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STRAIN FIELD THEORY FOR VISCOELASTIC CONTINUOUS HIGH-SPEED WEBS WITH PLANE STRESS BEHAVIOR

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Abstract. *In this paper, we address to the problem of the origin of in-plane stresses in continuous, high-speed webs. In the case of thin, slender webs a typical modeling approach is the application of static in-plane stress approximation without considering the effects of in-plane velocity field. In the case of one-dimensional equations, we will study the effects of material viscoelasticity and Eulerian non-linearity of the transport velocity. Finite element solutions of the non-linear equation are presented with both elastic and viscoelastic material assumptions. Despite the limitations of the Kelvin-Voigt material assumption, fundamental coupling effects between viscoelasticity and velocity are visible. The strain behavior in the span under study is examined, and from both analytical and numerical results it is seen that the web strain is not constant during the span length. Results also indicate that the viscous properties of the material are closely connected to the overall tension level behavior in the stretched web span. Material time-dependency changes the web stress behavior: the span length, material viscosity and the web velocity cause significant effects, which are observed in the in-plane dynamics of the web.*

1 Introduction

In the handling of continuous, high-speed webs the origin of in-plane stresses creates a scientific problem, which is not yet completely understood. Especially, the type of the web material has a significant effect on both qualitative and quantitative characteristics of the in-plane stresses. Web tension in the moving continuous web systems can usually be controlled in the direction of the transport velocity, the tension being generated by a velocity difference between the starting and ending lines of the span. With high transport velocities, both web stress and web stability are under concern not only in this longitudinal direction but also in the direction perpendicular to the velocity in the plane of the web.

Since axially moving materials, such as strings, belts, beams, membranes and plates, have many applications in industry, e.g. in paper production, their mechanics have been studied widely. In processing of different kinds of thin, laterally moving solid webs, such challenges as efficiency of production and effects caused by high processing speed are met.

Research history of vibrations of travelling elastic materials goes back to the 1950's, when Sack [40] and Archibald and Emslie [1] studied transverse vibrations in a traveling string. In the 1960s and 70's, many researchers continued studies on moving strings and beams concentrating mainly on various aspects of free and forced transverse vibrations [31, 33, 34, 35, 43, 46]. Stability of small transverse vibrations of travelling two-dimensional rectangular membranes and plates have been studied by Ulsoy and Mote [51], and Lin [26]. When the web is advancing through processes without an external support, the inertial forces depending on the web speed are coupled with web tension. Also the transverse behavior of the web and the response in the flowing fluid (air) surrounding the web are coupled (see e.g. [7, 38]).

Lin and Mote studied an axially moving membrane in a 2D formulation, predicting the equilibrium displacement and stress distributions under transverse loading [27]. Later, the same authors studied the wrinkling of axially moving rectangular webs with a small flexural stiffness [28]. They predicted the critical value of the non-linear component of the edge loading after which the web wrinkles and the corresponding wrinkled shape of a web. It is also known, that lack of web tension will result in loss of stability in the moving web, which from the application viewpoint, disturbs required smooth advancing of the web (see e.g. [3, 4]). From the other hand, web tension too high may cause web breaks, which deteriorate production efficiency and the strength properties of the processeed material (see e.g. [2, 39, 41, 44]).

Considering wet paper material, the viscoelastic properties play an important role in the behavior of the web and, thus, are to be included in the model. The first study on transverse vibration of travelling viscoelastic material was carried out by Fung et. al. using a string model [15]. Extending their work, they studied the material damping effect in their later research [16].

Viscoelastic strings and beams have been studied recently exceedingly, see e.g. [30, 53]. Oh et al. studied critical speeds, eigenvalues and natural modes of the transverse displacement of axially moving viscoelastic beams using the spectral element model [25, 36]. Chen and Zhao [12] represented a modified finite difference method to simplify a non-linear model of an axially moving string. They studied numerically the free transverse vibrations of both elastic and viscoelastic strings. Chen and Yang studied free vibrations of viscoelastic beams travelling between simple supports with torsion strings [11]. They studied the viscoelastic effect by perturbing the similar elastic problem and using the method of multiple scales. Very recently, Yang et al. studied vibrations, bifurcation, and chaos of axially moving viscoelastic plates using finite differences and a non-linear model for transverse displacements [52].

Marynowski and Kapitaniak studied differences between the Kelvin-Voigt and Bürgers models in modeling of internal damping of axially viscoelastic moving beams. They found out that both models gave accurate results with small damping coefficients, but with a large damping coefficient, the Bürgers model was more accurate [29]. In 2007, they compared the models with the Zener model studying the dynamic behavior of an axially moving viscoelastic beam [30]. They found out that the Bürgers and Zener model gave similar results for the critical transport speed whereas the Kelvin-Voigt model gave a greater transport speed compared to the other two models.

The origin and structure of the tension distribution in a moving solid web seems to be an exceptionally unknown area. The often used models with the web materials are based on assumptions of isotropic or orthotropic material properties (see e.g. [5, 48]). Also, the web materials are often considered as viscoelastic or viscoplastic but there is no coupling between in-plane strain and web velocity effects (see e.g. [19, 37, 50]). Time-dependent, in-plane vibrations of a moving continuous membrane were studied by Shin et al. [42]. In their work, in-plane vibration modes of an isotropic web were studied between the traction lines. Also Guan et. al. have studied viscoelastic web behavior in both steady state and unsteady state cases [17, 18].

Usually, the partial time derivative has been used instead of the material derivative in the viscoelastic constitutive relations. Mockensturm and Guo suggested that the material derivative should be used [32]. They studied non-linear vibrations and dynamic response of axially moving viscoelastic strings. Kurki and Lehtinen suggested, independently, that the material derivative in the constitutive relations should be used in their study concerning the in-plane displacement field of a travelling viscoelastic plate [23]. In the study by Chen et al., the material derivative was used in the viscoelastic constitutive relations [8]. They studied parametric vibration of axially accelerating viscoelastic strings. Chen and Ding studied stability of axially accelerating viscoelastic beams using the method of multiple scales and the material derivative in the viscoelastic constitutive relations [13]. Chen and Wang studied stability of axially accelerating viscoelastic beams using asymptotic perturbation analysis and the material derivative in the viscoelastic relations [10]. In a recent research by Chen and Ding, the material derivative was also used to study dynamic response of vibrations of axially moving viscoelastic beams [9]. In their study, a non-linear model was used taking into account the coupling of the transverse displacement with the longitudinal (in-plane) displacement. However, the transverse behavior of the beam was their main focus.

In this paper, we will represent a study where the effects of material viscoelasticity and Eulerian non-linearity of the transport velocity U in are considered in the following one-dimensional equation:

$$\eta U \frac{\partial^3 u}{\partial x^3} + (E - \rho U^2) \frac{\partial^2 u}{\partial x^2} - \rho U \frac{\partial U}{\partial x} \frac{\partial u}{\partial x} = 0 \tag{1}$$

where η is viscosity, ρ is density of material, x is axial coordinate and u is the in-plane displacement. One fundamental observation of this study is the significance of strain-based boundary conditions; in the case on one-dimensional model, the strain (Dirichlet) boundary condition affects throughout the web thickness isolating the span under observation from other preceding or succeeding web spans.

2 Continuous web flow phenomenon

Continuous, moving web creates a flow continuum, which may be considered as a solid flow medium. Due to its solid nature, web continuum is always under the stress state, which is caused by the strain state. Using the conservation of mass, we get the following equation:

$$\frac{\partial \rho}{\partial t} + \rho \nabla U = 0\,. \tag{2}$$

with web density ρ and longitudinal velocity U.

Assuming the density ρ to be constant, using Eq. (2) we may construct the mass conservation law for the situation described in Figure 1. Because there is a longitudinal strain component ε in the web span under observation, Eq. (2) can be represented as follows:

$$\rho A_1 U_1 - \rho A_2 U_2 = 0\,, \tag{3}$$

where

$$A_2 = \frac{A_1}{1 - \varepsilon_{\mathrm{T}}} \tag{4}$$

and ε_{T} is the strain at the end of the span, i.e at the area A_2 in Figure 1. From Eqs. (3) and (4), we obtain

$$\varepsilon_{\mathrm{T}} = \frac{U_2}{U_1} - 1\,. \tag{5}$$

Flowing solid continuum in the case above is assumed to be controlled only in the direction of the transport speed, i.e. in the longitudinal x-direction. Note that Eq. (3) can be applied only in the steady-state situation of the flow, i.e. the web is assumed to flow smoothly and without time-dependent disturbances [18]. Also, the traction lines at the cross-sectional areas A_1 and A_2 are assumed to affect only at the surfaces of the web, i.e. the stress and strain waves advancing inside the web thickness can cross the traction lines. Therefore the boundary conditions of the moving continuous webs in reality are consisting rather complicated friction-based force transmission phenomena at web-roll contact areas [22].

3 One-dimensional viscoelastic in-plane moving continuum equations

In this article, material assumption of the web continuum is based on viscoelasticity. With fibrous, composite-type materials, the elasticity properties are result of complicated material pre-processing, which further results in orthotropic anisotropy with material time-dependency (see e.g. [6, 20, 37, 50]). One can derive a vast number of different rheological models for the time-dependent material behavior but fundamental behavior of continuous flow of the solid viscoelastic web can be analyzed by using the simple Kelvin-Voigt model. The principle of the Kelvin-Voigt model is described in Figure 2.

Stress-strain behavior of one-dimensional Kelvin-Voigt material is (see e.g. [14])

$$\sigma = E\varepsilon + \eta \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\,, \tag{6}$$

where σ denotes the stress, ε the strain, E the Young's modulus, and η the viscosity coefficient.

In the following, we represent a description of the strains and deformations. A standard method to describe the structural deformations is to use *material* assumption with static medium according to the placement of observer. Longitudinal movement of material creates in-plane deformations, the modelling of which is a real challenge, since the actual deformation is to be handled using *spatial* or mixed Lagrange-Euler description[24, 45]. This description is standard

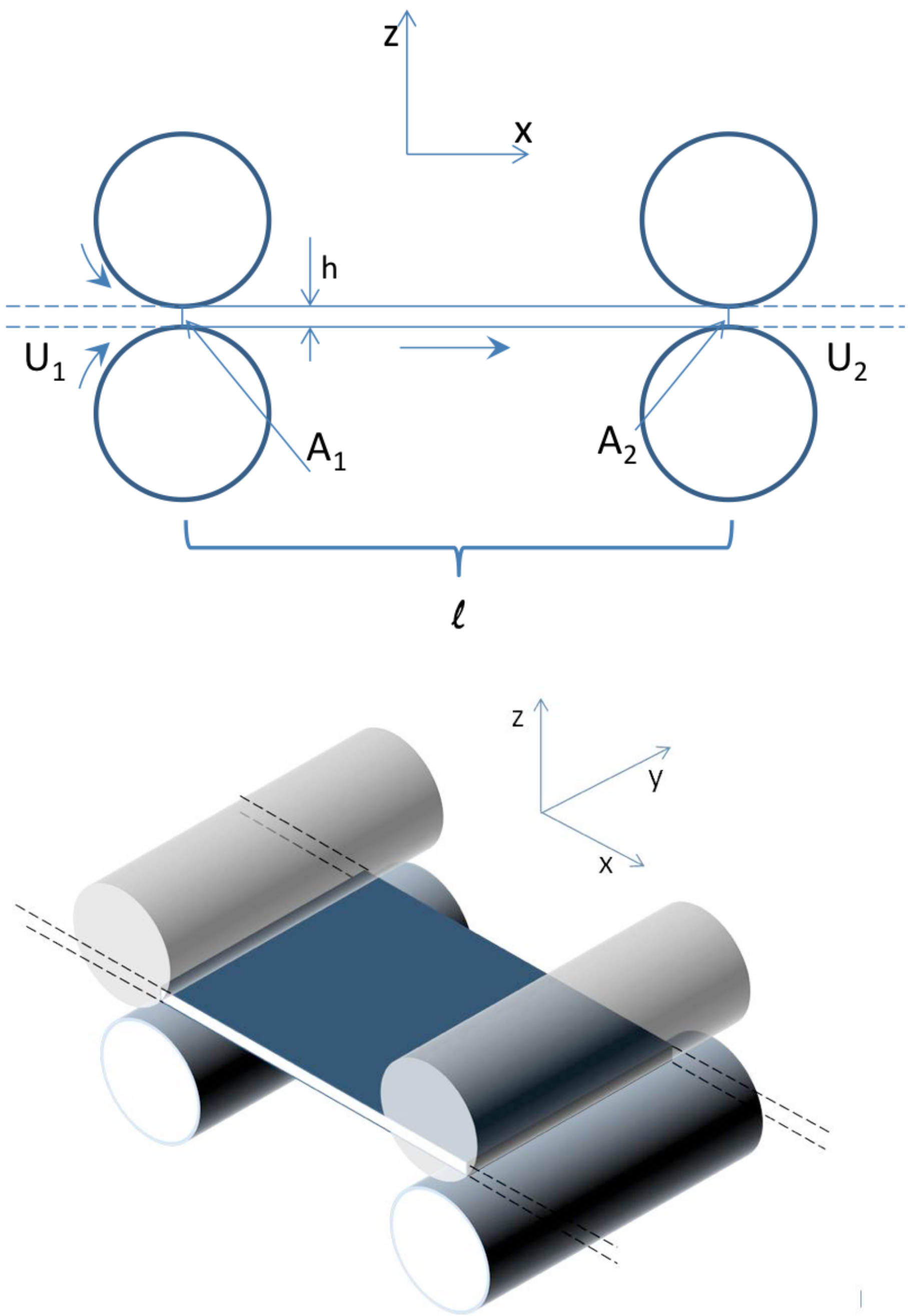


Figure 1: Solid web continuum flowing between the incoming and outgoing flow control areas A_1 and A_2 with longitudinal speeds U_1 and U_2 between the beginning and ending tractions lines, respectively.

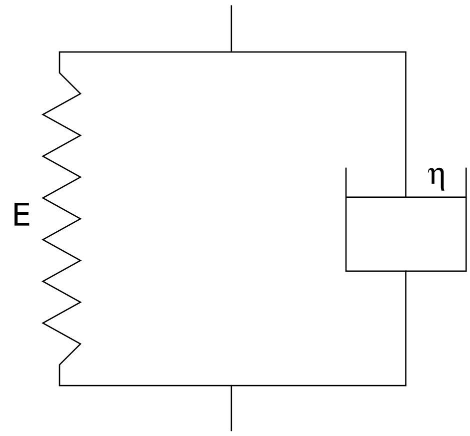


Figure 2: Kelvin-Voigt rheological model.

in fluid dynamics where the observer is *watching* a control volume where possible deformations will appear [21]. Using the same principle, we may construct a constitutive flow model for solid, anisotropic viscoelastic moving continuum. Therefore the strain ε is to be written in the Lagrange-Euler form:

$$\varepsilon = \varepsilon(x,t)\,. \tag{7}$$

The material derivative of strain ε is then

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\partial \varepsilon}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \varepsilon}{\partial t} = U\frac{\partial \varepsilon}{\partial x} + \frac{\partial \varepsilon}{\partial t}\,. \tag{8}$$

For time-dependent solid continuum flow, the following equation may be derived:

$$\rho\frac{\partial^2 u}{\partial t^2} + 2\rho U\frac{\partial^2 u}{\partial x \partial t} + \rho U^2\frac{\partial^2 u}{\partial x^2} + \rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right)\frac{\partial u}{\partial x} = E\frac{\partial^2 u}{\partial x^2} + \eta\left(\frac{\partial^3 u}{\partial x^2 \partial t} + U\frac{\partial^3 u}{\partial x^3}\right)\,. \tag{9}$$

If we assume that there is no time-dependent fluctuation in x-directional displacement u, we can represent a steady-state equation for ideal, undisturbed axial narrow web flow in the following form:

$$\eta U\frac{\partial^3 u}{\partial x^3} + (E - \rho U^2)\frac{\partial^2 u}{\partial x^2} - \rho U\frac{\partial U}{\partial x}\frac{\partial u}{\partial x} = 0\,. \tag{10}$$

With the assumption of linear Cauchy strains states

$$\varepsilon = \partial u/\partial x \tag{11}$$

and based on the linearized form of Eq. (10) ,we will get the equation

$$\eta U\frac{\partial^2 \varepsilon}{\partial x^2} + (E - \rho U^2)\frac{\partial \varepsilon}{\partial x} = 0\,. \tag{12}$$

The similarity between Eq. (12) and the heat convection equation

$$k_{\mathrm{T}}\frac{\partial^2 T}{\partial x^2} - \rho c_{\mathrm{p}} U\frac{\partial T}{\partial x} = 0 \tag{13}$$

in one dimension is apparent. In Eq. (13), T is temperature, U the spatial motion of the media surrounding the object under heat transfer, c_{p} the specific heat of the object, and k_{T} the heat diffusion coefficient [47].

4 Algebraic solution of the linearized steady-state case

The solution of Eq. (12) can be achieved using algebraic methods. If only pure elasticity is present, solution is of the form:

$$(E - \rho U^2)\varepsilon = C\,, \tag{14}$$

where C is constant. Thus, the solution obeys Hookean behavior, i.e. the strain ε is constant regardless of the level of the transport velocity U.

However, with nonzero viscosity, Eq. (12) becomes

$$\frac{\partial^2 \varepsilon}{\partial x^2} + \left(\frac{E - \rho U^2}{\eta U}\right)\frac{\partial \varepsilon}{\partial x} = 0\,. \tag{15}$$

With the boundary conditions $\varepsilon(0) = 0$, $\varepsilon(\ell) = \varepsilon_{\mathrm{T}}$, the algebraic solution of Eq. (15) is [22]

$$\varepsilon(x) = \varepsilon_{\mathrm{T}} \frac{1 - e^{-kx}}{1 - e^{-k\ell}}, \tag{16}$$

where

$$k = \frac{E - \rho U^2}{\eta U} \tag{17}$$

and ℓ is the length of the span under observation.

Analytical solution of the strain can be obtained only for the linearized one-dimensional case. Based on Eq. (8), we define the spatial strain in the steady-state case:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = U \frac{\partial \varepsilon}{\partial x}\,. \tag{18}$$

Now the x-directional stress σ appearing in the moving viscoelastic span based on the strain in Eq. (16) is a superposition of the elastic and viscous stress components:

$$\sigma = E\varepsilon_{\mathrm{T}} \frac{1 - e^{-kx}}{1 - e^{-k\ell}} + \eta U \varepsilon_{\mathrm{T}} \frac{k e^{-kx}}{1 - e^{-k\ell}}\,. \tag{19}$$

Substitution of (17) into (19), one gets

$$\sigma = \frac{\varepsilon_{\mathrm{T}}}{1 - e^{-k\ell}}(E - \rho U^2 e^{-kx})\,. \tag{20}$$

5 Numerical results by FEM

Numerical solution of the viscoelastic moving continuum problem is realized using the finite element method (FEM). The derivation of the FEM matrices is performed using the principle of virtual work. Virtual work δW can be calculated using the virtual strain $\delta\varepsilon^T$ as follows [54]:

$$\delta W = \int_V \delta\varepsilon^T \bar{\sigma}\,\mathrm{d}V\,. \tag{21}$$

In the finite element method, the connection between the strain vector ε and displacements u_e in the element nodes are defined by using strain-displacement approximation in a matrix B

$$\varepsilon = \mathbf{B} u_e \qquad \text{where} \qquad \mathbf{B} = \frac{\partial}{\partial x}\mathbf{N}_e\,. \tag{22}$$

In Eq. (22), $\mathbf{N}_e$ is a shape function matrix defining the displacement approximations inside the element. If the element is undergoing a virtual displacement δu_e, we can write using Eq. (21) (see e.g. [54]):

$$\delta W = \delta u_e \int_V \mathbf{B}^T \bar{\sigma} \, \mathrm{d}V \,. \tag{23}$$

However, the stress $\bar{\sigma}$ inside the volume is calculated using the stress strain behaviour of the viscoelastic in-plane moving continuum model. One-dimensional non-linear equation (10) will be solved using the finite element method.

While the velocity U is a function of x, the terms in Eq. (10) may be regrouped as follows:

$$\frac{\partial}{\partial x}\left[\eta U \frac{\partial^2 u}{\partial x^2} + (E - \rho U^2)\frac{\partial u}{\partial x}\right] + \rho U \frac{\partial U}{\partial x}\frac{\partial u}{\partial x} - \eta \frac{\partial U}{\partial x}\frac{\partial^2 u}{\partial x^2} = 0 \,. \tag{24}$$

We denote

$$\bar{\sigma} = \eta U \frac{\partial^2 u}{\partial x^2} + (E - \rho U^2)\frac{\partial u}{\partial x} \,. \tag{25}$$

Using the finite element method approximation presented in Eq. (22), the displacement operators in Eq. (25) can be written as

$$\bar{\sigma} = \left[\eta U \mathbf{B}_2 + (E - \rho U^2)\mathbf{B}_1\right] u_e \,, \tag{26}$$

where

$$\mathbf{B}_1 = \frac{\partial}{\partial x}\mathbf{N} \qquad \text{and} \qquad \mathbf{B}_2 = \frac{\partial^2}{\partial x^2}\mathbf{N}. \tag{27}$$

On the other hand, $\bar{\sigma}$ can be expressed with the help of strains (see Eq. (11))

$$\bar{\sigma} = \left[\eta U \mathbf{B}_1 + (E - \rho U^2)\mathbf{N}\right] \varepsilon_e \,, \tag{28}$$

where ε_e are the strains in the element nodes.

The substitution of Eq. (28) to Eq. (23) will result in

$$\delta W = \delta u_e \left[\int_V \mathbf{B}^T \left[\eta U \mathbf{B}_1 + (E - \rho U^2)\mathbf{N}\right] \mathrm{d}V\right] \varepsilon_e \,. \tag{29}$$

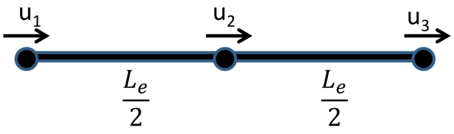
However, inside the element area the virtual energy $\delta W = \delta u_e F_e$, where F_e is the force vector, affecting on the element. The forces affecting the element can be represented as

$$F_e = \mathbf{K}_e \varepsilon_e \,,$$

where $\mathbf{K}_e$ is the following element stiffness matrix:

$$\mathbf{K}_e = A \int_0^{\ell_e} \left[\mathbf{B}^T \eta U \mathbf{B}_1 + \mathbf{B}^T (E - \rho U^2)\mathbf{N}\right] \mathrm{d}x \,. \tag{30}$$

The element used in the analysis is a 3-node quadratic rod element with three axial degrees of freedom. See Figure 3.



$$N_1 = 1 - \frac{3x}{L_e} + \frac{2\,x^2}{{L_e}^2}$$

$$N_2 = \frac{4x}{L_e} - \frac{4\,x^2}{{L_e}^2}$$

$$N_3 = -\frac{x}{L_e} + \frac{2\,x^2}{{L_e}^2}$$

Figure 3: Quadratic 3-node rod element and its corresponding shape functions.

The final global finite element equation is

$$F = \mathbf{K}u\,. \tag{31}$$

Vector u includes the displacements based on the boundary conditions presented in the Section 4. Boundary conditions for the axial system of Figure 1 are

$$u_{x=0} = 0 \qquad \text{and} \qquad u_{x=\ell} = \varepsilon_{\mathrm{T}}\ell = \left(\frac{U_2}{U_1} - 1\right)\ell\,. \tag{32}$$

The solution of Eq. (31) is now realized by substituting the displacements of Eq. (32) to appropriate places in the displacement vector u. The corresponding forces are calculated to force vector F by using elimination. Finally, the rest unsolved displacements are computed using $u = \mathbf{K}^{-1}F$. The non-linear term in Eq. (24) is handled as a body force applied to the element nodes. By this, the effect of the non-linear term can be solved as a non-linear force

$$F_{Be} = \int_V \mathbf{N}^T F_{nl}\,\mathrm{d}V\,. \tag{33}$$

The force F_{nl} originating from the non-linear term, will be calculated for each element:

$$F_{nl} = \rho U \frac{\partial U}{\partial x}\varepsilon - \eta \frac{\partial U}{\partial x}\frac{\partial \varepsilon}{\partial x}\,. \tag{34}$$

The final nodal forces F_{Be} for each element are individual and take into account the current displacement, velocity and velocity gradient inside of each element. The problem with these body forces is solved via the Newton–Raphson method.

Using Eqs. (16), (18), and (19), the strain and stress states of the one-dimensional viscoelastic beam can be calculated.We have used parameter values $E = 2.5 \cdot 10^7$ N/m^2, $\eta = 4.0 \cdot 10^5$ Ns/m^2, $U = 10$ m/s, span length $\ell = 1.0$ m and strain $\varepsilon_{\mathrm{T}} = 0.03$. The cross-directional area, which is under draw, is $A = 2.0$ m^2, and the web density is 0.16 kg/m^3.

For the exemplary parameter values above, the results obtained are shown in the Figures 4 – 7. The analytical solution (with a constant velocity U) is obtained from Eq. (12), and the FEM solutions from the discretized form of Eq. (10) with velocity depending on x. For the first Newton–Raphson iteration, the U was set constant, but U was updated during the iteration with the help of nodal strains. The number of nodal points used in the FEM was 600.

The strain distribution during the draw differs from the constant-strain presented in theory of elasticity [49]. See Figure 4. In this figure, one may also notice a slight difference between the analytical solution with constant U and the numerical iterated solution where the velocity U depends on x. However, the FEM solution from the first iteration (having constant U) and the analytical solution coincided as desired.

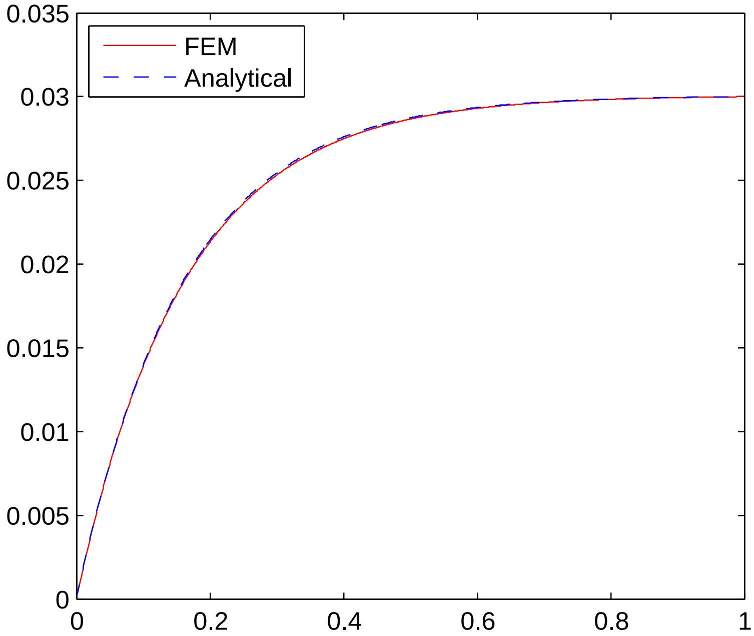


Figure 4: Analytical and numerical (FEM) solution of strain distribution during the length of the span.

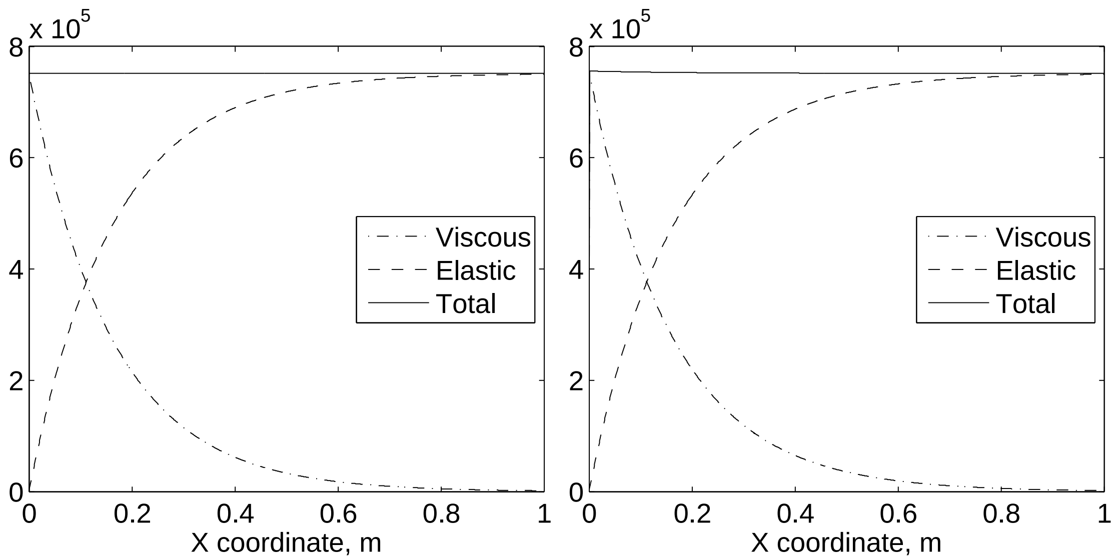


Figure 5: Analytical (left) and numerical (FEM) (right) solution of the stress distribution during the length of the span.

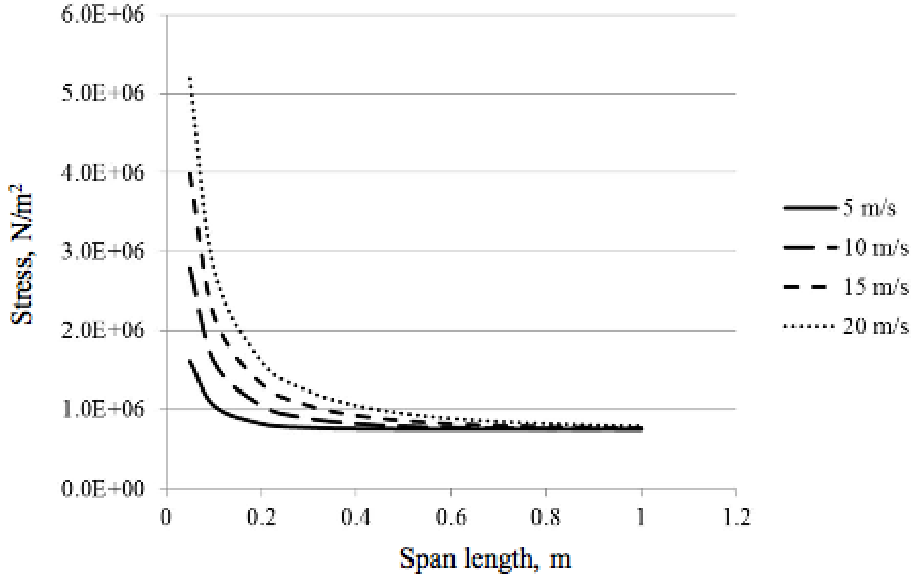


Figure 6: Effect of the span length to the web stress with different web speed levels.

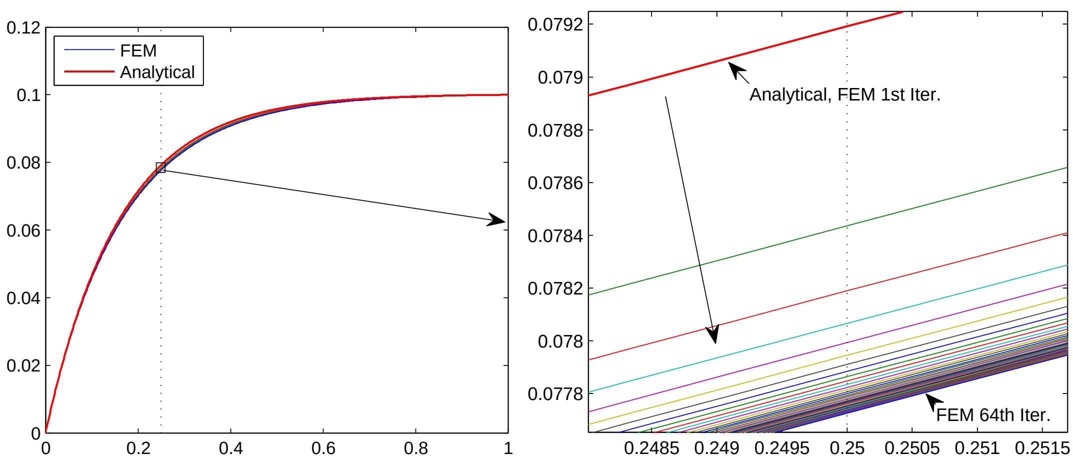


Figure 7: Strain distribution in the span and the effect of non-linearity. The strain at the end of the span is $\varepsilon_{\mathrm{T}} = 0.1$. On the right-hand side, the Newton–Raphson iteration is shown for some values of x.

Even though the strain distribution is not constant, the stress distribution is a combination of elastic and viscous forces based on Eq. (6) and it is *almost* constant. See Figure 5 and compare with the analytical solution in Eq. (20). The stress increases very slightly towards the traction line A_2.

The effect of the span length on the web stress state is visible in Figure 6. As seen, the shorter the processing time of the viscoelastic span, the higher the response of the time-dependent viscous force component. The effect of non-linearity is seen in Figure 7. However, the effect of non-linearity is relatively small.

In Table 1, numerical data in the case $\varepsilon_{\mathrm{T}} = 0.1$ is shown for some numbers of iterations. The value of the strain ε is collected for x = 0.25 m, 0.50 m, and 0.75 m (when the length of the span is $\ell = 1$ m). Also here, it is seen that the first iteration with constant velocity gives results that coincide with the analytical solution. This shows the accuracy of the FEM solution. When U is not constant, the strains seems to be slightly smaller than in the case when U is constant. See also Figure 7.

Table 1: Numerical data from the Newton–Raphson iteration. The value of the strain ε for some selected values of x and numbers of iterations (Iter.). At the first row, the analytical solution for the case of constant velocity is shown. The strain at the end of the span is $\varepsilon_{\mathrm{T}} = 0.1$. Compare with Figure 7.

Iter.	x (m)		
	0.25	0.50	0.75
Anal.	$7.9082{\cdot}10^{-2}$	$9.5768{\cdot}10^{-2}$	$9.9266{\cdot}10^{-2}$
1	$7.9081{\cdot}10^{-2}$	$9.5768{\cdot}10^{-2}$	$9.9266{\cdot}10^{-2}$
4	$7.7957{\cdot}10^{-2}$	$9.5119{\cdot}10^{-2}$	$9.9076{\cdot}10^{-2}$
16	$7.7683{\cdot}10^{-2}$	$9.4955{\cdot}10^{-2}$	$9.9026{\cdot}10^{-2}$
64	$7.7615{\cdot}10^{-2}$	$9.4914{\cdot}10^{-2}$	$9.9014{\cdot}10^{-2}$

6 Conclusions

In this paper, we presented models for handling of continuous, high-speed webs. We also took into consideration the type of the web material, which has a significant effect on both qualitative and quantitative characteristics of the in-plane stresses.

In this study, the effects of the material viscoelasticity and the Eulerian non-linearity were considered as a function of the transport velocity. Solutions of the one-dimensional non-linear equation were presented both with elastic and viscoelastic material assumptions. Finite element method (FEM) was used in the solution of the group of the second order PDEs.

Despite the limitations of the Kelvin-Voigt material assumption, fundamental coupling effects between viscoelasticity and the velocity field were visible. From the numerical solutions, the effect of the strain behavior in the span under study was seen: the web strain is not constant during the span length. In the case of pure elastic web material, the non-linear Euler term seemed to cause a qualitatively similar effect. The strain being non-constant originates from the velocity difference and the longitudinal strain wave velocity in the elastic material.

One fundamental observation on the significance of the strain-based boundary conditions was made. In the case of an one-dimensional model, the strain (Dirichlet) boundary condition affects throughout the web thickness isolating the span under observation from the (possibly) preceeding or succeeding web spans. Based on the Figure 1, this, however, is not the situation in reality. Even if the web was considered as slender, there would always be a possibility of the time-dependent strain waves advancing through the control areas A_1 and A_2. Therefore, one of the future challenges in developing realistic in-plane moving web models are the boundary conditions applied.

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