Improved Bounds for Rectangular and Guillotine Partitions†

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1. Introduction

Given a rectangular boundary S and a set Q of points inside S, we study the problem of partitioning S into rectangles in such a way that every point in Q lies on at least one of the partitioning line segments and the total length of the partitioning line segments is least possible. Such a partition is called an optimal rectangular partition. The proofs given by Lingas et al. (1982) can be trivially extended to show that finding an optimal rectangular partition is a computationally intractable problem (NP-hard). Since then, several approximation algorithms have been proposed, i.e. algorithms that guarantee for every problem instance I that \( L(E_{apx}(I)) \leq cL(E_{opt}(I)) \), where \( E_{apx}(I) \) is the set of partitioning line segments given by the approximation algorithm, \( E_{opt}(I) \) is the set of partitioning line segments in an optimal solution, \( c \) is some constant, and \( L(E(I)) \) is the sum of the length of the partitioning line segments in \( E(I) \).

Gonzalez & Zheng (1985a) present a divide-and-conquer approximation algorithm that generates solutions with \( L(E_{apx}(I)) \leq (3 + \sqrt{3})L(E_{opt}(I)) \). The time complexity for their algorithm is \( O(n^2) \), where \( n \) is the number of points in set Q. Levcopoulos (1986) showed that it is possible to implement this approximation algorithm in \( O(n \log n) \) time. Gonzalez & Zheng (1985b) give an \( O(n^4) \) approximation algorithm that guarantees solutions with \( L(E_{apx}(I)) \leq 3L(E_{opt}(I)) \). The approximation bound is smaller than the one for the algorithm given in Gonzalez & Zheng (1985a); however, there is a substantial difference between the time complexities of these two algorithms. The second algorithm (Gonzalez & Zheng, 1985b) is more efficient, but it does not provide the same guarantee on the approximation ratio.

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1985b) consists of two steps. In the first step, the original problem is transformed into a simpler optimization problem. In the second step, an existing $O(n^4)$ algorithm is employed to solve the new optimization problem.

Before we discuss other approximation algorithms for our problem, we need to develop additional notation. We say that the rectangular partition $E(I)$, where $I = (S, Q)$ is any problem instance, has a **guillotine cut** if there is a line segment in $E(I)$ that partitions $S$ into two rectangles (see Fig. 1(b)). We say that a rectangular partition $E(I)$ is a **guillotine partition** if either $E(I)$ is empty (note that $Q$ must be empty) or $E(I)$ has a guillotine cut that partitions $S$ into $S_1$ and $S_2$, and both $E(I_1)$ (the intersection of the partitioning line segments in $E(I)$ and rectangle $S_1$) and $E(I_2)$ (the intersection of the partitioning line segments in $E(I)$ and rectangle $S_2$) are guillotine partitions for $I_1 = (S_1, Q_1)$ and $I_2 = (S_2, Q_2)$, respectively (see Fig. 1(b)). An optimal guillotine partition is a guillotine partition whose partitioning line segments have least total length. It is simple to see that any guillotine partition is a rectangular partition, but the converse is not true (see Fig. 1).

An optimal guillotine partition can be found in $O(n^5)$ time (Shing, private communication). Du et al. (1986) show that the length of the line segments in an optimal guillotine partition is no more than twice the length of the line segments in an optimal rectangular partition. Therefore, finding a polynomial time approximation algorithm for the rectangular partition problem is reduced to the problem of finding an optimal guillotine partition. Gonzalez et al. (1986) present a simple proof for the approximation bound of 2 and point out that it is unlikely that the time complexity bound for this dynamic programming algorithm can be improved. The algorithm given in Gonzalez & Zheng (1985a) generates a guillotine partition; however, this is not true for the algorithm given in Gonzalez & Zheng (1985b). In this paper we improve the previous analyses for the optimal guillotine partition method and show that the length of the line segments in an optimal guillotine partition is within a factor of $1.75$ of the length of an optimal rectangular partition. In Table 1 we summarise the different approximation algorithms for our problem.

If, instead of a rectangle, we are given a rectilinear polygon, and instead of interior points the polygon contains holes (a hole is a rectilinear polygon without interior holes)

### Table 1. Comparison of algorithms

<table>
<thead>
<tr>
<th>Approximation Bound</th>
<th>Time Complexity Bound</th>
<th>Reference</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 + \sqrt{3}$</td>
<td>$O(n^2), O(n \log n)$</td>
<td>Gonzalez &amp; Zheng (1985a)</td>
<td>Divide and conquer</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Levcopoulos (1986)</td>
<td></td>
</tr>
<tr>
<td>$3$</td>
<td>$O(n^4)$</td>
<td>Gonzalez &amp; Zheng (1985b)</td>
<td>Transformation</td>
</tr>
<tr>
<td>$2$</td>
<td>$O(n^3)$</td>
<td>Du et al. (1986)</td>
<td>Dynamic programming</td>
</tr>
<tr>
<td>$1.75$</td>
<td>$O(n^3)$</td>
<td>This paper</td>
<td>Dynamic programming</td>
</tr>
</tbody>
</table>
the problem is called the GP problem. This problem has applications in computer-aided design of integrated circuits and systems for dividing routing regions into channels (Rivest, 1982). Gonzalez & Zheng (1985b) show how to modify their algorithm to generate approximation solutions to the GP problem when the sum of the length of the hole and boundary edges is less than \( L(E_{\text{opt}}(I)) \). The approximation bound obtained in Gonzalez & Zheng (1985b) for this restricted version of the GP problem is smaller than the one given by Lingas (1983). Several approximation algorithms for the GP problem exist (see Lingas, 1983; Levcopoulos, 1985, 1986; Du & Chen, 1986). The algorithms with the smallest worst-case approximation bound are the ones given in Levcopoulos (1985, 1986). The algorithm given in Levcopoulos (1986) uses as a subalgorithm the procedure given in Gonzalez & Zheng (1985a). Since the approximation algorithm given in this paper generates solutions of the same form as those in Gonzalez and Zheng (1985a), but with a solution value that is closer to the optimal solution value, we conjecture that a smaller approximation bound for the GP problem can be obtained by using the results reported in this paper.

For problem instance \( I = (S, Q) \), let \( E_{\text{ogp}}(I) \) be the set of partitioning line segments in an optimal guillotine partition and let \( E_{\text{opt}}(I) \) be the set of partitioning line segments in any optimal rectangular partition. In what follows we show that \( L(E_{\text{ogp}}(I)) < 1.75L(E_{\text{opt}}(I)) \).

Therefore, we have an \( O(n^5) \) approximation algorithm for the rectangular partitioning problem, such that \( L(E_{\text{ogp}}(I)) \leq 1.75L(E_{\text{opt}}(I)) \).

2. Definitions and Transformation Algorithm

We use P to denote the tuple \( I = (S, Q), E(I) \), where \( I \) is a problem instance and \( E(I) \) is any rectangular partition for \( I \). We present a transformation that generates a set of line segments \( E(I) \) such that \( E'(I) \cup E(I) \) forms a guillotine partition (of course \( E(I) \cap E'(I) = \emptyset \) [see, for example, Fig. 2]). The transformation is performed in such a way that \( L(E'(I) \cup E(I)) \leq 1.75L(E(I)) \). Applying this transformation to any optimal rectangular partition \( E_{\text{opt}}(I) \), we know that for the resulting guillotine partition \( E'(I) \cup E_{\text{opt}}(I) \), \( L(E_{\text{ogp}}(I)) \leq 1.75L(E_{\text{opt}}(I)) \). Therefore, \( L(E_{\text{ogp}}(I)) \leq 1.75L(E_{\text{opt}}(I)) \).

Let \( E_v(I) \) and \( E_h(I) \) represent the sets of vertical and horizontal line segments in \( E(I) \), respectively. In Fig. 3 we illustrate the terms that are formally defined below. For a vertical (horizontal) line segment \( l \), we use \( x(l)(y(l)) \) to denote the x-coordinate (y-coordinate) of \( l \). For a vertical line segment \( l \) we use \( B(l) \) and \( T(l) \) to denote the y-coordinate of the lower end point and the upper end point of line segment \( l \), respectively. Similarly, for a horizontal line segment \( l \) we use \( L(l) \) and \( R(l) \) to denote the x-coordinate of the left end point and the right end point of line segment \( l \), respectively. The y-coordinates of the bottom and top side of \( S \) are given by \( B(S) \) and \( T(S) \), respectively. The x-coordinates of the left and right side of \( S \) are given by \( L(S) \) and \( R(S) \), respectively. Let \( X = R(S) - L(S) \) represent the width of \( S \) and let \( Y = T(S) - B(S) \) represent the height of \( S \).

Since rotation of \( P \) by 90 degrees generates an equivalent problem, we may assume without loss of generality that \( L(E_v(I)) \leq L(E_h(I)) \). In what follows we claim that our

![Fig. 2. Dashed lines are the elements \( E'(I) \).](image-url)
transformation process introduces a set of vertical line segments $E'_v(I)$ such that $\bar{L}(E'_v(I)) \leq \overline{L}(E_v(I))$, and a set of horizontal line segments $E'_h(I)$ such that $\bar{L}(E'_h(I)) \leq 0.5\bar{L}(E_h(I))$. Therefore

$$\bar{L}(E_{v\text{eg}}(I)) \leq \bar{L}(E'(I) \cup E(I))$$

$$= \bar{L}(E_v(I)) + L(E_v(I)) + \bar{L}(E(I))$$

$$\leq 0.5\bar{L}(E_h(I)) + \bar{L}(E_v(I)) + \bar{L}(E(I))$$

$$= 0.5\bar{L}(E_v(I)) + 1.5\bar{L}(E(I))$$

$$\leq 1.75\bar{L}(E(I)).$$

We say that line segment $l$ is included by line segment $l'$ if every point in $l$ is in $l'$. The line segment $l$ is said to be included in $E(I)$ if there is a line segment $l'$ in $E(I)$ such that $l$ is included by $l'$. We use the (corrupted) notation $l \subseteq l'$ and $l \subseteq E(I)$ to indicate line inclusion.

The overlap of line segments $l$ and $l'$ is defined as the line segment $l \cap l'$. The overlap of two sets of line segments is defined similarly. A line segment $l$ is a vertical (horizontal) full cut of $S$ if $T(l) = T(S)$ ($L(l) = L(S)$) and $B(l) = B(S)$ ($R(l) = R(S)$). The dashed vertical line segment in Fig. 2 is part of a full cut. A vertical (horizontal) full cut $l$ of $S$ is said to be a vertical (horizontal) guillotine cut if $l \subseteq E(I)$. The rightmost vertical line segment in Fig. 1(b) is a guillotine cut. Note that this definition is equivalent to the one for guillotine cuts introduced in the previous section. When there is a guillotine cut $l$ of $S$ in $E(I)$, $P$ is partitioned into $P_1$ and $P_2$ without introducing any new line segment. At this point, we recursively transform $E(I_1)$ and $E(I_2)$.

If, at each step of this recursive transformation, we encounter an instance with a guillotine cut (see Fig. 1(b)), then $E_v(I) = E_h(I) = \emptyset$ and our claim for the 1.75 bound follows. However, when there is no guillotine cut of $S$ in $E(I)$ we must introduce a full cut. Selecting the full cut is the crucial part of the transformation.

When there is no guillotine cut of $S$ in $E(I)$ we either introduce a vertical full cut, or a set of horizontal and vertical full cuts, depending on the configuration of $E(I)$. The concept of separability, as we shall see later, plays an important role in this decision. Before we define this term we need to introduce additional notation. We say that a vertical full cut $l$ is left (right) covered by $E_v(I)$ if the line segments in the set $\{l\} - E_v(I)$ are not horizontally visible from the left (right) side of $S$, i.e. for every point $p$ in $l$ there exists a line segment $l' \in E_v(I)$ such that $x(l') \leq x(l)$, $(x(l') \geq x(l))$ and $B(l') \leq y(p) \leq T(l')$. A vertical through cut is a vertical full cut that is both left and right covered by $E_v(I)$. The only vertical through cuts in Fig. 4 appear in the region marked by vertical lines outside the rectangle. We say $P$ is vertically separable if there exists at least one vertical through cut in $P$. Figures 1(b) and 4 are separable, whereas Fig. 1(a) is not separable. When $P$ is vertically separable, $S$ may be partitioned along a vertical through cut. In this case, we mark all the line segments (or sections of line segments) in $E_v(I)$ that appear to the left of the vertical through cut and
which are horizontally visible from a point in the vertical through cut that is not part of $E_v(I)$ (see Fig. 4). Clearly, each time we introduce a vertical through cut the length of the new line segments introduced (those not in $E_v(I)$) is less than the length of the newly marked line segments. We claim that, if we repeat this process recursively, we can bound the length of the additional line segments by $E_v(I)$ (see, for example, Fig. 4). We formally prove this fact in Lemma 1. For the moment it is convenient to assume that, at each step of our recursive transformation, we either find a subproblem with no internal line segments, an instance with a guillotine cut, or an instance that is vertically separable. Under these conditions our transformation is defined as follows.

case $E(I) = \emptyset$: return;
$\vdash$P has a guillotine cut:
\begin{itemize}
  \item partition $I$ along a guillotine cut and recursively transform the resulting subproblems $P_1 = (I_1, E(I_1))$ and $P_2 = (I_2, E(I_2))$;
\end{itemize}
$\vdash$P has no guillotine cut, but it is vertically separable:
\begin{itemize}
  \item partition $I$ along a vertical through cut and recursively transform the resulting subproblems $P_1 = (I_1, E(I_1))$ and $P_2 = (I_2, E(I_2))$;
  \item mark all the line segments that appear to the left of the vertical through cut which are horizontally visible from a point in the through cut that is not in $E(I)$*
\end{itemize}
endcase

Since the transformation process does not introduce new horizontal line segments, we know that $E_h(I) = \emptyset$. The set $E_v(I) \neq \emptyset$ if in the recursive process we encounter a nonempty problem instance without a guillotine cut. In Lemma 1 we prove that for this case $\bar{L}(E_v(I)) \leq \bar{L}(E_v(I))$. This lemma also appears in Gonzalez et al. (1986). We include it for completeness.
LEMMA 1. For every $P = (I, E(I))$ our transformation process introduces a set of line segments $E_s(I)$ such that $\bar{L}(E_s(I)) = \bar{L}(E(I))$.

PROOF. Let $R = (R_1, R_2, \ldots, R_k)$ represent the subproblems generated by our procedure. Since every time we introduce additional line segments their length is bounded by the length of the line segments in $E_s(I)$ that we mark at that step, the proof of the lemma is straightforward if the following two statements hold at each step in our recursive process.

1. No point is marked more than once.
2. If a line segment inside the rectangle in subproblem $R_i$ is marked, then it is horizontally visible from the right boundary of $R_i$.

It is simple to see that (1) and (2) hold just before calling our procedure for the first time. Assume that (1) and (2) hold just before invoking our procedure for the $k$th time. Let us now show that (1) and (2) hold just before invoking our procedure for the $(k+1)$th time (or if the $k$th call is the last call, at the end of the $k$th call).

The proof for the induction step is trivial if $E(I) = \emptyset$ or $P$ has a guillotine cut during the $k$th call. So assume that the algorithm introduces a vertical through cut (that is not a guillotine cut) during the $k$th call that partitions $R_i$ into $R_i$ and $R_{i'}$. Since (2) holds just before the $k$th call and the algorithm introduces a vertical through cut (remember that through cuts are right covered), none of the previously marked segments inside $R_i$ will end up inside $R_{i'}$. Since no line segment inside $R_i$ is marked at this step and $R_i$ satisfies (1) and (2), it then follows that $R_{i'}$ satisfies (1) and (2). Since the only marked line segments in $R_i$ are the ones introduced at this step, it then follows that $R_{i'}$ satisfies (1) and (2). Hence, (1) and (2) hold at each step in our recursive process. This completes the proof of the lemma. Q.E.D.

For problem instances $P$ with the properties mentioned above, we know that the 1.75 bound is satisfied. For any arbitrary problem instance $P$ we cannot yet prove this bound. This is because our transformation process is not complete; there are nonempty and non-separable problem instances without a guillotine cut (see Fig. 7a). For those cases we apply a three-phase transformation carried out by procedure HVH_CUT. In the first phase of procedure HVH_CUT, we introduce a set of horizontal full cuts to partition $P$ into a set of vertically separable subproblems. Let $H(I)$ be the set of horizontal full cuts introduced in this phase. Let $H'_3(I) = H(I) \cap E_3(I)$ and let $H'_4(I) = H(I) - H'_3(I)$. Remember that there would be nothing left to prove if it were the case that $\bar{L}(H'_3(I)) \leq 0.5\bar{L}(H'_4(I))$. However, this bound does not necessarily hold. This is why we need to perform the following steps. In the second phase, each of the vertically separable problem instances constructed in phase one is partitioned by introducing a vertical through cut. The vertical through cuts are carefully selected so that in the third phase we can find a set of horizontal guillotine cuts. Let $H_4(I)$ be the set of horizontal guillotine cuts. Note that $H_4(I) \subseteq E_4(I)$. Our objective is to show that $\bar{L}(H'_3(I)) \leq 0.5(\bar{L}(H'_3(I)) + \bar{L}(H_4(I)))$. This is not obvious at this point. Our transformation process is formally defined below.

**procedure TRANS($P = (I = (S, Q), E(I))$)**

**case**

1. $E(I)$ is empty: return;
2. $P$ has a guillotine cut $l$:
   - partition $P$ along $l$ into $P_1$ and $P_2$;
   - recursively apply TRANS to $P_1$ and $P_2$;
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:P is vertically separable:
  let $l$ be any vertical through cut;
  partition $P$ (along $l$) into $P_1$ and $P_2$;
  recursively apply TRANS to $P_1$ and $P_2$;
:else: use procedure HVH_CUT to partition $P$ into $P_1, \ldots, P_u$;
  recursively apply TRANS to each $P_i$;
endcase
end of procedure TRANS

From procedure TRANS and our informal description of procedure HVH_CUT, we
know that every vertical line segment introduced is a vertical through cut. Therefore, a
proof similar to the one for Lemma 1 can be used to show that $\tilde{L}(E(I)) \leq \tilde{L}(E(I))$. To prove our 1.75 bound, it is only required to show that for every $P$ on which we invoke
procedure HVH_CUT, $\tilde{L}(H(I)) \leq 0.5(\tilde{L}(H_1(I)) + \tilde{L}(H_3(I)))$. Hereafter, we concentrate on
nonempty and nonseparable problem instances without guillotine cuts.

3. Procedure HVH_CUT and Bounds

As we mentioned before, to prove our 1.75 bound it is only required to show that for
every $P$ on which we invoke procedure HVH_CUT, $\tilde{L}(H(I)) \leq 0.5(\tilde{L}(H_1(I)) + \tilde{L}(H_3(I)))$.
Remember that we only need to concentrate on nonempty problem instances $P$ that do
not have a guillotine cut and are not separable. A nonempty and not separable problem
instance without guillotine cuts is given in Fig. 7a. Throughout this section we will use this
element to illustrate our procedure. The proof for the above bound is not simple. Before
proving it we need to introduce some additional notation and prove some intermediate
results.

We say that $P' = (I' = (S', Q'), E(I'))$ is a subproblem of $P = (I = (S, Q), E(I))$, written as
$P' \subseteq P$, if $S'$ is a subrectangle of $S$ (i.e. $T(S') \leq T(S), B(S') \geq B(S), L(S') \geq L(S), R(S') \leq R(S)$),
$Q'$ contains all the points in $Q$ located inside (not in the boundary of) $S'$, and $E(I')$ contains
all the line segments in $E(I)$ located inside $S'$ (i.e. the intersection of the line segments in
$E(I)$ and rectangle $S'$). We say that $P' = (I' = (S', Q'), E(I'))$ is empty if $E(I')$ is empty, i.e.
there are no line segments in $E(I')$. An important property of empty subproblems is given
by the following lemma. This property will be used in the remaining lemmas to show the
existence of a horizontal line segment above (below) an empty subproblem with a vertical
line segment above (below) it.

LEMMA 2. Given an empty subproblem $P' = (I', E(I')) \subseteq P = (I, E(I))$ with a vertical line
segment $l \in E(I)$ such that $L(S') < x(l) < R(S')$ and $B(l) = T(S')(T(l) = B(S'))$, there exists a
horizontal line segment $l' \in E_3(I)$ such that $y(l') = T(S')(y(l') = B(S'))$, $L(l') \leq L(S')$ and
$R(l') \geq R(S')$.

![Fig. 5. (a) Lines $l + l'$; (b) loose ends; (c) dangling corners.](image_url)
PROOF. Since the proof for both cases is similar, we only prove the case when \(B(I) = T(S')\). The proof is by contradiction. Suppose there is no line segment \(l'\) satisfying the above properties. Since \(E(I')\) is empty it must be that there is a horizontal line segment \(l'' \in E_\delta(I)\) with \(y(l'') = T(S')\), \(L(l'') < x(l)\) and \(R(l'') > x(l)\), otherwise \(l\) is a loose end (see Fig. 5(b)) or a dangling corner (see Fig. 5(c)), which implies that \(E(I)\) is not a rectangular partition. If \(L(l'') > L(S')\), then \(l''\) is either a loose end or a dangling corner. This contradicts the fact that \(E(I)\) is a rectangular partition. Similarly, if \(R(l'') < R(S')\) there is a contradiction. Since in each case there is a contradiction, there is a line segment \(l'\) with the properties mentioned in the statement of the lemma. This completes the proof of the lemma. Q.E.D.

Among all vertical through cuts in a vertically separable problem \(P\), the one with smallest \(x\)-coordinate and the one with largest \(x\)-coordinate are referred to as the leftmost vertical through cut \(lm(P)\) and the rightmost vertical through cut \(rm(P)\), respectively. Note that, for some \(P\), the leftmost vertical through cut could also be the rightmost vertical through cut. In what follows we identify some separable subproblems (via procedure ID), then examine some of their properties, and finally show how to use these subproblems and their properties to perform the three phases of procedure HVH_CUT.

Let \(y_1 < y_2 < \ldots < y_s\) be the distinct \(y\)-coordinates of the set of line segments in \(E_\delta(I)\). Let \(y_0 = B(S)\) and \(y_{s+1} = T(S)\). For \(0 \leq i \leq u \leq v \leq s+1\), let \(S_{i,u}\) denote the horizontal slice through \(S\) defined by \(((L(S), y_i), (L(S), y_u), (R(S), y_u), (R(S), y_i))\). Similarly, let \(P_{i,u}\) denote \(P\) restricted to \(S_{i,u}\). It is easy to see that if \(P_{i,u}\) and \(P_{u,v}\) are vertically separable but \(P_{i,v}\) is not vertically separable, then either \(x(lm(P_{i,v})) < x(lm(P_{u,v}))\) or \(x(lm(P_{i,v})) > x(lm(P_{u,v}))\). In the former case, we call \(P_{i,v}\) an LR-increasing problem; and in the latter case, we call \(P_{i,v}\) an LR-decreasing problem (see Fig. 6).

If \(P_{i,j}\) is separable, then \(P_{h,g}\), where \(i \leq h \leq g \leq j\), is also separable. Furthermore, \(x(lm(P_{h,g})) \leq x(lm(P_{i,j}))\) and \(x(rm(P_{h,g})) \geq x(rm(P_{i,j}))\). Note that \(P_{i,0}\) which is just a line segment, is separable for all \(i\); and for problem instances without guillotine cuts, \(P_{i,i+1}\) is always separable. Procedure ID finds a set of vertically separable subproblems. Later on we show how to use these subproblems.

**Procedure ID**

\[
\begin{align*}
&j \leftarrow 0; \ YI \leftarrow \emptyset; \\
&\text{for } i \leftarrow 1 \text{ to } s+1 \text{ do} \\
&\quad \text{if } P_{j,1} \text{ is not separable then } YI \leftarrow YI \cup \{y_{i-1}\}; j \leftarrow i-1; \\
&\text{endfor}
\end{align*}
\]

**Fig. 6.** LR problem: (a) LR-increasing; (b) LR-decreasing.
\[ I - Y(I) \]

\[ \{ 1, 2, \ldots, n \} \]

\[ \{ y_1, y_2, \ldots, y_k \} \]

\[ \{ y_1, y_2, \ldots, y_k \} \]

\[ i \leftarrow s + 1; \quad YJ \leftarrow \emptyset; \]

\[ \text{for } j \leftarrow s \text{ to } 0 \text{ by } -1 \text{ do} \]

\[ \quad \text{if } P_{j,i} \text{ is not separable then} \]

\[ \quad \quad \text{YJ} \leftarrow \text{YJ} \cup \{ y_{j+1} \}; \quad i \leftarrow j + 1; \]

\[ \text{end for} \]

end of procedure ID

Let \( y_1(1) < y_2(2) < \ldots < y_{k}(k) \), be the y-coordinates in \( YI \) and let \( y_1(1) < y_2(2) < \ldots < y_{k'}(k') \), be the y-coordinates in \( YJ \) defined by procedure ID. Figure 7(b) illustrates the sets \( YI \) and \( YJ \) for a rectangular partition given in Fig. 7(a). For convenience, let \( y_{j(0)} = y_{j(0)} = B(S) \) and \( y_{j(k+1)} = y_{j(k+1)} = T(S) \). Since the subproblem \( P_{j(p), j(p+1)} \) is separable and the algorithm selects the \( y_{j(p)}'s \) to represent maximal separable subproblems with respect to the previous \( y_{j(p)}'s \), we know that it is impossible for two \( y_{j(p)}'s \) to be the interval \( (y_{j(p)}, y_{j(p+1)}) \). Similarly, it is impossible for two \( y_{j(0)}'s \) to be in the interval \( [y_{j(0)}, y_{j(k+1)}] \). Hence, \( k = k' \) and

\[ y_{j(0)} = y_{j(0)} < y_{j(1)} < y_{j(2)} < y_{j(3)} < \ldots < y_{j(k)} < y_{j(k+1)} = y_{j(k+1)}. \]

In the next lemma we prove an important property of LR-decreasing and LR-increasing problems which will be useful in our transformation process.

**Lemma 3.** If \( P_{i(m-1), i(m+1)} \) is an LR-decreasing (LR-increasing) problem, then \( P_{j(m-1), j(m+1)} \) is also an LR-decreasing (LR-increasing) problem.

\[ y_{j(m+1)} \]

\[ y_{j(m)} \]

\[ y_{j(m-1)} \]

\[ y_{j(m+1)} \]

\[ y_{j(m)} \]

\[ y_{j(m-1)} \]

Fig. 8. LR-decreasing problem.
PROOF. Since the proof for both of these cases is similar, we only prove that if $P_{l(m-1), l(m+1)}$ is an LR-decreasing problem, then $P_{l(m-1), l(m+1)}$ is also an LR-decreasing problem (see Fig. 8). The proof is by contradiction. Suppose that $P_{l(m-1), l(m+1)}$ is an LR-decreasing problem, but $P_{l(m-1), l(m+1)}$ is an LR-increasing problem. Since $P_{l(m-1), l(m+1)}$ is an LR-increasing problem, $x(rm(P_{l(m-1), l(m+1)})) < x(lm(P_{l(m-1), l(m+1)}))$. Since $P_{l(m-1), l(m+1)} \subseteq P_{l(m), l(m+1)}$ (note that $P_{l(m), l(m+1)}$ is not necessarily equal to $P_{l(m), l(m+1)}$), we know that $x(lm(P_{l(m), l(m+1)})) \leq x(rm(P_{l(m), l(m+1)})) \leq x(lm(P_{l(m), l(m+1)}))$. Since $P_{l(m-1), l(m+1)}$ is an LR-decreasing problem and $P_{l(m-1), l(m+1)}$ is separable but $P_{l(m-1), l(m+1)}$ is not separable, $x(rm(P_{l(m), l(m+1)})) < x(lm(P_{l(m), l(m+1)}))$. Therefore, $x(rm(P_{l(m), l(m+1)})) < x(rm(P_{l(m), l(m+1)})) \leq x(rm(P_{l(m), l(m+1)})) \leq x(lm(P_{l(m), l(m+1)}))$. The vertical line segment with x-coordinate equal to $x(rm(P_{l(m), l(m+1)}))$ is right covered in $P_{l(m-1), l(m+1)}$ because it is to the left of $rm(P_{l(m-1), l(m+1)})$. This vertical line segment is also left covered in $P_{l(m-1), l(m+1)}$ because it is to the right of $rm(P_{l(m-1), l(m+1)})$ and by definition of $rm(P_{l(m), l(m+1)})$, it is left covered in $P_{l(m), l(m+1)}$. But, then $P_{l(m-1), l(m+1)}$ has a through cut with x-coordinate equal to $x(rm(P_{l(m), l(m+1)}))$ and $x(rm(P_{l(m), l(m+1)})) < x(lm(P_{l(m), l(m+1)}))$. This contradicts the definition of leftmost through cut. So it must be that $P_{l(m-1), l(m+1)}$ is an LR decreasing problem. Q.E.D.

Let LEFT = \{l \in E_S(I) and L(I) = L(S)\} and RIGHT = \{l \in E_R(I) and R(I) = R(S)\}. In the following lemma we show that for each $y_{l0}$ and $y_{l0}$ there is a distinct horizontal line segment from LEFT or RIGHT with the same y-coordinate value. For $x_{1} < x_{2}$ we use HLS($y$, $x_{1}$, $x_{2}$) to represent the horizontal line segment with end points ($x_{1}$, $y$) and ($x_{2}$, $y$).

**Lemma 4.**

(i) if $P_{l(m-1), l(m+1)}$ is an LR-decreasing problem, then the line segment $l = HLS(y_{l(m)}, x(rm(P_{l(m), l(m+1)})), R(S)) \in E_S(I)$;

(ii) if $P_{l(m-1), l(m+1)}$ is an LR-increasing problem, then the line segment $l = HLS(y_{l(m)}, L(S), x(lm(P_{l(m), l(m+1)}))) \in E_R(I)$;

(iii) if $P_{l(m-1), l(m+1)}$ is an LR-decreasing problem, then the line segment $l = HLS(y_{l(m)}, L(S), x(lm(P_{l(m-1), l(m+1)}))) \in E_R(I)$;

(iv) if $P_{l(m-1), l(m+1)}$ is an LR-increasing problem, then the line segment $l = HLS(y_{l(m)}, x(rm(P_{l(m-1), l(m+1)})), R(S)) \in E_S(I)$.

---

**Fig. 9.** Case (i) for Lemma 4.
PROOF. Since the proof of all four cases is similar, we only prove that if \( P_{t(m-1), (m+1)} \) is an LR-decreasing problem, then the line segment \( l = HS(y_{t(m)}, x(rm(P_{t(m), (m+1)})), R(S)) \subseteq E_{h}(I) \) (see Fig. 9). Since \( P_{t(m-1), (m+1)} \) is an LR-decreasing problem and \( P_{t(m-1), (m+1)} \) is separable but \( P_{t(m-1), (m+1)} \) is not separable, we know that \( x(rm(P_{t(m), (m+1)})) \) is not separable. Therefore, no line segment in \( E \) can be inside the rectangle formed by the points \( (x(rm(P_{t(m), (m+1)})), y_{t(m)}), (x(rm(P_{t(m), (m+1)})), y_{t(m)}+1), (R(S), y_{t(m)}), \) and \( (R(S), y_{t(m)}+1) \) (the shaded rectangle in Fig. 9). Since \( lm(P_{t(m-1), (m+1)}) \) is right covered in \( P_{t(m-1), (m+1)} \), there must exist a line segment \( T \subseteq E_{h}(I) \) such that \( T = y_{t(m)} \) and \( x(T) \geq x(lm(P_{t(m-1), (m+1)})) \). Therefore, the conditions of Lemma 2 are satisfied and we know that the horizontal line segment \( l = HLS(y_{t(m)}, x(rm(P_{t(m), (m+1)})), R(S)) \subseteq E_{h}(I) \). This completes the proof of the lemma. Q.E.D.

The line segment with \( y \)-coordinate value equal to \( y_{t(m)} \) identified by Lemma 4((i) and (ii)) is referred to by \( l_{t(m)} \), and the one with \( y \)-coordinate value equal to \( y_{t(m)} \) identified by Lemma 4((iii) and (iv)) is referred to by \( l_{t(m)} \). Let \( EI = \{l_{t(m)} \mid 1 \leq m \leq k \} \) and \( EJ = \{l_{t(m)} \mid 1 \leq m \leq k \} \). In Fig. 10 we identify these line segments for the partition \( E(I) \) given in Fig. 7(a).

Note that each line segment in \( EI \) (\( EJ \)) has a \( y \)-coordinate value equal to a value in \( YI \) (\( YJ \)). From Lemmas 3 and 4 we know that \( l_{t(m)} \in LEFT \) (\( RIGHT \)) if \( l_{t(m)} \in RIGHT \) (\( LEFT \)). If \( y(l_{t(m)}) = y(l_{t(m)}) \), then it is not possible for \( l_{t(m)} \) and \( l_{t(m)} \) to overlap because we are assuming there are no horizontal guillotine cuts. Therefore, for each \( y_{t(m)} \) and \( y_{t(m)} \) there is a distinct horizontal line segment in \( LEFT \) or \( RIGHT \) associated with it. In the next two lemmas we show that each of the line segments identified by the previous lemma can be associated with a distinct line segment in \( E_{h}(I) \) such that their total length is at least \( X \) (remember that \( X \) is the width of the rectangle). This is an important property needed to establish our 1.75 bound. In Lemma 5 we show that for each \( l_{t(m)} \) and \( l_{t(m)} \) there is another line segment in \( E_{h}(I) \) such that the sum of their length is at least \( X \). Since this does not necessarily guarantee a 1–1 association between line segments, we need Lemma 6.

LEMMA 5.

(i) If \( P_{t(m-1), (m+1)} \) is an LR-decreasing problem, then there exists at least one line segment \( l \) in \( E_{h}(I) \) such that

\[
l = HLS(y, L(S), x(lm(P_{t(m-1), (m+1)}))), \text{ where } y_{t(m)} < y \leq y_{t(m)};
\]

Fig. 10. Line segments \( l_{t(m)} \) and \( l_{t(m)} \).
Fig. 11. Case (i) in Lemma 5.

(ii) If $P_{j(m-1), j(m+1)}$ is an LR-increasing problem, then there exists at least one line segment $l$ in $E(I)$ such that
\[ l = \text{HLS}(y, x(r_m(P_{j(m-1), j(m)})), R(S)), \text{ where } y_{j(m-1)} < y \leq y_{j(m)}. \]

(iii) If $P_{j(m-1), j(m+1)}$ is an LR-decreasing problem, then there exists at least one line segment $l$ in $E(I)$ such that
\[ l = \text{HLS}(y, x(r_m(P_{j(m-1), j(m+1)})), R(S)), \text{ where } y_{j(m)} \leq y < y_{j(m+1)} \text{ and} \]

(iv) If $P_{j(m-1), j(m+1)}$ is an LR-increasing problem, then there exists at least one line segment $l$ in $E(I)$ such that
\[ l = \text{HLS}(y, L(S), x(l_m(P_{j(m-1), j(m)}))), \text{ where } y_{j(m)} \leq y < y_{j(m+1)}. \]

PROOF. Since the proof for all four cases is similar, we only prove that if $P_{j(m-1), j(m+1)}$ is an LR-decreasing problem, then there exists at least one line segment $l$ in $E(I)$ such that
\[ l = \text{HLS}(y, L(S), x(l_m(P_{j(m-1), j(m)}))), \text{ where } y_{j(m)} \leq y < y_{j(m+1)}. \]

Since $P_{j(m-1), j(m)}$ is a subproblem of $P_{j(m-1), j(m)}$, we know that $x(l_m(P_{j(m-1), j(m)})) < x(l_m(P_{j(m-1), j(m)}))).$ Suppose now that $x(l_m(P_{j(m-1), j(m)})) < x(l_m(P_{j(m-1), j(m)})).$ Then, since $x(r_m(P_{j(m), j(m+1)})) < x(l_m(P_{j(m-1), j(m)})))$ (otherwise $P_{j(m-1), j(m+1)}$ is separable) and $y_{j(m)} \leq y_{j(m)}$, it is easy to see that the through cut with $x$-coordinate value equal to $x(l_m(P_{j(m-1), j(m)})))$ is both left and right covered in $P_{j(m-1), j(m)}$. Therefore, it cannot be that $x(l_m(P_{j(m-1), j(m)})) < x(l_m(P_{j(m-1), j(m)}))).$ This is a contradiction. So it must be that $x(l_m(P_{j(m-1), j(m)})) = x(l_m(P_{j(m-1), j(m)}))).$

Since $y_{j(m)} < y_{j(m)}$ and $x(l_m(P_{j(m-1), j(m)}))) = x(l_m(P_{j(m-1), j(m)})))$, we know that there are no line segments from $E(I)$ inside a rectangle formed by the points $((L(S), y'), (L(S), y), x(l_m(P_{j(m-1), j(m)}))), y'), (x(l_m(P_{j(m-1), j(m)}))), y'))$, for some $y, y'$ such that $y_{j(m-1)} \leq y' \leq y_{j(m)}$. Let $S'$ be the rectangle that satisfies the above property for the largest value of $y$. If $y < y_{j(m)}$, then there is a vertical line segment that intersects the top side of $S'$ (but not the left or right sides of $S'$). On the other hand, if $y = y_{j(m)}$, then since $x(l_m(P_{j(m), j(m+1)})) < x(l_m(P_{j(m-1), j(m)}))$ and $x(l_m(P_{j(m-1), j(m)}))) = x(l_m(P_{j(m-1), j(m)})))$ (see previous paragraph) we know that there is a vertical line segment that intersects the top side of $S'$ (but not the left or right sides of $S'$). In either case the conditions of Lemma 2 hold, and we know that there exists a line segment $l \in E(I)$ such that $l = \text{HLS}(y, L(S), x(l_m(P_{j(m-1), j(m)}))).$ This completes the proof of the lemma. Q.E.D.
If all the subproblems are LR-decreasing or LR-increasing, the previous lemma would suffice for our transformation process, because it would associate each segment $l_{\text{inn}}$ and $l_{\text{inn}}'$ with a unique line segment from $E_s(I)$ in such a way that the sum of their lengths is greater than or equal to $X$ (this is a fundamental property required by our algorithm, as will be discussed shortly). However, in general, there are LR-decreasing problems adjacent to LR-increasing problems. For this case the previous lemma does not guarantee the existence of a distinct line segment that could be associated with each $l_{\text{inn}}$ and $l_{\text{inn}}'$. That is why we need to identify at least two line segments in some regions. Note that, in general, not all regions have these two line segments; however, the two line segments always exist when there is an LR-decreasing problem adjacent to an LR-increasing problem (or vice versa).

**Lemma 6.**

(i) If $P_{i(m-1),i(m+1)}$ is an LR-decreasing problem and $P_{i(m-2),i(m)}$ is an LR-increasing problem, then $x(lm(P_{i(m-1),i(m+1)})) = x(lm(P_{i(m-1),i(m)})) = x(lm(P_{i(m-1),i(m)}))$ and there are at least two distinct line segments $l, l' \in E_s(I)$ such that $R(I) = R(l') = x(lm(P_{i(m-2),i(m)})), y_{i(m-1)} \leq y(I) < y(l') \leq y_{i(m)}$, and $l, l' \in \text{LEFT}$.

(ii) If $P_{i(m-1),i(m+1)}$ is an LR-increasing problem and $P_{i(m-2),i(m)}$ is an LR-decreasing problem, then $x(rm(P_{i(m-1),i(m+1)})) = x(rm(P_{i(m-1),i(m)})) = x(rm(P_{i(m-1),i(m)}))$ and there are at least two distinct line segments $l, l' \in E_s(I)$ such that $L(l) = L(l') = x(rm(P_{i(m-1),i(m)})), y_{i(m-1)} \leq y(l) < y(l') \leq y_{i(m)}$, and $l, l' \in \text{RIGHT}$.

**Proof.** Let us consider the first case (see Fig. 12). The first part of the proof of the previous lemma can be used to show that $x(lm(P_{i(m-1),i(m)})) = x(lm(P_{i(m-1),i(m)}))$. Similarly, one can prove that $x(lm(P_{i(m-1),i(m)})) = x(lm(P_{i(m-1),i(m)}))$. Therefore, $x(lm(P_{i(m-1),i(m)})) = x(lm(P_{i(m-1),i(m)}))$.

Since $P_{i(m-1),i(m+1)}$ is an LR-decreasing problem and since $P_{i(m-2),i(m)}$ is an LR-increasing problem, it must be that $x(lm(P_{i(m-1),i(m+1)})) > \max\{x(rm(P_{i(m-1),i(m-1)})), x(rm(P_{i(m-1),i(m+1)}))\}$. Since $x(lm(P_{i(m-1),i(m+1)})) = x(lm(P_{i(m-1),i(m+1)})) = x(lm(P_{i(m-1),i(m+1)}))$, there exists an empty subproblem $P' = (S', E')$ of $P_{i(m-1),i(m)}$ such that $E'$ is empty, $T(S') \leq y_{i(m)}$, $B(S') \geq y_{i(m-1)}$, $L(S') = L(S)$, and $R(S') = x(rm(P_{i(m-1),i(m)}))$. Since $x(lm(P_{i(m-1),i(m+1)})) > \max\{x(rm(P_{i(m-2),i(m-1)})), x(rm(P_{i(m),i(m+1)}))\}$, there is at least one vertical line segment of
$E,(l)$ with its lower end point at the top side of $S'$ and there is at least one vertical line segment of $E,(l)$ with its upper end point at the bottom side of $S'$ (note that neither of these lines intersects the left or the right side of $S'$). The existence of $l$ and $l'$ is now established as in lemma 5. This completes the proof of this case. Since the proof for the remaining case is similar, it will be omitted. Q.E.D.

Remember that $EI = \{l_{(m)}|1 \leq m \leq k\}$, $EJ = \{l_{(m)}|1 \leq m \leq k\}$, and $l_{(m)} \in \text{LEFT (RIGHT)}$ iff $l_{(m)} \in \text{RIGHT (LEFT)}$. We partition $EI$ and $EJ$ into the crossing and noncrossing subsets $EI_n, EI_c, EJ_n$ and $EJ_c$ as follows:

$$EI_n = \{l_{(m)}|L(l_{(m)} \cup l_{(m)}) < X\}; \quad EI_c = \{l_{(m)}|L(l_{(m)} \cup l_{(m)}) \geq X\};$$
$$EJ_n = \{l_{(m)}|L(l_{(m)} \cup l_{(m)}) < X\}; \quad EJ_c = \{l_{(m)}|L(l_{(m)} \cup l_{(m)}) \geq X\}.$$

For the example given in Fig. 10, we have $EI_n = \{l_{(3)}, l_{(4)}\}$, $EJ_n = \{l_{(3)}, l_{(4)}\}$, $EI_c = \{l_{(1)}, l_{(2)}, l_{(3)}\}$, and $EJ_c = \{l_{(1)}, l_{(2)}, l_{(3)}\}$. Obviously, $l_{(m)} \in EI_n$ (or $l_{(m)} \in EI_c$) iff $l_{(m)} \in EJ_n$ (or $l_{(m)} \in EJ_c$). Let us now define matching pairs for the elements in set $EI_n \cup EJ_n$. If $l_{(m)} \in EI_n$, then we say the match for $l_{(m)}$ is $l_{(m)}$. Similarly, if $l_{(m)} \in EI_c$, the match for $l_{(m)}$ is $l_{(m+1)}$. If $l$ is the match for $l'$ and $l'$ is the match for $l$, then we say that $l$ and $l'$ form a matching pair. The following procedure finds a match for each of the elements in set $EI_n \cup EJ_n$.

procedure MATCH

$ES \leftarrow \emptyset$;
for $m \leftarrow 1$ to $k$ do
if $l_{(m)} \in EI_n$ and $P_{(m-1), (m+1)}$ is an LR-decreasing (LR-increasing) problem then
find a line segment $l \in \text{LEFT (RIGHT)}$ such that $l \notin ES \cup EI \cup EJ$, $y_{(m-1)} < y(l) \leq y_{(m+1)}$ and $R(l) \geq x(lm(P_{(m-1), (m)}))$ ($L(l) \leq x(lm(P_{(m-1), (m)}))$);
Let $l_{(m)} = \text{HLS}(y(l), L(S), x(lm(P_{(m-1), (m)})))$ (HLS$(y(l), L(S), x(lm(P_{(m-1), (m)})))$);
ES $\leftarrow ES \cup \{l_{(m)}\}$;
Let $l_{(m)}$ and $l_{(m)}$ form a matching pair;
endfor
for $m \leftarrow 1$ to $k$ do
if $l_{(m)} \in EJ_n$ and $P_{(m-1), (m+1)}$ is an LR-decreasing (LR-increasing) problem then
find a line segment $l \in \text{LEFT (RIGHT)}$ such that $l \notin ES \cup EI \cup EJ$, $y_{(m-1)} \leq y(l) < y_{(m+1)}$ and $L(l) \leq x(lm(P_{(m-1), (m+1)}))$ ($R(l) \geq x(lm(P_{(m-1), (m+1)}))$);
Let $l_{(m)} = \text{HLS}(y(l), x(lm(P_{(m-1), (m+1)})), R(S))$ (HLS$(y(l), L(S), x(lm(P_{(m-1), (m+1)})))$);
ES $\leftarrow ES \cup \{l_{(m)}\}$;
Let $l_{(m)}$ and $l_{(m)}$ form a matching pair;
endfor
end of procedure MATCH

See Fig. 13 to identify the $l_{(m)}$ and $l_{(m)}$ lines for the instance given in Fig. 7(a). Figure 14 illustrates all the matching pairs of these line segments. Lemma 5 and Lemma 6 can be used to prove that $ES$ can be constructed by procedure MATCH, i.e. the line segment with the desired properties can always be found. Let $T = EI \cup EJ \cup ES$. Let $p = |EI|$ (note that $|EJ| = |EI|$). Since $|EI| = |EJ| = k$ and $|ES| = 2 \cdot |EI|$ we know that $|T| = 2 \cdot k + 2 \cdot p$. It is simple to verify that $|T \cap \text{LEFT}| = |T \cap \text{RIGHT}| = k + p$ and $T$ contains $k + p$ matching
pairs. Since the length of each matching pair is at least \( X \), we know that \( L(T) \geq (k + p)X \).

Let us now order the line segments in \( T \). Note that it is possible that two line segments \( l \) and \( l' \) of \( T \) have the same \( y \)-coordinate. However, when this happens, the corresponding lines cannot overlap since we are assuming that there are no guillotine cuts. Also, one of the segments is \( y(l_{t(t)}) \) or \( y(l_{t(t)}) \) and the other is \( y(l_{t(t)}) \) or \( y(l_{t(t)}) \). When we compare two elements with the same \( y \)-coordinate value, \( l_{t(t)} \) or \( l_{t(t)} \) is considered smaller than \( l_{t(t)} \) or \( l_{t(t)} \). We sort all line segments in \( T \) by their \( y \)-coordinates and form the sequence \( l_{t(0)}, l_{t(2)}, \ldots, l_{t(2k+2p)} \) such that \( y(l_{t(0)}) \leq y(l_{t(2)}) \leq \ldots \leq y(l_{t(2k+2p)}) \). Let \( l_{t(0)} \) and \( l_{t(2k+2p+1)} \) be the bottom and top side of \( S \), respectively.

**Lemma 7.** \( P_{q(w),q(w+2)} \) is separable for \( 0 \leq w \leq 2k + 2p - 1 \).

**Proof.** Since every three adjacent \( l_{t(t)} \)'s have \( y \)-coordinate values located in the interval \([y_{t(m)}, y_{t(m+1)}]\) or in the interval \([y_{t(m)}, y_{t(m+1)}]\), and since both of these intervals are separable, we know that \( P_{q(w),q(w+2)} \) is separable for \( 0 \leq w \leq 2k + 2p - 1 \). Q.E.D.
Let $T_{\text{odd}} = \{l_{q(w)} | l_{q(w)} \in T \text{ and } w \text{ is odd} \}$ and $T_{\text{even}} = \{l_{q(w)} | l_{q(w)} \in T \text{ and } w \text{ is even} \}$. For each line segment $l_{q(w)}$ in $T$ we define its complement as the line segment $l_{q(w)}^c$, as follows:

- **If** $l_{q(w)} \in \text{LEFT}$ **then** its complement is the line segment $l_{q(w)}^c = \text{HLS}(y(l_{q(w)}), R(l_{q(w)}), R(S))$
- **Else** its complement is the line segment $l_{q(w)}^c = \text{HLS}(y(l_{q(w)}), L(S), L(l_{q(w)}))$.

Note that the complement of some line segment in $T$ may overlap with another line segment in $T$. This can happen only when two line segments in $T$ have the same y-coordinate value. Sets $T_{\text{odd}}$ and $T_{\text{even}}$ are defined to have the complements of the elements in sets $T_{\text{odd}}$ and $T_{\text{even}}$, respectively. Figure 15 shows the sets $T_{\text{odd}}$ and $T_{\text{even}}$ for the instance in Fig. 7(a).

Given any nonempty nonseparable problem $P$ without guillotine cuts, we use the following procedure to partition it.

**procedure HVH\_CUT**

1. Use procedure ID and MATCH to construct $T$;
   - Order the line segments in $T$ following the rules mentioned above;
   - Partition $T$ into $T_{\text{odd}}$ and $T_{\text{even}}$ as defined above;
   - if $\bar{L}(T_{\text{odd}}) \leq \bar{L}(T_{\text{even}})$ then introduce all the line segments in $T_{\text{odd}}$; $g \leftarrow 1$;
     - else introduce all the line segments in $T_{\text{even}}$; $g \leftarrow 0$;
2. /* for each resulting partition after step (1), introduce a vertical through cut as follows: */
   - for $w = g$ to $2(k+p) - 1$ by 2 do
     - case $l_{q(w+1)}$ is $l_{i(m)}$ or $l_{j(m)}$ for some $m$:
       - if $l_{q(w+1)} \in \text{LEFT}$ then introduce the leftmost vertical through cut in $P_{q(w), q(w+2)}$
         - else introduce the rightmost vertical through cut in $P_{q(w), q(w+2)}$;
     - case /* later on we show the existence of the through cuts we introduce at this step */
Improved Bounds for Partitions

(1) if \( l_{q(w+1)} \) is \( l_{r(m)} \) for some \( m \):
   - if \( l_{q(w+1)} \) is LEFT then introduce a vertical through cut with \( x \)-coordinate value equal to \( x(\text{im}(P_{l(m-1)}, l(m))) \) in \( P_{q(w), q(w+2)} \)
   - else introduce a vertical through cut with \( x \)-coordinate value equal to \( x(\text{rm}(P_{l(m-1)}, l(m))) \) in \( P_{q(w), q(w+2)} \);

(2) if \( l_{q(w+1)} \) is \( l_{r(m)} \) for some \( m \):
   - if \( l_{q(w+1)} \) is LEFT then introduce a vertical through cut with \( x \)-coordinate value equal to \( x(\text{im}(P_{l(m)}, l(m+1))) \) in \( P_{q(w+1), q(w+2)} \)
   - else introduce a vertical through cut with \( x \)-coordinate value equal to \( x(\text{rm}(P_{l(m)}, l(m+1))) \) in \( P_{q(w+1), q(w+2)} \);

endcase
endcase
endfor

(3) for each resulting partitions of step (2), introduce a horizontal guillotine cut if possible.

end of procedure HVH_CUT

In phase one of procedure HVH_CUT we introduce a set of horizontal full cuts to partition \( P \) into \( k+p+1 \) separable subproblems (Lemma 7). Remember that set \( H(I) \) contains the set of horizontal full cuts introduced in phase one, \( H(I) = H(I) \cap E_h(I) \), and \( H_1(I) = H(I) - H_1(I) \). We will show that \( \overline{\mu}(H_1(I)) \leq \frac{(k+p)X}{2} \). In phase three of procedure HVH_CUT we identify a set of guillotine cuts \( H_3(I) \). Remember that \( H_3(I) \leq E_3(I) \). We will prove that \( \overline{\mu}(H_1(I) \cup H_3(I)) \geq (k+p)X \). Therefore, \( \overline{\mu}(H_1(I)) \leq 0.5 \overline{\mu}(H_1(I) \cup H_3(I)) \) and the 0.5 bound is satisfied. Let us now establish these important bounds for \( \overline{\mu}(H_1(I) \cup H_3(I)) \) and \( \overline{\mu}(H_3(I)) \).

**Lemma 8.** Let \( H_1(I) \), \( H_3(I) \) and \( H_3(I) \) be defined as above. Then,

\( \overline{\mu}(H_1(I)) \) \leq \min\{ \overline{\mu}(T_{even}^u), \overline{\mu}(T_{odd}) \} \leq 0.5(k+p)X; \)

\( \overline{\mu}(H_1(I) \cup H_3(I)) \geq \overline{\mu}(T_{even}) + \overline{\mu}(T_{odd}) \geq (k+p)X; \) and

\( \overline{\mu}(H_3(I)) \leq 0.5 \overline{\mu}(H_1(I) \cup H_3(I)) \).

**Proof.** Since (iii) follows from (i) and (ii), we only prove (i) and (ii). First let us prove (i). From step (1) of procedure HVH_CUT, we know that \( \overline{\mu}(H_1(I)) \leq \min\{ \overline{\mu}(T_{even}^u), \overline{\mu}(T_{odd}) \} \). Since every matching pair has total length at least \( X \), we know that \( \overline{\mu}(T_{even}^u) + \overline{\mu}(T_{odd}) \geq (k+p)X \) and thus \( \overline{\mu}(T_{even}) + \overline{\mu}(T_{odd}) \leq (k+p)X \). So, it must be that \( \min\{ \overline{\mu}(T_{even}^u), \overline{\mu}(T_{odd}) \} \leq 0.5(k+p)X \). Hence, \( \overline{\mu}(H_1(I)) \leq \min\{ \overline{\mu}(T_{even}^u), \overline{\mu}(T_{odd}) \} \leq 0.5(k+p)X \). This completes the proof of part (i).

Let us now prove part (ii). Again, since every matching pair has total length greater than or equal to \( X \), we know that \( \overline{\mu}(T_{even}) + \overline{\mu}(T_{odd}) \geq (k+p)X \). Therefore, to complete the proof of part (ii) it is only required to show that \( \overline{\mu}(H_1(I) \cup H_3(I)) \geq \overline{\mu}(T_{even}) + \overline{\mu}(T_{odd}) \). Assume that \( \overline{\mu}(T_{even}^u) \leq \overline{\mu}(T_{odd}) \). The proof for the other case is similar. The algorithm introduces a set \( H(I) \) of full cuts at the \( y \)-coordinate values \( y(l(2), \ldots, y(l(2(k+p))) \) and \( H_1(I) = H(I) \cap E_h(I) \). Therefore, \( T_{even} \subseteq H_1(I) \) and to complete the proof of part (ii) it is only required to show that \( \overline{\mu}(H_1(I) \cup H_3(I)) - T_{even} \geq \overline{\mu}(T_{odd}) \). We prove this by identifying a set \( T_{odd} \) of
horizontal line segments such that \( T_{\text{odd}} \subseteq H_1(I) \cup H_3(I), T'_{\text{odd}} \cap T_{\text{even}} = \emptyset \) and showing that there is a 1-1 correspondence between elements \( l' \in T_{\text{odd}} \) and \( l \in T_{\text{odd}} \) with the property that \( L(l') \geq L(l) \).

During the \( w \)th iteration of procedure HVH_CUT, if \( y(l_{q(w+1)}) = y(l_{q(w+1)}) \) or \( y(l_{q(w+1)}) = y(l_{q(w+1)}) \), then \( l_{q(w+1)} \in H_1(I) \) since it is part of a horizontal full cut introduced in step 1 of procedure HVH_CUT. In this case, we let the line segment in \( T_{\text{odd}} \) corresponding to \( l_{q(w+1)} \) be \( l_{q(w+1)} \) itself. So assume that \( y(l_{q(w)}) < y(l_{q(w+1)}) < y(l_{q(w+2)}) \). Let us now find a line segment \( l' \) in \( H_3(I) \) that will be associated with \( l_{q(w+1)} \) such that \( y(l_{q(w)}) < y(l) < y(l_{q(w+2)}) \) and \( L(l) \geq L(l_{q(w+1)}) \). The line segment \( l' \) is added to \( T'_{\text{odd}} \). There are four cases depending on the type of \( l_{q(w+1)} \).

**Case 1.** \( l_{q(w+1)} \) is \( l_{i(m)} \) for some \( m \).

We only prove the case when \( l_{i(m)} \in \text{LEFT} \) is similar. From Lemma 4 we know that the rectangle, defined by the points

\[
(x(rm(P_{i(m)}, l_{i(m)+1})), y(l_{i(m)}), (x(rm(P_{i(m)}, l_{i(m)+1})), y(l_{i(m)})), (R(S), y(l_{i(m)})), (R(S), y(l_{i(m)})))
\]

has no interior line segments from \( E \). Hence, the right most vertical through cut in \( P_{q(w), q(w+2)} \) must be located not to the right of \( x(rm(P_{i(m)}, l_{i(m)+1})) \). Procedure HVH_CUT introduces in step (2) the rightmost vertical through cut in \( P_{q(w), q(w+2)} \). Therefore, there exists a horizontal guillotine cut with length greater than or equal to \( L(l_{q(w+1)}) \) which is introduced in step (3) of procedure HVH_CUT. This line segment is added to \( T_{\text{odd}} \) and \( l_{q(w+1)} \) is associated with it.

**Case 2.** \( l_{q(w+1)} \) is \( l_{j(m)} \) for some \( m \).

Since the proof is similar to case (1), it will be omitted.

**Case 3.** \( l_{q(w+1)} \) is \( l_{j(m)} \) for some \( m \).

We only prove the case when \( l_{j(m)} \in \text{LEFT} \) since the proof for the case when \( l_{j(m)} \in \text{LEFT} \) is similar. From procedure MATCH we know that \( l_{j(m)} = \text{HLS}(y, L(S), x(lm(P_{i(m)-1}, j(m)))) \), where \( y_{i(m)-1} \leq y \leq y_{j(m)} \). Since \( y(l_{q(w)}) < y(l_{q(w+1)}) < y(l_{q(w+2)}) \), we know that \( y_{i(m)-1} < y < y_{j(m)} \). In step (2) of procedure HVH_CUT a vertical through cut with x-coordinate value \( x(lm(P_{i(m)-1}, j(m))) \) is introduced (remember that from Lemmas 5 and 6 we know that \( x(lm(P_{i(m)-1}, l_{j(m)})) = x(lm(P_{i(m)-1}, j(m))) \)) and a subproblem with rectangular boundary

\[
((L(S), y(l_{q(w)})), (L(S), y(l_{q(w+2)})), (x(lm(P_{i(m)-1}, j(m))), y(l_{q(w+2)})), (x(lm(P_{i(m)-1}, j(m))), y(l_{q(w)}))
\]

is generated. A line segment of length equal to \( L(l_{j(m)}) \) is the horizontal guillotine cut introduced in step (3) for this subproblem (note that \( l_{j(m)} \) is a possible horizontal guillotine cut). Thus we can include this line segment in \( T_{\text{odd}} \).

**Case 4.** \( l_{q(w+1)} \) is \( l_{r(m)} \) for some \( m \).

Since the proof is similar to case (3), it will be omitted.

From the above discussion, we can see that for every \( l_{q(w+1)} \in T_{\text{odd}} \), there is a distinct line segment \( l' \in T_{\text{odd}} \) where \( l' \in H_1(I) \cup H_3(I), l' \neq T_{\text{even}} \) and \( L(l') \geq L(l_{q(w+1)}) \). Therefore \( \bar{L}(H_1(I) \cup H_3(I)) \geq \bar{L}(T_{\text{even}}) + \bar{L}(T_{\text{odd}}) \). This completes the proof of the lemma. Q.E.D.

**Lemma 9.** \( \bar{L}(E_a(I)) \leq 0.5 \bar{L}(E_a(I)) \) and \( \bar{L}(E_a(I)) \leq \bar{L}(E_a(I)) \).
PROOF. Since every time the algorithm introduces horizontal cuts is in procedure HVH CUT and since those segments satisfy Lemma 8(iii), we know that $L(E_{opt}(I)) \leq 0.5L(E_{h}(I))$. Since every time a vertical full cut is introduced, either it is a guillotine cut or a through cut that overlaps with some segment in $E(I)$ a proof similar to the one for Lemma 1 can be used to show that $L(E_{v}(I)) \leq L(E_{a}(I))$. This completes the proof of this lemma. Q.E.D.

THEOREM 1. $\tilde{L}(E_{ogp}(I)) \leq 1.75\tilde{L}(E_{opt}(I))$.

PROOF. The proof follows from Lemma 9 and the comments at the beginning of this section. Q.E.D.

Figure 16 shows the guillotine partition obtained by applying our transformation to the rectangular partition given in Fig. 7(a).

4. Discussion

As pointed out in Du et al. (1986), there is a problem instance $I$ such that $\tilde{L}(E_{opt}(I)) = 1.5\tilde{L}(E_{opt}(I))$. In this paper we established the bound $\tilde{L}(E_{ogp}(I)) \leq 1.75\tilde{L}(E_{opt}(I))$. We believe that our upper bound cannot be improved by following our proof technique. However, there might be some other way of proving a smaller approximation bound. Using the techniques in Gonzalez & Zheng (1985b) and the bound obtained in this paper, one can easily find a 2.75 approximation algorithm for partitioning rectilinear polygons with interior points. In Levcopoulos (1986) it is shown how to apply the algorithm, given in Gonzalez & Zheng (1985a), that constructs a rectangular partition for a rectangle with interior points, to solve the GP problem. We believe that a similar technique based on finding an optimal guillotine partition can be used to obtain a smaller approximation bound for the GP problem. The major research question is to develop a faster approximation algorithm that achieves the 1.75 approximation bound or a smaller bound, and to incorporate it to solve the more general problems.
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References


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