SYNCHRONIZING CHAOTIC ATTRACTORS OF CHUA’S CANONICAL CIRCUIT. THE CASE OF UNCERTAINTY IN CHAOS SYNCHRONIZATION

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Received November 15, 2004; Revised May 10, 2005

In this paper, we have studied the dynamics of two identical resistively coupled Chua’s canonical circuits and have found that it is strongly affected by initial conditions, coupling strength and the presence of coexisting attractors. Depending on the coupling variable, chaotic synchronization has been observed both numerically and experimentally. Anti-phase synchronization has also been studied numerically clarifying some aspects of uncertainty in chaos synchronization.

Keywords: Chua’s canonical circuit; coupled systems; chaos synchronization.

1. Introduction

Electric circuits have emerged as a simple yet powerful experimental and analytical tool in studying chaotic behavior in nonlinear dynamics. Most chaotic and bifurcation effects cited in the literature have been observed in electric circuits e.g. the period-doubling route to chaos [Linsay, 1981; Van Buskirk & Jeffries, 1985; Kennedy, 1993; Chua et al., 1993a; Murali & Lakshmanan, 1991], intermittency route to chaos [Jeffries & Perez, 1982; Huang & Kim, 1987; Richetti et al., 1986; Fukushima & Yamada, 1988; Chua & Lin, 1991], quasiperiodicity route to chaos [Matsumoto et al., 1987; Chua et al., 1986b; Petrani et al., 1994], crisis [Jeffries & Perez, 1983; Ikezi et al., 1983; Rollins & Hunt, 1984; Kyprianidis et al., 1995], antimonotonicity [Kocarev et al., 1993; Kyprianidis et al., 2000] and hyperchaos [Matsumoto et al., 1986; Mitsubori & Saito, 1994; Kapitaniak & Chua, 1994; Kapitaniak et al., 1994c; Tamaševičius et al., 1997; Tamaševičius & Čenys, 1998; Kyprianidis & Stouboulos, 2003a].

It has been shown, that it is possible to construct a set of chaotic systems, so that their common signals will have identical or synchronized behavior [Pecora & Carroll, 1990; Chua et al., 1993b; Murali & Lakshmanan, 1993]. Generally, there are two methods of chaos synchronization available in the literature. In the first method due to Pecora and Carroll [1990], a stable subsystem of a chaotic system could be synchronized with a separate chaotic system under certain suitable conditions. The second method to achieve chaos...
synchronization between two identical nonlinear systems is due to the effect of resistive coupling without requiring to construct any stable subsystem [Murali & Lakshmanan, 1994; Kapitaniak et al., 1994b; Murali et al., 1995; Kyprianidis & Stouboulos, 2003a, 2003b]. For all these coupled systems there exists a critical value, or synchronization threshold, that corresponds to a blowout bifurcation [Ott & Sommerer, 1994]. Two interesting phenomena, namely, on-off intermittency and attractors with riddled basins of attraction are expected to take place in the vicinity of the bifurcation point.

In contrast to the studies before, Zhong et al. [2001] investigated the uncertainty of chaos synchronization in the sense that coupled systems, which satisfy the standard synchronization criterion, do not necessarily operate in a synchronization regime. They proved that depending on the initial states of the coupled systems, distinct behaviors of synchronization, such as in-phase synchronization, anti-phase synchronization, oscillation-quenching, and attractor bubbling, may take place. Their study was focused on the dynamical behaviors of two resistively coupled identical Chua’s circuits.

Chua’s circuit is a paradigm for chaos [Madan, 1993]. Among the members of Chua’s circuit family, the autonomous canonical Chua’s circuit introduced by Chua and Lin [1990] is of considerable importance. This is because it is capable of realizing the behavior of every member of the Chua’s circuit family [Chua & Lin, 1990, 1991; Kyprianidis et al., 1995]. It consists of two active elements, one linear negative conductor, and one nonlinear resistor with odd-symmetric piecewise linear $v$–$i$ characteristic.

In this paper we have studied chaotic dynamics of two resistively, unidirectionally and bidirectionally coupled, identical canonical Chua’s circuits. We have focused on the cases of existence of synchronization and of uncertainty in chaos synchronization.

### 2. Dynamics of Coupled Systems

It is well known that two coupled nonlinear systems can exhibit a variety of distinct performances of dynamical behavior, among which chaotic synchronization is of special importance. To achieve synchronization between two systems, which are chaotic, the linear coupling technique is commonly used [Chua et al., 1993b; Kapitaniak et al., 1994b; Pecora & Carroll, 1991]. In particular, for a system with two coupled subsystems $\mathbf{u}$ and $\mathbf{v}$, the state equation of the system in the case of bidirectional coupling can be written as:

\[
\begin{align*}
\frac{d\mathbf{u}}{dt} &= f_1(\mathbf{u}) + E(\mathbf{v} - \mathbf{u}) \\
\frac{d\mathbf{v}}{dt} &= f_2(\mathbf{v}) - E(\mathbf{v} - \mathbf{u})
\end{align*}
\]

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $E = \text{diag}[\varepsilon_1, \ldots, \varepsilon_n]^T$.

In the case of unidirectional coupling, we have the following state equation

\[
\begin{align*}
\frac{d\mathbf{u}}{dt} &= f_1(\mathbf{u}) \\
\frac{d\mathbf{v}}{dt} &= f_2(\mathbf{v}) - E(\mathbf{v} - \mathbf{u})
\end{align*}
\]

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $E = \text{diag}[\varepsilon_1, \ldots, \varepsilon_n]^T$.

The synchronization problem is formulated in the following way: find $E$ such that $s = ||\mathbf{v}(t) - \mathbf{u}(t)|| \rightarrow 0$ as $t \rightarrow \infty$. This means that in the case of bidirectional coupling the desired limit (synchronous) solution cannot be predetermined. In the case of unidirectional coupling, the submanifold $\mathbf{v}(t)$ tends to coincide with the submanifold $\mathbf{u}(t)$ of the system.

In this study we have focused on the case that the two subsystems are identical, that is, $f_1 = f_2 = f$. We have studied the influence of initial conditions on chaos synchronization, especially the cases when the two subsystems have: (a) equal initial conditions with opposite signs, (b) arbitrary initial conditions, which create chaotic coexisting attractors.

Zhong et al. [2001] have proved that, when two identical bidirectionally coupled systems with odd symmetry have equal initial conditions with opposite signs, i.e. $\mathbf{u}(t = 0) = -\mathbf{v}(t = 0)$, and, if there is a constant $K > 0$ such, that $f'(\mathbf{q}) \leq K \mathbf{I} < 0$ uniformly for all $\mathbf{q} \in \mathbb{R}^n$, where $\mathbf{I}$ is the identity matrix, then $\mathbf{u}(t) = -\mathbf{v}(t)$ for all $t \geq 0$, independent of the value of $E > 0$. They named this type of dynamical behavior as anti-phase synchronization.

### 3. The Chua’s Canonical Circuit

We have adopted Chua’s canonical circuit [Chua & Lin, 1990, 1991] as a paradigm to study the dynamical behaviors referred to in Sec. 2. Chua’s canonical circuit is shown in Fig. 1(a). It has a nonlinear resistor $N_R$ with an odd symmetric $v$–$i$ characteristic of type-N [Fig. 1(b)]. $G_n$ is a negative linear conductance.
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The state equations of the circuit are
\[
\begin{align*}
\frac{dv_{C1}}{dt} &= \frac{1}{C_1}\{i_L - i_N\} \\
\frac{dv_{C2}}{dt} &= \frac{1}{C_2}\{-G_nv_{C2} - i_L\} \\
\frac{di_L}{dt} &= \frac{1}{L}\{-v_{C1} + v_{C2} - Ri_L\}
\end{align*}
\]
while the current \(i_N\) through the nonlinear resistor is
\[
i_N = g(v_{C1}) = G_b v_{C1} + 0.5(G_a - G_b) \\
\times \{|v_{C1} + B_p| - |v_{C1} - B_p|\}
\]
These state equations take the dimensionless form
\[
\begin{align*}
\frac{dx}{d\tau} &= \alpha\{z - h(x)\} \\
\frac{dy}{d\tau} &= -\beta y - z \\
\frac{dz}{d\tau} &= \gamma(y - x - z)
\end{align*}
\]
where
\[
h(x) = m_b x + 0.5(m_a - m_b)\{|x + 1| - |x - 1|\}
\]
is the dimensionless function of the current through the nonlinear resistor. The dimensionless variables and parameters of Eqs. (5) and (6) are defined as follows:
\[
\begin{align*}
x &= \frac{v_{C1}}{B_p}, \quad y = \frac{v_{C2}}{B_p}, \quad z = \frac{Ri_L}{B_p} \\
\alpha &= \frac{C_2}{C_1}, \quad \beta = RG_n, \quad \gamma = \frac{R^2C_2}{L}, \\
m_a &= RG_a, \quad m_b = RG_b
\end{align*}
\]
with the dimensionless time variable
\[
\tau = \frac{t}{RC_2}
\]
We have chosen the following parameter values for the Chua canonical circuit throughout our study: \(L = 100\) mH, \(R = 330\) Ω, \(C_1 = 33.0\) nF, \(G_n = -0.5\) mS, \(G_a = -0.105\) mS, \(G_b = 4.70\) mS, \(B_p = 0.68\) V. \(C_2\) can take different values giving a double-scroll strange attractor or a spiral strange attractor. In Fig. 2, the bifurcation diagram \(v_{C1}\) versus \(C_2\) is shown for the chosen parameter values.

4. Linearly Coupled Chua’s Canonical Circuits

Based on the Chua’s canonical circuit presented above, the state equation of the unidirectionally-coupled system (2), in the case of \(f_1 = f_2\), is written
\[
\begin{align*}
\frac{dx_1}{d\tau} &= \alpha\{z_1 - h(x_1)\} \\
\frac{dy_1}{d\tau} &= -\beta y_1 - z_1 \\
\frac{dz_1}{d\tau} &= \gamma(y_1 - x_1 - z_1)
\end{align*}
\]
\[
\frac{dx_2}{d\tau} = \alpha \{z_2 - h(x_2)\} + \varepsilon_x (x_1 - x_2)
\]
\[
\frac{dy_2}{d\tau} = -\beta y_2 - z_2 + \varepsilon_y (y_1 - y_2)
\]
\[
\frac{dz_2}{d\tau} = \gamma (y_2 - x_2 - z_2) + \varepsilon_z (z_1 - z_2)
\] (9)

where
\[
h(x_1) = m_b x_1 + 0.5(m_a - m_b) \{|x_1 + 1| - |x_1 - 1|\}
\] (10)

\[
h(x_2) = m_b x_2 + 0.5(m_a - m_b) \{|x_2 + 1| - |x_2 - 1|\}
\] (11)

and \(\varepsilon_j (j = x, y, z)\) are the coupling factors.

The state equations of the bidirectionally-coupled system (1), in the case of \(f_1 = f_2\), on the other hand, take the form
\[
\frac{dx_1}{d\tau} = \alpha \{z_1 - h(x_1)\} - \varepsilon_x (x_1 - x_2)
\]
\[
\frac{dy_1}{d\tau} = -\beta y_1 - z_1 - \varepsilon_y (y_1 - y_2)
\]

Fig. 3. Experimental phase portraits \(\psi_{C_2}\) versus \(\psi_{C_1}\): (a) the spiral attractor \((C_2 = 64.0 \text{nF})\); horizontal axis 0.5 V/div, vertical axis 1.0 V/div, (b) the double-scroll attractor \((C_2 = 58.0 \text{nF})\); horizontal axis 0.5 V/div, vertical axis 1.0 V/div, (c) the double-scroll attractor \((C_2 = 43.0 \text{nF})\); horizontal axis 0.5 V/div, vertical axis 1.0 V/div, (d) period-1 limit cycle \((C_2 = 41.0 \text{nF})\); horizontal axis 0.5 V/div, vertical axis 5.0 V/div.
\[
\begin{align*}
\frac{dz_1}{d\tau} &= \gamma (y_1 - x_1 - z_1) - \varepsilon_z (z_1 - z_2) \\
\frac{dx_2}{d\tau} &= \alpha \{ z_2 - h(x_2) \} + \varepsilon_x (x_1 - x_2) \\
\frac{dy_2}{d\tau} &= -\beta y_2 - z_2 + \varepsilon_y (y_1 - y_2) \\
\frac{dz_2}{d\tau} &= \gamma (y_2 - x_2 - z_2) + \varepsilon_z (z_1 - z_2)
\end{align*}
\] (12)

where
\[
\begin{align*}
h(x_1) &= m_b x_1 + 0.5(m_a - m_b)\{|x_1 + 1| - |x_1 - 1|\} \\
h(x_2) &= m_b x_2 + 0.5(m_a - m_b)\{|x_2 + 1| - |x_2 - 1|\}
\end{align*}
\] (13)

and \(\varepsilon_j (j = x, y, z)\) are the coupling factors.

4.1. \textit{x-Coupled system}

Figure 4 shows the experimental set up of the unidirectionally-coupled system. By removing the voltage follower from the coupling branch, we obtain the bidirectionally \(x\)-coupled system. In this case, only the \(x\) state variable is coupled. This means that \(\varepsilon_y = \varepsilon_z = 0\), while \(\varepsilon_x = \alpha R/R_c\).

For \(C_2 = 60.0\,\text{nF}\), the uncoupled circuits are in a chaotic state having a \textit{double scroll} strange attractor. In Fig. 5, the bifurcation diagrams \(y_2 - y_1\) versus \(\varepsilon_x\) are shown for unidirectional and bidirectional coupling. Synchronization, \(y_2 - y_1 = 0\) is observed in both cases. Clearly, in the second case, synchronization is established for a lower critical value of the coupling factor \(\varepsilon_x\).

When the strange attractor of the canonical Chua’s circuit is a \textit{spiral} one, then coexisting attractors are observed, as shown in Fig. 6. We have

![Diagram](image-url)
studied the dynamics of such a coupled system in the following cases:

(a) the two sub-circuits have initial conditions belonging to the same basin of attraction,
(b) the two sub-circuits have initial conditions belonging to different basin of attraction. In both cases, synchronization is not observed (Figs. 7 and 8) independently of the way of coupling (unidirectional or bidirectional).

4.2. \textit{y}-Coupled system

Figure 9 shows the experimental set up of a unidirectionally-coupled system with \textit{y}-coupling. By removing the voltage follower from the coupling branch, we obtain the bidirectionally \textit{y}-coupled system. Since only the \textit{y} state variable is coupled, this means that $\varepsilon_x = \varepsilon_z = 0$, while $\varepsilon_y = R/R_c$.

For $C_2 = 60.0$ nF, the uncoupled circuits are in a chaotic state having a double scroll strange

![Fig. 6. Coexisting spiral attractors for $C_2 = 63.0$ nF.](image)

![Fig. 7. Bifurcation diagrams $y_2 - y_1$ versus $\varepsilon_x$, for (a) unidirectional, and (b) bidirectional coupling, in the case $C_2 = 63.0$ nF. Synchronization is not observed in any case. The initial conditions of each individual circuit belong to different basins of attraction.](image)
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Fig. 8. Bifurcation diagrams $y_2 - y_1$ versus $\varepsilon_x$, for (a) unidirectional, and (b) bidirectional coupling, in the case $C_2 = 70.0 \text{ nF}$. Synchronization is not observed in any case. The initial conditions of each individual circuit belong to the same basin of attraction.

Fig. 9. The unidirectionally-coupled Chua's canonical circuits ($y$-coupling).

attractor. In Fig. 10, the bifurcation diagrams $y_2 - y_1$ versus $\varepsilon_y$ are shown for unidirectional and bidirectional $y$-coupling. Synchronization, $y_2 - y_1 = 0$, is now observed in both cases. In the second case, however, synchronization is established for a lower critical value of the coupling factor $\varepsilon_y$.

When the strange attractor of the canonical Chua's circuit is a spiral one, synchronization is always observed even though the initial conditions of the $y$-coupled circuits belong to different basin of attraction (Fig. 11).

Fig. 10. Bifurcation diagrams $y_2 - y_1$ versus $\varepsilon_y$, for (a) unidirectional, and (b) bidirectional $y$-coupling, in the case $C_2 = 60.0 \text{ nF}$. Synchronization is observed in both cases.
5. Experimental Results

In the following figures, (Figs. 12 and 13), we now exhibit some characteristic experimental phase space portraits of the coupled system for different values of the coupling factor. In Figs. 12(a) and 13(a) the system is out of synchronization, while in Figs. 12(b) and 13(b) the system is in chaotic synchronization.

The results of Fig. 13(b) demonstrate that the system can experimentally achieve synchronization, even at parameter values not predicted by the simulations. That the converse may also occur is discussed in the next section.
6. Uncertainty in Chaos Synchronization

According to the bifurcation diagram of Fig. 5(a), in the case of unidirectional $x$-coupling for $C_2 = 60.0 \text{nF}$, chaotic synchronization must be observed for $\varepsilon_x > 0.315$. The experimental realization of the coupled system revealed that a different situation can also be observed (Fig. 14), in which the system, for a sufficiently high value of $\varepsilon_x$, can be out of synchronization.

To see how this can occur, let us examine more closely the experimental attractors of the driving (master) and the driven (slave) circuit shown here in Fig. 15. The attractor of the driving circuit is the well-known double scroll attractor [Fig. 15(a)], while the attractor of the driven circuit is an unknown chaotic attractor [Fig. 15(b)].

In the absence of coupling, for $C_2 = 60.0 \text{nF}$, there are two coexisting attractors. The chaotic double scroll attractor (see also Fig. 15(a) and a limit-cycle of period-1 (Fig. 16)). The initial conditions of the chaotic double scroll attractor are $x_0 = -1.0$, $y_0 = 0.40$, and $z_0 = 0.50$, while the initial conditions of the limit-cycle of period-1 are $x_0 = -1.0$, $y_0 = 1.0$, and $z_0 = 2.0$.

For $\varepsilon_x = 1.20$, the computer simulated phase portraits, $x_1$ versus $x_2$ of the $x$-coupled system, and $y_2$ versus $x_2$ of the driven system, are shown in Fig. 17. The driven circuit oscillates in the basin of attraction of the big limit cycle giving a new type of chaotic attractor, similar to the one observed experimentally in Fig. 15(b).

In contrast to this property of the double scroll attractor, the chaotic spiral attractor does not
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Fig. 15. Experimental phase portrait, for the unidirectionally $x$-coupled system, when $C_2 = 60.0 \text{nF}$. The coupling factor is $\varepsilon_x = 1.20$. (a) $v_{C21}$ versus $v_{C11}$, horizontal axis 0.5 V/div, vertical axis 0.5 V/div; (b) $v_{C22}$ versus $v_{C12}$, horizontal axis 5.0 V/div, vertical axis 0.5 V/div.

Fig. 16. The coexisting attractors for $C_2 = 60.0 \text{nF}$. The initial conditions are (a) $x_0 = -1.0$, $y_0 = 0.40$, $z_0 = 0.50$, for the chaotic double scroll attractor, (b) $x_0 = -1.0$, $y_0 = 1.0$, $z_0 = 2.0$, for the limit-cycle of period-1.

have a coexisting periodic attractor, so an uncertainty of chaos synchronization of this kind is not observed in that case.

6.1. **Anti-phase synchronization**

As we have mentioned in Sec. 2, anti-phase synchronization can be observed, when two identical bidirectionally coupled systems with odd symmetry have equal initial conditions with opposite sign.

For $C_2 = 60.0 \text{nF}$, each Chua’s canonical circuit of the system with the parameter values listed above operated in a double-scroll attractor before being coupled. We have chosen the following sets of initial conditions for the computer simulation:

\[
\begin{align*}
    x_1(0) &= -1.0, \quad y_1(0) = 0.4, \quad z_1(0) = 0.5, \\
    x_2(0) &= 1.0, \quad y_2(0) = -0.4, \quad z_2(0) = -0.5.
\end{align*}
\]

6.1.1. **$x$-Coupled system**

As the coupling factor $\varepsilon_x$ is increased, starting from zero (the uncoupled case), a change of the type of the attractor of each subsystem is observed. The double-scroll attractor gives its place to a chaotic spiral attractor and then a reverse period doubling sequence is observed, while anti-phase synchronization is conserved.

The reverse period doubling sequence as the coupling factor $\varepsilon_x$ increases is shown in Fig. 20.

For $0.215 \leq \varepsilon_x \leq 0.680$, the oscillations in the two Chua’s canonical circuits were quenched and each circuit operated in a point attractor located at $(0, 0)$.

For $\varepsilon_x \geq 0.690$ oscillations were present again and each Chua’s canonical circuit operated in a period-1 attractor with anti-phase.

6.1.2. **$y$-Coupled system**

In the case of the $y$-coupled system, the reverse period-doubling sequence is also observed, as the
Fig. 17. The computer simulated phase portraits, (a) $x_1$ versus $x_2$ of the $x$-coupled system, and (b) $y_2$ versus $x_2$ of the driven system, for $C_2 = 60.0 \text{nF}$ and $\varepsilon_x = 1.20$.

Fig. 18. Phase portrait $y_2$ versus $y_1$ in the case of anti-phase synchronization ($y$-coupled system, $\varepsilon_y = 0.001$) and the computational environment.
Fig. 19. $y_2$ versus $x_2$ and $y_2$ versus $y_1$ phase portraits, in the case of anti-phase synchronization of the bidirectionally $x$-coupled identical Chua’s canonical circuits with opposite identical initial conditions, for $C_2 = 60.0 \text{nF}$: (a) $\varepsilon_x = 0.010$. Double-scroll attractor. (b–d) $\varepsilon_x = 0.020, 0.130, 0.142$. Spiral attractors. (e) $\varepsilon_x = 0.146$. Period-4 attractor. (f) $\varepsilon_x = 0.150$. Period-2 attractor. (g) $\varepsilon_x = 0.180$. Period-1 attractor.
Fig. 19. (Continued)
coupling factor $\varepsilon_y$ is increased. For $0.030 \leq \varepsilon_y \leq 0.390$ the oscillations in the two Chua’s canonical circuits were quenched and each circuit operated in a stable point attractor. The coordinates of the point attractor are dependent on the value of $\varepsilon_y$. For $\varepsilon_y \geq 0.40$, the point attractor is located at $(0, 0, 0)$.

In the case of $y$-coupled identical Chua’s canonical circuits with opposite identical initial conditions, the transition of the system from the point attractor to the period-1 attractor was not observed.

7. Concluding Remarks

In the present work, using two identical Chua’s canonical circuits resistively coupled, we have studied the process of synchronization, both in unidirectional and bidirectional coupling forms, using computer simulation and experiment. The results have shown that the dynamics of the coupled system is strongly affected by the initial states and the coupling strength. Furthermore, although the experimental results generally agree with the predictions of the simulations, there are some differences, which demonstrate, that the actual electrical circuits may include additional effects, which are not present in our systems of differential equations.

Our observations suggest that the synchronization state of the $y$-coupled system is the most robust. When each individual circuit operates in a double-scroll or a spiral chaotic attractor, chaotic synchronization is always observed, when the coupling strength exceeds a critical value.

In the case of the $x$-coupled system, however, when each individual circuit operates in a spiral attractor mode, chaotic synchronization is not observed. On the other hand, computer simulations show that when the circuits display double-scroll attractor dynamics, chaotic synchronization can be observed. Generally, the convergence of the system to the state of synchronization is very slow. Our experimental studies have also demonstrated that the state of synchronization is very sensitive to the unavoidable mismatches of the components of the
coupled circuits, leading to an experimental uncertainty of chaotic synchronization.

The study of anti-phase synchronization by computer simulation verified the reverse period-doubling scenario and oscillation-quenching observed by Zhong et al. [2001]. A new feature we noted, in the case of the $x$-coupled system, is the period-1 oscillation, which become possible after the state of oscillation-quenching, as the coupling factor is further increased.

References


