APPROXIMATIONS TO $m$-COLOURED COMPLETE INFINITE SUBGRAPHS

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Abstract. Given an edge colouring of a graph with a set of $m$ colours, we say that the graph is exactly $m$-coloured if each of the colours is used. In 1996, Stacey and Weidl, partially resolving a conjecture of Erickson from 1994, showed that for a fixed natural number $m > 2$ and for all sufficiently large $k$, there is a $k$-colouring of complete graph on $\mathbb{N}$ such that no complete infinite subgraph is exactly $m$-coloured. In the light of this result, we consider the question of how close we can come to finding an exactly $m$-coloured complete infinite subgraph. We show that for a natural number $m$ and any finite colouring of the edges of the complete graph on $\mathbb{N}$ with $m$ or more colours, there is an exactly $m'$-coloured complete infinite subgraph for some $m'$ satisfying $|m - m'| \leq \sqrt{\frac{2}{m}} + \frac{1}{2}$; this is best possible up to an additive constant. We also obtain analogous results for this problem in the setting of $r$-uniform hypergraphs. Along the way, we also prove a recent conjecture of the second author and investigate generalisations of this conjecture to $r$-uniform hypergraphs.

1. Introduction

The classical problem of Ramsey theory is to find a large monochromatic structure in a larger coloured structure; for a host of results, see [4]. On the other hand, the objects of interest in anti-Ramsey theory are large “rainbow coloured” or “totally multicoloured” structures; see, for example, the paper of Erdős, Simonovits and Sós [2]. Between these two ends of the spectrum, one could consider the question of finding structures which are coloured with exactly $m$ different colours: this was first done by Erickson [3] and this is the line of enquiry that we pursue here.

Our notation is standard. Thus, following Erdős, for a set $X$, we write $X^{(r)}$ for the family of all subsets of $X$ of cardinality $r$; equivalently, $X^{(r)}$ is the complete $r$-uniform hypergraph on the vertex set $X$. We write $|n|$ for $\{1,\ldots,n\}$, the set of the first $n$ natural numbers. By a colouring of a hypergraph, we will always mean a colouring of its edges.

Let $\Delta : \mathbb{N}^{(r)} \to [k]$ be a surjective $k$-colouring of the edges of the complete $r$-uniform hypergraph on the natural numbers. We say that a subset $X \subset \mathbb{N}$ is exactly $m$-coloured if $\Delta[X^{(r)}]$, the set of values attained by $\Delta$ on the edges induced by $X$, has size exactly $m$. Let $\gamma_{\Delta}(X)$, or $\gamma(X)$ in short, denote the size of the set $\Delta[X^{(r)}]$; in other words, every set $X$ is

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γ(X)-coloured. In this paper, we will study for fixed r and large k, the set of values m for which there exists an infinite m-coloured set with respect to a colouring $\Delta : \mathbb{N}^r \rightarrow [k]$. (It is also interesting to study what happens when we also consider finite sets or allow colourings with infinitely many colours; see [5] for some results of this flavour.) With this in mind, let us define

$$F_\Delta := \{ \gamma(X) : X \subset \mathbb{N} \text{ such that } X \text{ is infinite} \}.$$  

Clearly, $k \in F_\Delta$ as $\Delta$ is surjective. Ramsey’s theorem tells us that $1 \in F_\Delta$. In the case of graphs, i.e., when $r = 2$, Erickson [3] noted that a fairly straightforward application of Ramsey’s theorem enables one to show that $2 \in F_\Delta$ for any surjective $k$-colouring $\Delta$ with $k \geq 2$. He also conjectured that with the exception of $1, 2$ and $k$, no other elements are guaranteed to be in $F_\Delta$ and that if $k > m > 2$, then there is a surjective $k$-colouring $\Delta$ of $\mathbb{N}^2$ such that $m \notin F_\Delta$. Stacey and Weidl [7], partially resolving this conjecture, showed using a probabilistic construction that there is a constant $C_m$ such that if $k > C_m$, then there is a surjective $k$-colouring $\Delta$ of $\mathbb{N}^2$ such that $m \notin F_\Delta$.

Since an exactly $m$-coloured complete infinite subgraph is not guaranteed to exist, we are naturally led to the question of whether we can find complete infinite subgraphs that are exactly $m'$-coloured for some $m'$ close to $m$. In this paper, we establish the following result.

**Theorem 1.** For any $\Delta : \mathbb{N}^r \rightarrow [k]$ and any natural number $m \leq k$, there exists $m' \in F_\Delta$ such that $|m - m'| \leq c_r m^{1-1/r} + O(m^{1-2/r})$, where $c_r = r/2(\Gamma(1))^1/r$.

Theorem 1 is tight up to the $O(m^{1-2/r})$ term. To see this, let $k = \binom{n}{a} + 1$ for some natural number $a$. We consider the “small-rainbow colouring” $\Delta$ which colours all the edges induced by $[a]$ with $\binom{n}{a}$ distinct colours and all the remaining edges with the one colour that has not been used so far. In this case, we see that $F_\Delta = \{ \binom{n}{a} + 1 : n \leq a \}$. Now let $m = \binom{l}{r} + \binom{l+1}{r} + 2$ for some natural number $l$ such that $l < a$. It follows that $|m - m'| \leq \binom{l}{r-1}$ for each $m' \in F_\Delta$; furthermore, it is easy to check that $\binom{l}{r-1}/2 = (c_r - o(1)) m^{1-1/r}$.

In the case of graphs where $r = 2$, Theorem 1 tells us that for any finite colouring of the edges of the complete graph on $\mathbb{N}$ with $m$ or more colours, there is an exactly $m'$-coloured complete infinite subgraph for some $m'$ satisfying $|m - m'| \leq \sqrt{m} + O(1)$; a careful analysis of the proof of Theorem 1 in this case allows us to replace the $O(1)$ term with an explicit constant, $1/2$.

We know from Theorem 1 that $F_\Delta$ cannot contain very large gaps. Another natural question we are led to ask is if there are any sets, and in particular, intervals that $F_\Delta$ is guaranteed to intersect. Making this more precise, the second author conjectured in [6] that the small-rainbow colouring described above is extremal for graphs in the following sense.

**Conjecture 2.** Let $\Delta : \mathbb{N}^2 \rightarrow [k]$ and suppose $n$ is a natural number such that $k > \binom{n}{2} + 1$. Then $F_\Delta \cap \left( \binom{n}{2} + 1, \binom{n+1}{2} + 1 \right] \neq \emptyset$. 

In this paper, we shall prove this conjecture. There are two natural generalisations of this conjecture to \( r \)-uniform hypergraphs which are equivalent Conjecture 2 in the case of graphs.

The first comes from considering small-rainbow colourings; indeed we can ask whether \( F_{\Delta} \cap I_{r,n} \neq \emptyset \) when \( k > \binom{n}{r} + 1 \), where \( I_{r,n} \) is the interval \((\binom{n}{r} + 1, \binom{n+1}{r} + 1]\).

The second comes from considering a different type of colouring, namely the “small-set colouring”. Let \( k = \sum_{i=0}^{r} (a_i) \) and consider the surjective \( k \)-colouring \( \Delta \) of \( N^{(r)} \) defined by \( \Delta(e) = e \cap [a] \). Note that in this case, \( F_{\Delta} = \{ \sum_{i=0}^{r} (a_i) : n \leq a \} \). Consequently, we can ask whether \( F_{\Delta} \cap J_{r,n} \neq \emptyset \) when \( k > \sum_{i=0}^{r} (n-1) \), where \( J_{r,n} \) is the interval \((\sum_{i=0}^{r} (n-1), \sum_{i=0}^{r} (n)\)]. Note that both these questions are identical when \( r = 2 \); indeed \( (n^2) + (n^1) + (n^0) = (n^2 + 1) + 1 \) and so \( I_{2,n} = J_{2,n} \).

We shall demonstrate that the correct generalisation is the former. We shall first prove that the answer to the first question is in the affirmative, provided \( n \) is sufficiently large.

**Theorem 3.** For every \( r \geq 2 \), there exists a natural number \( n_r \geq r - 1 \) such that for any \( \Delta : N^{(r)} \to [k] \) and any natural number \( n \geq n_r \) with \( k > \binom{n}{r} + 1 \), we have \( F_{\Delta} \cap I_{r,n} \neq \emptyset \).

Using a result of Baranyai [1] on factorisations of uniform hypergraphs, we shall exhibit an infinite family of colourings that answer the second question negatively for every \( r \geq 3 \).

**Theorem 4.** For every \( r \geq 3 \), there exist infinitely many values of \( n \) for which there exists a colouring \( \Delta : N^{(r)} \to [k] \) with \( k > \sum_{i=0}^{r} (n-1) \) such that \( F_{\Delta} \cap J_{r,n} = \emptyset \).

In the next section, we shall prove Theorems 1, 3 and 4 and deduce Conjecture 2 from the proof of Theorem 3. We then conclude by mentioning some open problems.

### 2. PROOFS OF THE MAIN RESULTS

We start with the following lemma which we will use to prove both Theorems 1 and 3.

**Lemma 5.** Let \( m \geq 2 \) be an element of \( F_\Delta \). Then there exists a natural number \( a = a(m, \Delta) \) such that

1. \( \sum_{i=0}^{r} (a_i) \geq m \), and
2. \( F_{\Delta} \cap \left[m - \min \left(\sum_{i=0}^{r-1} (a_i-1), \frac{r(m-1)}{a}\right), m\right) \neq \emptyset \).

Furthermore, if \( m = \sum_{i=t+1}^{r} (\binom{a}{i}) + s + 1 \) for some \( s \geq 0 \) and \( 0 \leq t + 1 \leq r \), then

\[
F_{\Delta} \cap \left[\sum_{i=t+1}^{r} \binom{a-1}{i} + \left(1 - \frac{t}{a}\right)s + 1, m\right) \neq \emptyset.
\]
Proof. We start by establishing the following claim.

**Claim A.** There is an infinite $m$-coloured set $X \subset \mathbb{N}$ with a finite subset $A \subset X$ such that

(i) the colour of every edge of $X$ is determined by its intersection with $A$, i.e., $e_1 \cap A = e_2 \cap A \Rightarrow \Delta(e_1) = \Delta(e_2)$, and

(ii) $\gamma(X\{v\}) < m$ for all $v \in A$.

Proof. To see this, let $W \subset \mathbb{N}$ be an infinite $m$-coloured set. For each colour $c \in \Delta [W^{(r)}]$, pick an edge $e_c$ in $W$ of colour $c$ and let $A$ be the set of vertices incident to some edge $e_c$. So $A \subset W$ is a finite $m$-coloured set. Let $A_1, A_2, \ldots, A_l$ be an enumeration of subsets of $A$ of size at most $r$. Note that this is the complete list of possible intersections of an edge with $A$. We now define a descending sequence of infinite sets

$$B_0 = W \setminus A.$$ 

Having defined the infinite set $B_{i-1}$, we induce a colouring of the $(r - |A_i|)$-tuples $T$ of $B_{i-1}$, by giving $T$ the colour of the edge $A_i \cup T$. By Ramsey’s theorem, there is an infinite monochromatic subset $B_i \subset B_{i-1}$, whose intersection with $A$ is $A_i$, have the same colour.

Hence, $X = A \cup B_l$ is an infinite $m$-coloured set satisfying property (i). Observe that any subset of $X$ also satisfies property (i). Now, if we have a vertex $v \in A$ such that $\gamma(X\{v\}) = m$, we delete $v$ from $A$. We repeat this until we are left with an $m$-coloured set $X$ satisfying (i) and (ii).

Let $X$ and $A$ be as guaranteed by Claim A. Note that $A$ is nonempty since $m \geq 2$. We shall prove the lemma with $a(m, \Delta) = |A|$. From the structure of $X$ and $A$, we note that

$$\sum_{i=0}^{r-1} \binom{a}{i} \geq m.$$ 

That $\mathcal{F}_\Delta \cap \left[ m - \min \left( \sum_{i=0}^{r-1} \binom{a-1}{i}, \frac{r(m-1)}{a} \right), m \right] \neq \emptyset$ is a consequence of the following claim.

**Claim B.** There exist infinite sets $X_1, X_2 \subset X$ such that $m - \sum_{i=0}^{r-1} \binom{a-1}{i} \leq \gamma(X_1) < m$ and $m - \frac{r(m-1)}{a} \leq \gamma(X_2) < m$.

Proof. Let $X_1 = X \setminus \{v\}$ for any $v \in A$. We know from Claim A that $\gamma(X_1) < m$. We shall now prove that $\gamma(X_1) \geq m - \sum_{i=0}^{r-1} \binom{a-1}{i}$; that is, the number of colours lost by removing $v$ from $X$ is at most $\sum_{i=0}^{r-1} \binom{a-1}{i}$. Since the colour of an edge is determined by its intersection with $A$, the number of colours lost is at most the numbers of subsets of $A$ containing $v$ of size at most $r$, which is precisely $\sum_{i=0}^{r-1} \binom{a-1}{i}$.

Next, we shall prove that there is a subset $X_2 \subset X$ such that $m - \frac{r(m-1)}{a} \leq \gamma(X_2) < m$. Let $A = \{v_1, v_2, \ldots, v_a\}$ and let

$$C_i = \Delta \left[ X^{(r)} \right] \setminus \Delta \left[ (X \setminus \{v_i\})^{(r)} \right]$$

be the set of colours lost by removing $v_i$ from $X$; since $\gamma(X \setminus \{v_i\}) < m$ for all $v_i \in A$, we have $C_i \neq \emptyset$. For each colour $c \in \Delta \left[ X^{(r)} \right]$, pick an edge $e_c$ of colour $c$, and let $A_c = e_c \cap A$;
in particular, we take $A_{c_0} = \emptyset$, where $c_0$ is the colour corresponding to an empty intersection with $A$. Since every edge of colour $c \in C_i$ contains $v_i$, we have

$$\sum_{i=1}^{a} |C_i| \leq \sum_{c \neq c_0} |A_c| \leq r(m - 1),$$

and so there exists an $i$ such that $0 < |C_i| \leq \frac{r(m-1)}{a}$; the claim follows by taking $X_2 = X\backslash\{v_i\}$. \hfill \Box

We finish the proof of the lemma by establishing the following claim.

**Claim C.** If we can write $m = \sum_{i=t+1}^{r} \binom{a}{i} + s + 1$, then

$$\mathcal{F}_\Delta \cap \left[ \sum_{i=t+1}^{r} \binom{a-1}{i} + \left(1 - \frac{t}{a}\right)s + 1, m \right] \neq \emptyset.$$

**Proof.** As in the proof of Claim B, for each colour $c \in \Delta \left[ X^{(r)} \right]$, pick an edge $e_c$ of colour $c$, and let $A_c = e_c \cap A$; in particular, let $A_{c_0} = \emptyset$. We know from Claim A that edges of $X$ of distinct colours cannot have the same intersection with $A$. Consequently, all the $A_c$’s are distinct subsets of $A$, each of size at most $r$. Hence,

$$\sum_{c \neq c_0} |A_c| \leq \sum_{i=t+1}^{r} \binom{a}{i} + ts.$$

Arguing as in the proof of Claim B, we conclude that there exists a vertex $v \in A$ such that the number of colours lost by removing $v$ from $X$ is at most $\frac{1}{a} \left( \sum_{i=t+1}^{r} \binom{a}{i} + ts \right)$. Therefore,

$$\gamma(X\backslash\{v\}) \geq m - \frac{1}{a} \left( \sum_{i=t+1}^{r} \binom{a}{i} + ts \right) = m - \left( \sum_{i=t+1}^{r} \binom{a-1}{i-1} + \frac{ts}{a} \right) = \sum_{i=t+1}^{r} \binom{a-1}{i} + \left(1 - \frac{t}{a}\right)s + 1,$$

and so

$$\mathcal{F}_\Delta \cap \left[ \sum_{i=t+1}^{r} \binom{a-1}{i} + \left(1 - \frac{t}{a}\right)s + 1, m \right] \neq \emptyset.$$

as required. \hfill \Box

The lemma follows from Claims A, B and C. We are done. \hfill \Box

Having established Lemma 5, it is easy to deduce both Theorem 1 and 3 from the lemma.
Proof of Theorem 1. Let \( t = m + c_r m^{1-1/\alpha} \). We may assume that \( m > r^{\alpha}/\alpha! \) since otherwise \( m = O(1) \) and there is nothing to prove. Let \( m'' \) be the smallest element of \( F_\Delta \) greater than \( t \). Applying Lemma 5 to \( m'' \), we find an \( m' \in F_\Delta \) such that \( m' \leq t \) and

\[
m' \geq m'' - \min \left( \sum_{i=0}^{r-1} \binom{a-1}{i}, \frac{r(m'' - 1)}{a} \right)
\]

for some natural number \( a \). Now if \( a \geq (r!m)^{1/\alpha} > r \), then

\[
m' \geq m'' - \frac{r(m'' - 1)}{a} \geq m'' \left( 1 - \frac{r}{a} \right) \geq t \left( 1 - \frac{r}{a} \right)
\]

and so it follows that \( m' \geq m - c_r m^{1-1/\alpha} - O(m^{1-2/\alpha}) \). If on the other hand, \( a < (r!m)^{1/\alpha} \), then using the fact that

\[
m' \geq m'' - \sum_{i=0}^{r-1} \binom{a-1}{i} \geq t - \frac{a^{r-1}}{(r-1)!} - O(a^{r-2}) \geq t - \frac{(r!m)^{1-1/\alpha}}{(r-1)!} - O(m^{1-2/\alpha}),
\]

it follows once again that \( m' \geq m - c_r m^{1-1/\alpha} - O(m^{1-2/\alpha}) \).

Proof of Theorem 3. If \( k \leq \binom{n+1}{r} + 1 \), we are done since \( k \in F_\Delta \). So suppose that \( k > \binom{n+1}{r} + 1 \). Let \( m \) be the smallest element of \( F_\Delta \) such that \( m > \binom{n+1}{r} + 1 \); hence, \( F_\Delta \cap \binom{n+1}{r} + 1, m \) = \( \emptyset \). Now, since \( m \geq 2 \), there exists by Lemma 5, a natural number \( a \) such that

\[
F_\Delta \cap \left[ m - \frac{r(m - 1)}{a}, \binom{n+1}{r} + 1 \right] \neq \emptyset.
\]

To prove the theorem, it is sufficient to show that \( m - \frac{r(m - 1)}{a} > \binom{n}{r} + 1 \). We know from Lemma 5 that \( \sum_{i=0}^{r} \binom{a}{i} \geq m > \binom{n+1}{r} + 1 \). If \( n \) is sufficiently large, we must have \( a \geq n \).

If \( a \geq n + 1 \), then

\[
m - \frac{r(m - 1)}{a} = (m - 1) \left( 1 - \frac{r}{a} \right) + 1 > \binom{n+1}{r} \left( 1 - \frac{r}{n+1} \right) + 1 = \binom{n}{r} + 1
\]

since \( m > \binom{n+1}{r} + 1 \) and \( n \geq r - 1 \).

We now deal with the case \( a = n \). First, we write \( m = \binom{n}{r} + \binom{n}{r-1} + s + 1 \). Since \( m > \binom{n+1}{r} + 1 \) and \( \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r-1} \), we see that \( s > 0 \). By Lemma 5, we have

\[
F_\Delta \cap \left[ \binom{n}{r} + \left( 1 - \frac{r-2}{n} \right) s + 1, m \right] \neq \emptyset.
\]

Since \( n \geq r - 1 \) and \( s > 0 \), the result follows.

A careful inspection of the proof of Theorem 3 shows that when \( r = 2 \), the statement holds for all \( n \in \mathbb{N} \). We hence obtain a proof Conjecture 2. By constructing a sequence of highly structured subgraphs, the second author [6] proved that for any \( \Delta : \mathbb{N}(2) \rightarrow [k] \) with \( k \geq \binom{n}{2} + 1 \) for some natural number \( n \), \( |F_\Delta| \geq n \); Conjecture 2 gives a short proof of this lower bound.
Theorem 3 also yields a generalisation of this lower bound for \( r \)-uniform hypergraphs, albeit with a constant additive error term (which depends on \( r \)).

We now turn to the proof of Theorem 4. We will need a result of Baranyai’s [1] which states that the set of edges of the complete \( r \)-uniform hypergraph on \( l \) vertices can be partitioned into perfect matchings when \( r \mid l \).

**Proof of Theorem 4.** We shall show that if \( n \) is sufficiently large and \( (r - 1) \mid (n + 1) \), then there is a surjective \( k \)-colouring \( \Delta \) of \( \mathbb{N}^{(r)} \) with \( k > \sum_{i=0}^{r} \binom{n-i}{r-1} \) and \( \mathcal{F}_\Delta \cap J_{r,n} = \emptyset \). We shall define a colouring of \( \mathbb{N}^{(r)} \) such that the colour of an edge \( e \) is determined by its intersection with a set \( A \) of size \( n + 1 \), say \( A = [n + 1] \). Let \( \mathcal{B} \) be the family of all subsets of \( A \) of size at most \( r \). For \( B \in \mathcal{B} \), we denote the colour assigned to all the edges \( e \) such that \( e \cap A = B \) by \( c_B \).

To define our colouring, we shall construct a partition \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) with \( \emptyset \in \mathcal{B}_2 \). Then, for every \( B \in \mathcal{B}_2 \), we set \( c_B \) to be equal to \( c_\emptyset \). Finally, we take the colours \( c_B \) for \( B \in \mathcal{B}_1 \) to all be distinct and different from \( c_\emptyset \). Hence, the number of colours used is \( k = |\mathcal{B}_1| + 1 \). It remains to construct this partition of \( \mathcal{B} \).

Since \( (r - 1) \mid (n + 1) \), by Baranyai’s theorem we have an ordering \( B_1, B_2, \ldots, B_{\binom{n+1}{r-1}} \) of the subsets of \( A \) of size \( r - 1 \) such that for all \( 0 \leq t \leq \binom{n}{r-2} \), the family

\[
\left\{ B_{\binom{n+1}{r-1}+1}, B_{\binom{n+1}{r-1}+2}, \ldots, B_{\binom{n+1}{r-1}(t+1)} \right\}
\]

is a perfect matching. Let \( \mathcal{B}_1 = \{ B_1, B_2, \ldots, B_s \} \cup \{ B \in \mathcal{B} : |B| = r \} \), where

\[
s = \sum_{i=0}^{r} \binom{n}{i} - \binom{n+1}{r};
\]

our colouring is well defined because \( 0 \leq s \leq \binom{n+1}{r-1} \) for all sufficiently large \( n \).

Observe that

\[
k = |\mathcal{B}_1| + 1 = \binom{n+1}{r} + s + 1 = \sum_{i=0}^{r} \binom{n}{i} + 1.
\]

We shall show that the second largest element of \( \mathcal{F}_\Delta \) is at most \( \sum_{i=0}^{r} \binom{n-1}{i} \). Note that any \( X \subset \mathbb{N} \) with \( \gamma(X) < k \) cannot contain \( A \). As before, let \( C_i \) be the set of colours lost by removing \( i \in A \) from \( \mathbb{N} \), i.e.,

\[
C_i = \Delta \left[ \mathbb{N}^{(r)} \right] \setminus \Delta \left[ (\mathbb{N} \setminus \{i\})^{(r)} \right].
\]

We shall complete the proof by showing that \( k - |C_i| \leq \sum_{i=0}^{r} \binom{n-1}{i} \) for all \( i \in A \). Note that our construction ensures that \( ||C_i| - |C_j|| \leq 1 \) for all \( i, j \in A \). Now, observe that

\[
\sum_{i=1}^{n+1} |C_i| = \sum_{B \in \mathcal{B}_1} |B| = r \binom{n+1}{r} + (r-1)s,
\]
and so $|C_i| \geq \frac{1}{n+1} \left( r\left( \frac{n+1}{r} \right) + (r-1)s \right) - 1$ for all $i \in A$. Hence,

$$k - |C_i| \leq \left( \binom{n+1}{r} + s + 1 \right) - \frac{1}{n+1} \left( r\left( \frac{n+1}{r} \right) + (r-1)s \right) + 1$$

$$= \binom{n}{r} + \left( 1 - \frac{r-1}{n+1} \right) s + 2$$

$$= \binom{n}{r} + \left( 1 - \frac{r-1}{n+1} \right) \left( \sum_{i=0}^{r} \binom{n}{i} - \binom{n+1}{r} \right) + 2$$

$$\leq \sum_{i=0}^{r} \binom{n-1}{i},$$

where the last inequality holds when $r \geq 4$ for all sufficiently large $n$.

When $r = 3$, it is easy to check that $s = n+1$ and so $s$ is divisible by $(n+1)/(r-1) = (n+1)/2$. Consequently, in this case, $|C_i| = |C_j|$ for $i, j \in A$. Hence, we have

$$k - |C_i| \leq \left( \binom{n+1}{3} + s + 1 \right) - \frac{1}{n+1} \left( r\left( \binom{n+1}{3} + 2s \right) \right)$$

$$= \binom{n}{3} + \left( 1 - \frac{2}{n+1} \right) (n+1) + 1$$

$$= \sum_{i=0}^{3} \binom{n-1}{i}.$$ 

This completes the proof. \qed

3. Concluding Remarks

We conclude by mentioning two open problems. We proved that for any $\Delta : \mathbb{N}^{(r)} \to [k]$ and every sufficiently large natural number $n$, we have $\mathcal{F}_\Delta \cap I_{r,n} \neq \emptyset$ provided $k > \binom{n}{r} + 1$. A careful analysis of our proof shows that the result holds when $n \geq \left( \frac{5}{2} + o(1) \right) r$; we chose not to give details to keep the presentation simple. However, we suspect that the result should hold as long as $n \geq r - 1$ but a proof eludes us.

To state the next problem, let us define

$$\psi_r(k) := \min_{\Delta : \mathbb{N}^{(r)} \to [k]} |\mathcal{F}_\Delta|.$$ 

A consequence of Theorem 3 is that $\psi_r(k) \geq (r!k)^{1/r} - O(1)$. Turning to the question of upper bounds for $\psi_r$, the small-rainbow colouring shows that the lower bound that we get from Theorem 3 is tight infinitely often, i.e., when $k$ is of the form $\binom{a}{r} + 1$ for some natural number $a$. However, when $k$ is not of the form $\binom{a}{r} + 1$, the obvious generalisations of the small-rainbow colouring fail to give us good upper bounds for $\psi_r(k)$. In [6], the second author
proved that

\[ \psi_2(k) = O \left( \frac{k}{(\log \log k)^9 (\log \log \log k)^{3/2}} \right) \]

for almost all natural numbers \( k \) and some absolute constant \( \delta > 0 \). The same construction can be extended to show that \( \psi_r(k) = o(k) \) for almost all natural numbers \( k \). It would be very interesting to decide if, in fact, \( \psi_r(k) = o(k) \) for all \( k \in \mathbb{N} \).

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