Abstract—In previous work, we studied the structure and performance of optimum maximum-likelihood receivers for binary antipodal and orthogonal signals in the presence of Gaussian-distributed channel estimation error and additive white Gaussian noise for flat Rayleigh fading channels. In this paper, we investigate the structure and performance of these receivers for an application where Gaussian-distributed channel estimation error arises: minimum mean-square error channel estimation in quasi-static Rayleigh fading channels. Exact closed-form analytical expressions are derived for the average bit error probability (BEP). We quantify the impact of number of pilot symbols in each frame as well as the ratio of the power of the pilot symbol to the power of data on the average BEP. We derive conditions under which orthogonal signalling results in a lower average BEP compared with binary antipodal signalling.

I. INTRODUCTION

In wireless communication systems, received signals are distorted versions of transmitted signals due to fading channels and noise. Two types of receivers, coherent and noncoherent, are conventionally studied in the literature to detect transmitted symbols [1], [2]. For coherent reception, channel information is precisely known at the receiver while for noncoherent no channel estimate is available at the receiver. The performance of these conventional receivers has been extensively analyzed in the literature for several types of fading channels [1]-[6].

In practical applications, channel information can not be estimated without error at the receiver. Gaussian-distributed channel estimation error is a common model, which generally emerges when a minimum mean-square error (MMSE) estimator is used for estimation in the Bayesian linear model [7]. The performance of MMSE channel estimators for M-ary phase-shift keying (M-PSK) signals as well as M-ary quadrature amplitude (M-QAM) signals are studied in [8] for Rayleigh fading channels and in [9] for Rician fading channels. The results of [8]-[9] are generalized in [10]-[11] for arbitrary two-dimensional signalling.

Orthogonal modulation, such as orthogonal frequency-shift keying (FSK), is another important type of modulation. In [12], we investigated the structure and performance of binary antipodal and orthogonal modulations in the presence of Gaussian-distributed channel estimation error. By using the statistical information of the channel estimation error, we derived the structure of the optimum receiver, based on the maximum-likelihood (ML) criterion, for orthogonal signals and single-antenna reception. We found that in the presence of Gaussian-distributed channel estimation error and additive white Gaussian noise (AWGN) the optimum receiver for orthogonal signals is a linear combination of a matched filter and a square-law detector, which are optimum receivers for purely coherent and totally noncoherent receptions, respectively. Then, we derived exact theoretical expression for the average BEP of the the proposed optimum receiver in Rayleigh fading channels.

In [12], the optimum receiver structure and average BEP expressions are derived for the general case of Gaussian-distributed channel estimation error. In this paper, by using the results of [12], we study the performance and structure of the optimum receivers for MMSE channel estimators in quasi-static Rayleigh fading channels, as a special application where Gaussian-distributed channel estimation error arises. We quantify the effect of number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data on the average BEP. Analytical conditions are derived under which orthogonal modulation has a better performance than binary antipodal signalling in terms of average BEP.

The remainder of the paper is organized as follows: Section II describes the system model, and in section III, we review the results of [12] on the structure and performance of optimum receivers for binary antipodal and orthogonal signalings in the presence of Gaussian channel estimation error. In section IV, the structure and performance of the optimum receivers are analyzed for MMSE channel estimation. Numerical results are presented in section V, and the conclusion is provided in section VI.

II. SYSTEM MODEL

We consider transmitting of binary signals over flat fading channels. For binary antipodal signalling, the received baseband signal $r$ can be written as

$$r = hs + n$$

where $s$ is the transmitted signal which is either $s_1 = \sqrt{E_b}$ or $s_2 = -\sqrt{E_b}$, and $E_b$ is the energy per bit. The fading
channel is denoted by \( h \), and complex AWGN \( n \) is distributed as \( n \sim CN(0, 2\sigma_n^2) \) where \( CN(m, 2\sigma^2) \) denotes a circularly symmetric Gaussian distribution for a complex random variable with mean \( m \) and variance \( 2\sigma^2 \).

For binary orthogonal signalling, e.g. orthogonal frequency-shift keying, the baseband received vector \( r \) is

\[
r = hs + n
\]  

where the transmitted signal vector \( s \) is either \( s_1 = (\sqrt{E_b}h, 0) \) or \( s_2 = (0, \sqrt{E_b}) \), and \( h \) denotes the fading channel again. The vector \( n = (n_1, n_2) \) is the complex AWGN vector, where independently distributed \( n_1 \) and \( n_2 \) are the first and second elements of vector \( n \), respectively, and \( n_1, n_2 \sim CN(0, 2\sigma_n^2) \).

Throughout this paper, channel \( h \) is assumed to be Rayleigh fading with variance \( 2\sigma_h^2 \), i.e. \( h \sim CN(0, 2\sigma_h^2) \). The received SNR per bit \( \gamma_b \) for both antipodal and orthogonal modulations is defined as \( \gamma_b = \frac{E_b|h|^2}{\sigma_n^2} \), and thus, the average SNR per bit \( \bar{\gamma}_b \) is equal to

\[
\bar{\gamma}_b = \frac{E_b}{2\sigma_h^2} E(|h|^2) = \frac{E_b\sigma_h^2}{\sigma_n^2} \tag{3}
\]

where \( E(\cdot) \) denotes the statistical expectation.

To detect the transmitted bits, the channel \( h \) must be estimated at the receiver. In practice channel information cannot be estimated without error. The channel estimation error is defined as

\[
e = h - \hat{h}
\]  

where \( \hat{h} \) is the channel estimate. Channel estimation error \( e \) is assumed to be circularly symmetric Gaussian with variance \( 2\sigma_e^2 \), i.e. \( e \sim CN(0, 2\sigma_e^2) \). Channel estimate \( \hat{h} \) is also circularly symmetric Gaussian distributed with variance \( 2\sigma_{\hat{h}}^2 \), i.e. \( \hat{h} \sim CN(0, 2\sigma_{\hat{h}}^2) \). Channel estimation error \( e \) and channel estimate \( \hat{h} \) are assumed to be mutually independent which is valid for MMSE estimation in which the estimate and the error are orthogonal [7]. We will also verify in section IV that for MMSE channel estimation in fading channels channel estimation error \( e \) and channel estimate \( \hat{h} \) are independent.

Note that since \( \hat{h} \) and \( e \) are independent, from (4) we have \( \sigma_e^2 = \sigma_h^2 - \sigma_{\hat{h}}^2 \), and \( h \) and \( \hat{h} \) are jointly circularly symmetric Gaussian distributed with the correlation coefficient \( \rho \) which can be written as

\[
\rho = \frac{E(h^*\hat{h})}{(\sqrt{2}\sigma_h) (\sqrt{2}\sigma_{\hat{h}})} = \frac{\sigma_h}{\sigma_{\hat{h}}} \sqrt{\frac{\sigma_h^2}{\sigma_{\hat{h}}^2} - \frac{\sigma_e^2}{\sigma_h^2}} \tag{5}
\]

Throughout this paper it is assumed that channel estimation error \( e \) is independent from the noise during transmission of data symbols, i.e. error \( e \) is independent from noise \( n \) in (1) for antipodal signalling and from noise \( n \) in (2) for orthogonal signalling. This assumption will be verified in section IV for MMSE channel estimation.

**III. OPTIMUM RECEIVER STRUCTURE AND PERFORMANCE ANALYSIS**

In this section, we review the results in [12], which were derived for the structure and performance of ML optimum receivers for binary antipodal and orthogonal modulations in the presence of Gaussian channel estimation error.

**A. Antipodal Signalling**

For binary antipodal signalling and single-antenna reception, based on the ML criterion, the optimum receiver is [12, eq. (14)]

\[
\text{Re}\{\hat{h}^*r\} \begin{array}{c}
\hat{s}_1 \\
\hat{s}_2
\end{array} \begin{array}{c}
\geq 0 \\
\geq 0
\end{array}
\]  

where \( \text{Re}\{\cdot\} \) indicates the real part of a complex number.

The receiver in (6) is a coherent receiver that treats the estimated channel as the true channel, i.e. just a matched filter that is matched to the channel estimate. The knowledge of \( \sigma_e^2 \) is not needed to implement the optimum receiver in (6).

The average BEP of the receiver in (6) can be expressed as [12, (35)]

\[
P_b(E) = \frac{1}{2} \left( 1 - \sqrt{\frac{(\sigma_h^2 - \sigma_e^2) \bar{\gamma}_b}{\sigma_h^2 (1 + \bar{\gamma}_b)}} \right)
\]

where \( \bar{\gamma}_b \) is defined in (3).

**B. Orthogonal Signalling**

In orthogonal signalling, the ML decision rule can be expressed as [12, eq. (9)]

\[
\frac{|r_1 - \sqrt{E_b}|h|_1|^2}{E_b\sigma_e^2 + \sigma_h^2} + \frac{|r_2|^2}{\sigma_e^2} \begin{array}{c}
\hat{s}_1 \\
\hat{s}_2
\end{array} \begin{array}{c}
\leq \frac{\bar{\gamma}_b}{\sigma_h^2} r_2 \\
\leq \frac{\bar{\gamma}_b}{\sigma_h^2}
\end{array}
\]

which can be simplified to

\[
\sqrt{E_b\sigma_e^2} |r_1|^2 + \text{Re}\{\hat{h}^*r_1\} \begin{array}{c}
\hat{s}_1 \\
\hat{s}_2
\end{array} \begin{array}{c}
\geq \frac{\bar{\gamma}_b}{\sigma_h^2} E_b\sigma_e^2 |r_2|^2 + \text{Re}\{\hat{h}^*r_2\}
\end{array}
\]

Therefore, in contrast to the optimum receiver for binary antipodal modulations in (6), the optimum receiver for binary orthogonal signals in (9) is not just a coherent receiver that treats the estimated channel as the true channel. In fact, the optimum receiver in (9) is a linear combination of the matched filter receiver and the square-law detector, which are the optimum receivers for coherent and noncoherent receptions, respectively. The combining weights depend on \( \sigma_e^2 \). Note that in order to implement (9), \( \sigma_e^2 \) should be available at the receiver.

The average BEP of the receiver in (9) can be expressed as [12, eq. (32)]

\[
P_b(E) = \frac{1}{2} \left( 1 - \sqrt{\frac{\bar{\gamma}_b(\bar{\gamma}_b\sigma_e^2 + 2(\sigma_h^2 - \sigma_e^2))}{(\bar{\gamma}_b + 2)(\bar{\gamma}_h\sigma_e^2 + 2\sigma_h^2)}} \right)
\]

**C. Performance Comparison**

As mentioned in [12, eq. (36)], by comparing (10) with (7), orthogonal signalling has smaller average BEP than antipodal signalling if

\[
\sigma_e^2 > \frac{\sigma_h^2}{1 + 0.5\bar{\gamma}_h}
\]  

otherwise, antipodal modulation has smaller average BEP.
Expression (11) gives the threshold of $\sigma_r^2$ beyond which orthogonal signalling leads to a better performance.

IV. MMSE CHANNEL ESTIMATION

Analytical expressions of the last section are derived for the general case of Gaussian-distributed channel estimation error when the variance of Gaussian-distributed error is given by $2\sigma_r^2$. In other words, the receiver structure in (9) and average BEP expressions in (7) and (10) are expressed in terms of $\sigma_r^2$.

In this section, we study the structure and performance of optimum receivers of the last section for a special cases where Gaussian-distributed channel estimation error arises: MMSE channel estimation in quasi-static channels. We will show that in this case the estimation error is Gaussian-distributed. Then, we find the variance of error analytically in terms of the average SNR, number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data. Subsequently, by substituting the derived variance of error into the expressions (9), (7) and (10), we will obtain the structure of the optimum receiver for orthogonal signalling as well as the average BEP for antipodal and orthogonal signals, respectively.

In a quasi-static fading channel, the channel remains constant during a whole frame of symbols but varies independently from frame to frame. In order to estimate this type of channel, predetermined symbols (pilot symbols) are transmitted over the channel. Note that in quasi-static fading channels the channel during transmission of pilot symbols in a frame is the same as the channel during transmission of data symbols in that frame. The received signals during this training period are used to estimate the channel, and the estimate is used in the rest of the frame in which data symbols are transmitted. In this part, we use an MMSE channel estimator based on received signals during transmission of pilot symbols. The number of pilot symbols and data symbols in each frame is denoted by $K$ and $M$, respectively.

We study the performance of the optimum receivers in (6) and (9) when multiple pilot symbols are transmitted during the training period. For antipodal signalling, we assume that the pilot symbols are $\sqrt{\beta}s_1 = \sqrt{\beta E_b}e$ where the scalar $\beta$ is the ratio of the power of the pilot symbol to the power of data symbol. The pilot symbols are assumed to be transmitted over $K$ symbol intervals, and hence, the $K \times 1$ observed vector during transmission of $K$ pilot symbols can be expressed as

$$x = \sqrt{\beta E_b}h1_K + \eta$$

where the channel $h$ is fixed during the whole frame, $1_K$ denotes a $K \times 1$ vector of ones, and the vector $\eta$ is the AWGN during transmission of pilot symbols. The vector $\eta$ is assumed to be independent from channel $h$ and also independent from $n$ in (1), the noise during transmission of data symbols. We assume that $\eta \sim CN_K(0, 2\sigma_r^2I_K)$ where $CN_p(\theta, \Phi)$ denotes a complex Gaussian distribution for a $p$-dimensional complex random vector with mean $\theta$ and covariance matrix $\Phi$, and $I_K$ is a $K \times K$ identity matrix.

Note that the model (12) can also be used for orthogonal signalling when pilot symbols are $\sqrt{\beta}s_1 = (\sqrt{\beta E_b}, 0)$ and the observed vector $x$ consists of first elements of the received signal vectors.

In the following, by using an MMSE channel estimator we will derive the channel estimate as a function of the observed vector $x$ in (12) and then calculate the variance of the channel estimation error. We will show that the channel estimate $\hat{h}$ and estimation error $e$ satisfy the conditions mentioned for the model in (4), i.e. $\hat{h}$ and $e$ are circularly symmetric Gaussian distributed and mutually independent.

The observed vector in (12) is of the form of the Bayesian linear model described in chapter 10 of [7]. Therefore, by using theorems 10.3 and 11.1 of [7], the MMSE channel estimate is

$$\hat{h} = E(h|x) = C_{hx}C_{xx}^{-1}x$$

where matrices $C_{hx}$ and $C_{xx}$ are defined as

$$C_{hx} = E(hx^H); \quad C_{xx} = E(xx^H)$$

By substituting (12) in (14), we get

$$C_{hx} = 2\sqrt{\beta E_b}\sigma_h^21_K^H$$

and

$$C_{xx} = 2\beta E_b\sigma_h^2(1_K \times K + 2\sigma_r^2I_K)$$

where $1_K \times K$ denotes a $K \times K$ matrix of ones.

By applying the matrix inversion Lemma [13], the inverse of the matrix $C_{xx}$ in (16) can be written as

$$C_{xx}^{-1} = \frac{\bar{\gamma}_b}{2\beta E_b}K\beta(1 + 1/K\beta_b)^{-1}(1_K \times K)$$

Now, by substituting (15) and (17) in (13) we get

$$\hat{h} = \frac{\sqrt{\beta\gamma_b}}{\sqrt{E_b}(1 + K\beta_b)}1_K^Hx$$

Expression (18) shows how the channel estimate $\hat{h}$ should be computed at the receiver as a function of the observed vector $x$ in (12).

To facilitate calculating the variance of channel estimation error and to show that the derived channel estimate and the channel estimation error are independent and Gaussian distributed, we rewrite the channel estimate $\hat{h}$ as a function of $h$ and $\eta$ by substituting (12) in (18) as

$$\hat{h} = \frac{K\beta\gamma_b}{1 + K\beta_b}h + \frac{\sqrt{\beta\gamma_b}}{\sqrt{E_b}(1 + K\beta_b)}1_K^H\eta$$

From (19), the channel estimation error is equal to

$$e = h - \hat{h} = \frac{1}{1 + K\beta_b}h - \frac{\sqrt{\beta\gamma_b}}{\sqrt{E_b}(1 + K\beta_b)}1_K^H\eta$$

Equation (20) shows that $e$ is caused by $\eta$ (the noise during transmission of pilot symbols), and hence, $e$ is independent of the AWGN during transmission of data signals. Note that
\[ \frac{\bar{\gamma}_b}{\sqrt{E_b} (1 + K \bar{\beta} \gamma_b)} |r_1[m]|^2 + \text{Re} \left\{ \tilde{h}^* r_1[m] \right\} \]
\[ \frac{\bar{\gamma}_b}{\sqrt{E_b} (1 + K \bar{\beta} \gamma_b)} |r_2[m]|^2 + \text{Re} \left\{ \tilde{h}^* r_2[m] \right\} \]

(25)

\( \hat{h} \) and \( e \) in (19) and (20) are circularly symmetric Gaussian distributed since \( \hat{h} \) and the elements of vector \( \eta \) are circularly symmetric Gaussian distributed and mutually independent. It can also be seen from (19) and (20) that \( E (\hat{h}^* e) = 0 \), i.e. random variables \( \hat{h} \) and \( e \) are orthogonal, and hence independent. Therefore, \( \hat{h} \) and \( e \) satisfy the conditions mentioned for the model in (4).

Now, from (20), \( \sigma^2_e \) is equal to

\[ \sigma^2_e = \frac{1}{2} E \left( |e|^2 \right) = \frac{1}{2} \left( \frac{2 \sigma_h^2}{(1 + K \bar{\beta} \gamma_b)^2} + \frac{2 K \bar{\beta} \gamma_b \sigma_n^2}{E_b (1 + K \bar{\beta} \gamma_b)^2} \right) \]
\[ = \frac{\sigma_h^2}{1 + K \bar{\beta} \gamma_b} \] (21)

From (21), the variance of channel estimation error decreases as \( K \) or \( \beta \) or \( \bar{\gamma}_b \) increases, as expected.

Now, from (6), for antipodal signals the optimum detector for \( s[m] \), the \( m \)th data symbol, is

\[ \text{Re} \left\{ \hat{h}^* r[m] \right\} \]
\[ \hat{s}[m] = \begin{cases} s_1 & \text{for } m = 1, \ldots, M \end{cases} \] (22)

where \( r[m] \) is the received signal when \( s[m] \) is transmitted, \( \hat{s}[m] \) is the \( m \)th estimated data symbol and \( \hat{h} \) is given by (18).

By substituting (21) in (7), the average BEP of the optimum receiver (22) during a whole frame of data for antipodal signals is equal to

\[ P_b(E) = \frac{1}{2} \left( 1 - \frac{\sqrt{K \bar{\beta} \gamma_b}}{\sqrt{1 + (K \bar{\beta} + 1) \gamma_b + K \bar{\beta} \gamma_b^2}} \right) \] (23)

For \( K = 0 \), the average BEP in (23) is equal to 0.5, i.e. the BEP for noncoherent reception, as expected. For one-symbol observation \( (K = 1) \), when \( \beta = 1 \) the expression (23) simplifies to

\[ P_b(E) = \frac{1}{2} \left( 1 - \frac{\bar{\gamma}_b}{1 + \bar{\gamma}_b} \right) = \frac{1}{2} (1 + \bar{\gamma}_b) - \frac{1}{2} \] (24)

which is equal to the average BEP of binary differential phase-shift keying (DPSK) \([1, \text{eq. (14.3-10)}]\). In other words, with only one pilot symbol the BEP of DPSK can be achieved. Therefore, for \( K > 1 \) a better performance than DPSK modulation is expected, which shows how efficient the MMSE estimator is. Note that as \( K \to \infty \), the expression (23) reduces to \([1, \text{eq. (14.3-7)}]\), the average BEP of purely coherent reception for antipodal signalling.

For binary orthogonal signals, by substituting (21) in (9), the structure of the optimum receiver for the \( m \)th data symbol, \( s[m] \), can be written as shown in (25) at the top of this page where \( r_1[m] \) and \( r_2[m] \) are the first and second elements of the received vector, respectively, \( \hat{s}[m] \) is the estimate of \( s[m] \), and \( \hat{h} \) is given by (18).

Now, by substituting (21) in (10), the average BEP of the optimum receiver in (25) over a whole frame of data symbols for orthogonal signals is equal to

\[ P_b(E) = \frac{1}{2} \left( 1 - \frac{\sqrt{2K \beta + 1} \bar{\gamma}_b}{\sqrt{4 + 4(K \beta + 1) \gamma_b + (2K \beta + 1) \gamma_b^2}} \right) \] (26)

For \( K = 0 \), expression (26) reduces to \([1, \text{eq. (14.3-12)}]\), the average BEP for noncoherent reception. On the other hand, as \( K \to \infty \), (26) converges to \([1, \text{eq. (14.3-8)}]\), the average BEP for purely coherent reception, as expected.

To compare antipodal modulation with orthogonal modulation, by substituting (21) in (11) we find that orthogonal signalling has a better performance than antipodal if

\[ \frac{\sigma_h^2}{1 + K \bar{\beta} \gamma_b} > \frac{\sigma_n^2}{1 + 0.5 \bar{\gamma}_b} \] (27)

which can be simplified to

\[ K \beta < 0.5 \] (28)

The result in (28) can also be obtained by comparing (23) and (26).

V. NUMERICAL RESULTS

In this section, we study the performance of the optimum receivers presented in the last section for flat Rayleigh fading channels with MMSE channel estimation.

Figs. 1 and 2 show the average BEP of the optimum receivers in (22) and (25) versus the average SNR \( \bar{\gamma}_b \) for binary antipodal and orthogonal signals, respectively, by both theory (expressions (23) and (26)) and Monte Carlo simulations. The curves are plotted for different numbers of pilot symbols \( K \) when \( \beta \), the ratio of the power of the pilot symbol to the power of data symbol, is equal to one. The Monte Carlo simulations are performed until \( 10^4 \) samples of errors are observed for each data point. It is evident that the analytical results match precisely the Monte Carlo simulations. The average BEP of optimum coherent receiver for antipodal signals in \([1, \text{eq. (14.3-7)}]\) as well as the average BEP of optimum coherent and non-coherent receivers for orthogonal signals in \([1, \text{eq. (14.3-8)}]\) and \([1, \text{eq. (14.3-12)}]\) have also been added for comparison in Figs. 1 and 2, respectively. It is clear that the performance of the optimum receivers improve as the number of pilot symbols \( K \) increases, as expected.

By using theoretical expressions (23) and (26), the average BEP of antipodal and orthogonal modulations are compared in Fig. 3 for different values of \( K \beta \). As can be seen, for \( K \beta < 0.5 \) orthogonal signalling has a better performance while for \( K \beta > 0.5 \) antipodal modulation outperforms or-
In this paper, we investigated the structure and performance of the optimum receivers for binary antipodal and orthogonal signals for MMSE channel estimation in quasi-static Rayleigh fading channels. Exact closed-form analytical expressions were derived for the average BEP of optimum receivers in terms of number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data. We obtained conditions under which orthogonal signalling results in a better performance than binary antipodal signalling in terms of average BEP. We found that orthogonal signalling results in a lower average BEP compared with antipodal modulation if the product of number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data is less than 0.5.

VI. CONCLUSION

In this paper, we investigated the structure and performance of the optimum receivers for binary antipodal and orthogonal signals for MMSE channel estimation in quasi-static Rayleigh fading channels. Exact closed-form analytical expressions were derived for the average BEP of optimum receivers in terms of number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data. We obtained conditions under which orthogonal signalling results in a better performance than binary antipodal signalling in terms of average BEP. We found that orthogonal signalling results in a lower average BEP compared with antipodal modulation if the product of number of pilot symbols in each frame and the ratio of the power of the pilot symbol to the power of data is less than 0.5.

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