICI.

In Fig. 5, the BER performance of ML sub-carrier detector using these imperfect estimates from the proposed estimation algorithm is presented under various SNR values. For the case of CFO = 0.2123, relied on Group A of BER values, it is observed that the use of the ML CFO-SFO estimator is mandatory to perform symbol detection at receiver.

![MSE and CRLB of CIR estimates.](image2)

**Fig. 2:** MSE and CRLB of CIR estimates.

![MSEs and CRLBs of CFO and SFO estimates.](image3)

**Fig. 3:** MSEs and CRLBs of CFO and SFO estimates.

![BER performance of the ML sub-carrier detector with QPSK constellation under various SFO values.](image6)

**Fig. 6:** BER performance of the ML sub-carrier detector with QPSK constellation under various SFO values.
Furthermore, without the TD CFO-SFO compensation (Group B of BER values), the BER degradation increases significantly and the error floor appears as SNR value is greater than 25dB since ICI is completely preserved and becomes dominant (when the receiver noise power decreases). As can be seen from Group C of BER values, based on the use of the ML CFO-SFO estimator and the pilot-aided joint estimation approach with ICI reduction, the resulting BER values of ML sub-carrier detector is remarkably close to the BER values under the ideal case of perfect channel estimation and synchronization (CFO = 0 and SFO = 0). In other words, the TD CFO-SFO compensation is able to suppress ICI effectively in term of BER performance.

Fig. 6 exhibits the robustness of the pilot-aided joint estimation approach against a wide range of SFO values under CFO = 0.2123. It is noted the practical SFO values in IEEE 802.11a are only about 40 ppm while the BER values of Group D at SFO values up to 1000 ppm are still very close to the BER values under the ideal case of perfect channel estimation and synchronization (CFO = 0 and SFO = 0). Also, the detrimental effect of ICI is illustrated by Group B of BER values which are considerably greater than the BER values under the ideal case of perfect channel estimation and synchronization (Group E of BERs). In other words, in the absence of the TD CFO-SFO compensation, the ICI components are completely preserved and in turn degrade significantly the BER performance of the ML sub-carrier detector. Finally, Group C of BER values demonstrates the detrimental effect of SFO on the BER performance. As can be seen, in the presence of CFO and SFO, even under the use of perfect CIR and CFO estimates with ICI reduction, the BER performance still degrade dramatically if SFO effect is neglected at receiver side.

VII. CONCLUSION
In the paper, a pilot-aided approach is proposed to jointly estimate and track the CIR, CFO and SFO in burst-mode OFDM transmissions. Based on the hybrid FD-TD fashion, the proposed tracking algorithm possesses the inherent merits of both FD-based and TD-based estimation techniques. As a result, by using the CIR, CFO and SFO estimates from the proposed estimation approach, the BER performance of the ML-based symbol detector is remarkably close to that of the ideal case of perfect synchronization and channel estimation. In addition, a simple ML CFO-SFO estimator is proposed to further widen the allowable range of residual CFO and SFO values in the pilot-aided joint estimation approach. Finally, to mitigate ICI in FD, we propose the use of the TD CFO-SFO compensation, which is able to eliminate ICI effectively in term of the BER performance.

APPENDIX

DERIVATION OF THE GRADIENT VECTOR
The gradient vector in (9) with respect to the weight vector of estimated parameters $\hat{\phi}$ can be determined by

$$
\nabla f \left( X_m(k), \hat{\phi} \right) = \left[ \frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Re} \left[ h_k \right]}, \ldots, \frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Re} \left[ h_{L-1} \right]}, \frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Im} \left[ h_k \right]}, \ldots, \frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Im} \left[ h_{L-1} \right]}, \frac{\partial f(X_m(k), \hat{\phi})}{\partial \delta e}, \frac{\partial f(X_m(k), \hat{\phi})}{\partial \delta \eta} \right]^T.
$$

(A.1)

where

$$
\frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Re} \left[ h_k \right]} = X_m(k) e^{-\frac{2\pi}{N} l k \hat{\phi}} e^{\frac{2\pi}{N} \sum_{n=0}^{N-1} \frac{1}{N} \left( a_n + e^{2\pi \frac{\delta \varphi}{\pi} \hat{\varphi} \omega k \epsilon \right)},
$$

(A.2)

$$
\frac{\partial f(X_m(k), \hat{\phi})}{\partial \text{Im} \left[ h_k \right]} = X_m(k) e^{-\frac{2\pi}{N} l k \hat{\phi}} e^{\frac{2\pi}{N} \sum_{n=0}^{N-1} \frac{1}{N} \left( a_n + e^{2\pi \frac{\delta \varphi}{\pi} \hat{\varphi} \omega k \epsilon \right)},
$$

(A.3)

$$
\frac{\partial f(X_m(k), \hat{\phi})}{\partial \delta e} = X_m(k) \hat{H}(k) \left[ j \frac{2\pi}{N} N_m(1+\eta) e^{\frac{2\pi}{N} \sum_{k=1} k \hat{\phi}} - e^{\frac{2\pi}{N} \sum_{k=1} k \hat{\phi}} \right],
$$

(A.4)

$$
\frac{\partial f(X_m(k), \hat{\phi})}{\partial \delta \eta} = X_m(k) \hat{H}(k) \left[ j \frac{2\pi}{N} N_m(1+\eta) e^{\frac{2\pi}{N} \sum_{k=1} k \hat{\phi}} - e^{\frac{2\pi}{N} \sum_{k=1} k \hat{\phi}} \right].
$$

(A.5)

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