Defining and Computing Topological Persistence for 1-cocycles

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Abstract

The concept of topological persistence, introduced recently in computational topology, finds applications in studying a map in relation to the topology of its domain. Since its introduction, it has been extended and generalized in various directions. However, no attempt has been made so far to extend the concept of topological persistence to a generalization of ‘maps’ such as cocycles which are discrete counterparts of closed differential forms, a well known concept in differential geometry. We define a notion of topological persistence for 1-cocycles in this paper and show how to compute its relevant numbers. It turns out that, instead of the standard persistence, one of its variants which we call level persistence can be leveraged for this purpose. It is worth mentioning that 1-cocycles appear in practice such as in data ranking or in discrete vector fields.

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1 Introduction

The persistent homology, introduced recently \[7\] in computational topology, studies the longevity of topological features as one sweeps a topological space with growing sublevel sets of a real-valued function. Since its introduction, the concept has been generalized in many directions, including its study for homology groups under different coefficient rings \[12\], its stability under function perturbations \[3\], its multidimensional version \[2\], and other variants \[1, 4, 6, 9\].

In a discrete setting, which is prevalent in practice, the topological space in question is usually taken as the one defined by a simplicial complex \(X\) whereas the function in question is taken as a linear map \(f: X \to \mathbb{R}\) whose restriction to each simplex is linear. Such a pair \((X, f)\) provides what is called a 0-cochain in algebraic topology. A 0-cochain can be specified by providing only a \(n\)-dimensional vector of \(f\)-values on \(n\) vertices of \(X\). Such data appear frequently in applications involving shape and data analysis. Similarly, a more general input may specify real values on edges of the simplicial complex by presenting a skew symmetric \(n \times n\) matrix for a simplicial complex \(X\) with \(n\) vertices. Such a data, in reverence to the term 0-cochain, is called a 1-cochain. An interesting situation, which occurs in practice, is when the 1-cochain is a 1-cocycle (definition in section 2). For example, in data ranking, pairwise orderings which are locally consistent represent a 1-cocycle \[13, 14\]. A sampling of a curl-free vector field on a Riemannian manifold provides interesting examples of 1-cocycles.

It is natural to study the qualitative complexity of 0-cochains and 1-cocycles in analogy with Morse theory and Novikov theory \[11\] respectively. In particular, one would like to relate the dynamics of the 0-cochain and 1-cocycle with the topology of \(X\). For 0-cochains, this is precisely achieved by studying persistent topology. Although the concept has been generalized in many directions as we mentioned, it has not yet been extended to 1-cocycles.

In this paper, we provide a definition of persistence for almost integral 1-cocycles\[^1\] defined on a simplicial complex and an algorithm to compute relevant numbers. First, we show that a 1-cocycle when almost integral\[^2\] can be interpreted as a \(S^1\)-valued (circle valued) map. The standard persistence \[7, 12\] does not work for such maps since sublevel sets are not defined. We observe that level persistence originally introduced as interval persistence in \[6\] can be leveraged to define a notion of persistence for circle-valued continuous maps. This persistence uses a lifting of the circle-valued map which becomes a linear map on a lifted space of \(X\) which, in topologists’ terminology, is the infinite cyclic cover associated with the cohomology class of the almost integral cocycle. We show how one can calculate the relevant numbers for level persistence for this lifted map by adapting the algorithm for standard persistence. This adaptation is simplified by our observation that the incidence structure of a cell complex involved in level persistence computation can be derived easily from the given simplicial complex with a filtration.

It is appropriate to mention that the level persistence considered here can be approached from the Zigzag persistence introduced in \[11\] based on quiver representations. However, the outcome is not the same. In particular, our relevant level persistence numbers are not the same as invariants in \[11\]. It would be interesting to recover our numerical invariants from the barcode type invariants of \[11\]. This and the Morse Novikov theory for circle valued maps from the perspective of quiver representations will be topics of further research.

2 1-cocycle

Let \(X = (X_0, \mathcal{X})\) be a simplicial complex where \(X_0\) is the set of vertices, and \(\mathcal{X}\) is a set of finite subsets of \(X_0\) which satisfy the following properties :

1. \(X_0 \subseteq \mathcal{X}\) and

\[^1\]the general case can be reduced to almost integral case
\[^2\]values on integral 1-cycles are integer multiples of a fixed real number
2. If $\sigma \in \mathcal{X}$ then $\tau \subset \sigma \Rightarrow \tau \in \mathcal{X}$.

The subsets in $\mathcal{X}$ of cardinality $k + 1$ are called $k$-simplexes. These subsets, considered as ordered $k$-tuples up to an even permutation, represent oriented $k$-simplexes. We denote sets of oriented $k$-simplexes by $\mathcal{X}_k$. We continue to denote by $X$ the canonical topological space (the geometric realization) associated with the combinatorial structure $X$.

Broadly speaking, cochains are real valued functions that assign numbers to oriented simplexes. Instead of introducing the most general definition of cochains, we only define 0- and 1-cochains which are sufficient for this exposition. A 0-cochain is a function restricted to vertices, $f : \mathcal{X}_0 \rightarrow \mathbb{R}$. A 0-cochain can be identified with a continuous map $f : X \rightarrow \mathbb{R}$ whose restriction to each simplex is linear (a linear map). The 0-cochain $f$ is generic if $f : \mathcal{X}_0 \rightarrow \mathbb{R}$ is injective.

The 1-cochains are a generalization of 0-cochains in that their domain is the set $\mathcal{X}_1$ of oriented edges of $X$. Let $f : \mathcal{X}_1 \rightarrow \mathbb{R}$ be a 1-cochain defined on $X$. Recall that $\mathcal{X}_1$ can be regarded as the subset of $\mathcal{X}_0 \times \mathcal{X}_0$ consisting of pairs $(x, y)$ which are vertices of a 1-simplex. The map $f$ is a 1-cocycle if it satisfies:

1. $f(x, y) = -f(y, x)$ for any ordered pair $(x, y) \in \mathcal{X}_1$, and
2. if $(x, y, z) \in \mathcal{X}_2$ then $f(x, y) + f(y, z) + f(z, x) = 0$; equivalently $f(x, y) + f(y, z) = f(x, z)$.

Clearly, a 0-cochain $f$, or equivalently a linear map, provides a 1-cocycle $\delta f$ defined by

$$\delta f(x, y) = f(y) - f(x).$$

Any 1-cocycle $f$ represents a cohomology class $\langle f \rangle \in H^1(X; \mathbb{R})$ and any such cohomology class is represented by a 1-cocycle. Two 1-cocycles $f_1$ and $f_2$ represent the same cohomology class iff $f_1 - f_2 = \delta f$ for some 0-cochain $f$.

An almost integral 1-cocycle is a pair $(f, \alpha)$ where $f$ is a 1-cocycle whose values on integral 1-cycles are integer multiple of a fixed positive real $\alpha$.

If $St(x)$ denotes the star of the vertex $x \in \mathcal{X}_0$ (the star of any simplex is a sub complex), a 1-cocycle $f$ defines a unique function $f_x : St(x) \rightarrow \mathbb{R}$ by the formulae $f_x(x) = 0$ and $f_x(y) = f(x, y)$ for any vertex $y \neq x$ in $St(x)$. Clearly, $(f_x - f_y)(z)$ is constant in $z$ for any $z$ in a connected component of $St(x) \cap St(y)$.

Thus, a 1-cocycle can be thought as a collection of linear maps $\{f_x : St(x) \rightarrow \mathbb{R}\}$ for each vertex $x$, such that the difference $f_x - f_y$ is constant on each connected component of $St(x) \cap St(y)$. Therefore, one can regard $f \equiv \{f_x, x \in \mathcal{X}_0\}$ as a multivalued linear map. A 1-cocycle $f$ is generic if all linear maps $f_x$ are generic, i.e., injective when restricted to vertices of $St(x)$.

### 2.1 1-cocycles and circle valued maps

Consider a continuous circle valued map $f : X \rightarrow S^1$. Let $p : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\alpha \mathbb{Z}$ be the map defined by $p(t) = t(\text{mod} \alpha)$, $\alpha$ a positive real number. For any simplex $\sigma \in X$, the restriction $f|_{\sigma}$ admits liftings $\tilde{f} : \sigma \rightarrow \mathbb{R}$, i.e., $\tilde{f}$ is a continuous map which satisfies $p \circ \tilde{f} = f|_{\sigma}$. The map $f : X \rightarrow S^1$ is called linear if, for any simplex $\sigma$, at least one of the liftings (and then any other) is linear. Any linear map $f : X \rightarrow S^1 = \mathbb{R}/\alpha \mathbb{Z}$ defines an almost integral 1-cocycle $(f, \alpha)$ by assigning to each edge $e$ with vertices $x, y$,

$$f(x, y) = \tilde{f}(y) - \tilde{f}(x)$$

where $\tilde{f}$ is a lift of $f$ restricted to $e$. Now, we show the opposite, that is,

- an almost integral 1-cocycle $(f : \mathcal{X}_1 \rightarrow \mathbb{R}, \alpha)$ can be associated to a linear map $f : X \rightarrow S^1 = \mathbb{R}/\alpha \mathbb{Z}$.

\[^3\text{in the language of algebraic topology the cohomology class } \langle f \rangle \in H^1(X; \mathbb{R}) \text{ has degree of rationality 1, or equivalently the image of the induced homomorphism } H_1(X; \mathbb{Z}) \rightarrow \mathbb{R} \text{ is } \alpha \mathbb{Z} \subset \mathbb{R}.\]
Covering associated with a cohomology class \(< f > \in H^1(X; \mathbb{R})\): Regard \(X\) as a topological space. Choose a base point \(x \in X\) and consider the space of continuous paths \(\gamma : [0, 1] \to X\) with \(\gamma(0) = x\), equipped with compact open topology. Make two continuous paths \(\gamma_1\) and \(\gamma_2\) equivalent iff the \(\gamma_1(1) = \gamma_2(1)\) and the closed path \(\gamma_1 \ast \gamma_2^{-1}\) satisfies \(< f > ([(\gamma_1 \ast \gamma_2^{-1})]) = 0\). Here \(\ast\) denotes the concatenation of the paths \(\gamma_1\) and \(\gamma_2^{-1}\) defined by \(\gamma_2^{-1}(t) = \gamma_2(1-t)\), and \([\gamma_1 \ast \gamma_2^{-1}]\) denotes the homology class of \(\gamma_1 \ast \gamma_2^{-1}\). The quotient space \(\tilde{X}\), whose underlying set is the set of equivalence classes of paths, is equipped with the canonical map \(\pi : \tilde{X} \to X\) induced by assigning to \(\gamma\) the point \(\gamma(1) \in X\). The map \(\pi\) is a local homeomorphism and \(X\) is the total space of a principal covering with group \(G\) where

\[
G = \text{img}(< f > : H_1(X; \mathbb{Z}) \to \mathbb{R}).
\]

When \(f\) is almost integral, \(G\) is isomorphic to \(\mathbb{Z}\). If \(X\) is equipped with a triangulation, then \(\tilde{X}\) gets a triangulation whose simplexes, when viewed as subsets of \(X\) are homeomorphic by \(\pi\) to simplexes of \(X\) (when viewed as subsets of \(X\)).

Construction of \(\tilde{f}\) and \(f\). We construct the circle valued map \(f\) via its cyclic cover \(\tilde{f}\) as follows.

Step 1. Consider \(\pi : \tilde{X} \to X\) the principal \(\mathbb{Z}\)–covering associated with the cohomology class \(< f >\) defined by \(f\). This means that \(\tilde{X}\) is a simplicial complex equipped with a free simplicial action \(\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}\) whose quotient space, \(\tilde{X}/\mu\), is the simplicial complex \(X\). The map \(\pi\) identifies to \(\tilde{X} \to X\) and satisfies \(\pi^* < f > = 0\).

Choose a vertex \(x\) of \(X\) and call it a base point. Notice that the vertices \(\tilde{x}_0\) of \(\tilde{X}\) can be also described as equivalence classes of sequences \(\{x = x_0, x_1, \ldots \}\) with \(x_i\)'s being consecutive vertices of \(X\) (i.e. \(x_i, x_{i+1}\) are vertices of an edge). Two such sequences, \(\{x = x_0, x_1, \ldots \}\) and \(\{x = y_0, y_1, \ldots \}\) are equivalent if \(x_N = y_L\) and the sequence \(\{y = z_0, \ldots , z_{N+L}\}\) with \(z_i = x_i\) if \(i \leq N\) and \(z_{j+N} = y_{L-j}\) if \(j \leq L\), satisfies

\[
\sum_{0 \leq i \leq L+N-1} f(z_i, z_{i+1}) = 0.
\]

Step 2. Define the map \(\tilde{f} : \tilde{X}_0 \to \mathbb{R}\) by \(\tilde{f}(\tilde{y}) := \sum_{0 \leq i \leq L-1} f(y_i, y_{i+1})\) where \(\tilde{y} \in \tilde{X}_0\) is the vertex corresponding to the equivalent class of \(\{y = y_0, \ldots , y_L\}\). The description of \(\tilde{X}\) given above guarantees that \(\tilde{f}\) is well defined. Extend \(\tilde{f}\) to a linear map \(\tilde{f} : \tilde{X} \to \mathbb{R}\). Observe that if \(\tilde{y}_1\) and \(\tilde{y}_2\) satisfy \(\pi(\tilde{y}_1) = \pi(\tilde{y}_2)\) then \(\tilde{f}(\tilde{y}_1) = \tilde{f}(\tilde{y}_2) \in \alpha \mathbb{Z}\). In addition if \(\tilde{e}_1\) and \(\tilde{e}_2\) are two edges of \(X\) from \(\tilde{y}_1\) to \(\tilde{y}_1'\) and \(\tilde{y}_2\) to \(\tilde{y}_2'\) respectively with \(\pi(\tilde{e}_1) = \pi(\tilde{e}_2)\), then \(\tilde{f}(\tilde{y}_1') = \tilde{f}(\tilde{y}_2') = \tilde{f}(\tilde{y}_1) - \tilde{f}(\tilde{y}_2)\). This implies that if \(\tilde{\sigma}_1\) and \(\tilde{\sigma}_2\) are two simplexes with \(\pi(\tilde{\sigma}_1) = \pi(\tilde{\sigma}_2) = \sigma\) and \(\pi_i\)'s are the restrictions of \(\pi\) to \(\tilde{\sigma}_i\) (are bijections on their image), then \(\tilde{f} \cdot \pi^{-1}_1 - \tilde{f} \cdot \pi^{-1}_2 : \sigma \to \mathbb{R}\), is constant and this constant is an integer multiple of the fixed real number \(\alpha\).

Step 3.

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\
\downarrow{\pi} & & \downarrow{p} \\
\tilde{X}/\mu = X & \xrightarrow{f} & S^1
\end{array}
\]

Observe that the map \(p : \tilde{X} \to S^1\) factors through \(\tilde{X}/\mu = X\) inducing a map from \(X\) to \(S^1\). This is our linear map \(f\) whose associated \(1\)-cocycle is \(f\). The relation between \(f\) and \(\tilde{f}\) can be summarized by the commutative diagram on left.

3 Definitions of persistence

This section defines persistence for almost integral \(1\)-cocycles via circle valued maps. First, we expose the definitions for level persistence in analogy to standard persistence \([8][12]\), and then extend them to circle valued maps.
Let \( f : X \to \mathbb{R} \) be a continuous map. For simplicity suppose \( X \) is a nice compact space and \( f \) a nice map. This means that the homology \( H_r(\cdot), r \geq 0, \) of all levels and sub levels are finitely generated. Define

\[
X_{\leq t} := f^{-1}((-\infty, t]), \quad X_t := f^{-1}(t), \quad \text{and} \quad X_{t_1, t_2} := f^{-1}([t_1, t_2]), t_2 \geq t_1.
\]

### Persistence of a \( \mathbb{R} \)-valued map.
Standard persistent homology with coefficients in a field \([12]\) is the collection of vector spaces \( B_r(t, t') := \text{img}(H_r(X_{\leq t}) \to H_r(X_{\leq t'})) \), \( t \leq t' \). The corresponding Betti numbers are \( \beta_r(t, t') := \dim B_r(t, t') \). Equivalently, one can define persistent homology as the collection of vector spaces \( K_r(t) := H_r(X_{\leq t}) \) each of which is equipped with a filtration provided by subspaces:

\[
K_r(t, t') := \ker(H_r(X_{\leq t}) \to H_r(X_{\leq t'})) \text{ with } K_r(t, t') \subseteq K_r(t, t'') \subseteq K_r(t) \text{ for } t \leq t' \leq t''.
\]

The corresponding Betti numbers are \( \kappa_r(t) := \dim K_r(t) \) and \( \kappa_r(t, t') := \dim K_r(t, t') \). The two type of informations are equivalent. Indeed, \( B_r(t, t) = K_r(t) \) giving \( \beta_r(t, t) = \kappa_r(t) \) and \( B_r(t, t') \) is isomorphic to \( K_r(t)/K_r(t, t') \) implying \( \beta_r(t, t'') = \kappa_r(t) - \kappa_r(t, t') \).

### Definition 3.1
Let \( c \in H_r(X_{\leq t}) \). One says that

(i) \( c \) is born at \( t' \), \( t' \leq t \), if \( c \in \text{img}(H_r(X_{\leq t'+\epsilon}) \to H_r(X_{\leq t})) \) but \( c \notin \text{img}(H_r(X_{\leq t'-\epsilon}) \to H_r(X_{\leq t})) \) for any positive \( \epsilon \) with \( t - t' > \epsilon > 0 \).

(ii) \( c \) dies at \( t'' \), \( t'' \geq t \), if its image is nonzero in \( \text{img}(H_r(X_{\leq t''-\epsilon}) \to H_r(X_{\leq t})) \) but zero in \( \text{img}(H_r(X_{\leq t}) \to H_r(X_{\leq t''-\epsilon})) \) for any positive \( \epsilon \) with \( t'' - t > \epsilon > 0 \).

For any \( c \in H_r(X_{\leq t}) \), let \( t^+(c) \) and \( t^-(c) \) denote the death and birth times of \( c \) respectively where \( t^+(c) \geq t, t^-(c) \leq t \). For \( t' \leq t \leq t'' \), one may introduce the numbers

- \( \mu_r(t', t, t'') := \text{the maximal number of linearly independent elements in } H_r(X_{\leq t}) \text{ so that } t^+(x) = t'' \) and \( t^-(x) = t' \). This also means that no linear combination of such elements is born before \( t' \), or dies before \( t'' \).

For tame maps (defined in section 3.4), the numbers \( \mu_r(t', t, t'') \) do not depend on \( t \) and hence can be written as \( \mu_r(t', t'') \). The numbers \( \beta_r(t', t'') \) determine and are determined by the numbers \( \mu_r(t', t'') \). The set of numbers \( \beta_r(t, t') \) or \( \kappa_r(t) \) or \( \kappa_r(t, t') \) or \( \mu_r(t, t') \) are the relevant persistent numbers.

### 3.1 Level Persistence

Level sets instead of sublevel sets define level persistence. Let

\[
L^+_r(t; \tau) := \ker(H_r(X_t) \to H_r(X_{t,t+\tau})) \quad \text{and} \quad L^-_r(t; \tau) := \ker(H_r(X_t) \to H_r(X_{t-\tau,t})).
\]

### Definition 3.2
The level persistent homology with coefficients in a field is the collection of vector spaces \( L_r(t) := H_r(X_t) \) equipped with two filtrations:

\[
L^+_r(t; \tau) \subseteq L^+_r(t; \tau') \subseteq L_r(t), \tau \leq \tau' \quad \text{and} \quad L^-_r(t; \tau) \subseteq L^-_r(t; \tau') \subseteq L_r(t), \tau \leq \tau'.
\]
Consequently we have the relevant level persistence numbers:

\[ l_r(t) := \dim L_r(t), \quad l_r^\pm(t; \tau) := \dim L_r^\pm(t; \tau), \quad \text{and} \]
\[ e_r(t; \tau', \tau'') := \dim(L_r^-(t; \tau') \cap L_r^+(t; \tau'')). \]

**Definition 3.3** Let \( c \in H_r(X_t) \). One says that

(i) \( c \) dies downward at \( t', t' \leq t \), if its image is nonzero in \( \text{img}(H_r(X_t) \to H_r(X_{t'-\epsilon,t})) \) but is zero in \( \text{img}(H_r(X_t) \to H_r(X_{t'-\epsilon,t})) \) for any \( \epsilon \) with \( t - t' > \epsilon > 0 \),

(ii) \( c \) dies upward at \( t''' \), \( t'' \geq t \), if its image is nonzero in \( \text{img}(H_r(X_t) \to H_r(X_{t''-\epsilon,t''})) \) but is zero in \( \text{img}(H_r(X_t) \to H_r(X_{t,t''+\epsilon})) \) for any \( \epsilon \) with \( t'' - t > \epsilon > 0 \).

For \( c \in H_r(X_t) \), let \( \tau^+(c) \) and \( \tau^-(c) \) be the upward and downward life time of \( c \) respectively, if \( c \) dies upward and downward at \( t + \tau^+(c) \) and \( t - \tau^-(c) \) respectively. For \( \tau^-, \tau^+ \geq 0 \), and \( t \), we introduce the numbers:

1. \( \nu_r^+(t; \tau^+) := \) the maximal number of linearly independent elements in \( H_r(X_{t,t}) \) so that \( \tau^+(c) = \tau^+, \)

2. \( \nu_r^-(t; \tau^-) := \) the maximal number of linearly independent elements in \( H_r(X_{t,t}) \) so that \( \tau^-(c) = \tau^-.

As before, this also means that no linear combination \( c \) of such elements has \( \tau^+(c) \) smaller than \( \tau^\pm \). For tame maps the numbers \( l_r^\pm(t; \tau) \) determine and are determined by the numbers \( \nu_r^\pm(t; \tau) \). The numbers \( \nu_r^\pm(t; \tau) \) can be calculated from standard persistence numbers \( \mu_r(\cdot, \cdot) \) for another space \( Y \) equipped with a tame map \( g : Y \to \mathbb{R} \) derived from \( (X, f : X \to \mathbb{R}) \). Similarly, the numbers \( e_r(t; \tau', \tau'') \) can be derived from \( \mu_r(\cdot, \cdot) \) calculated for another space \( Y \) with \( g : Y \to [0,1] \) derived canonically from \( (X, f) \).

1. \( \nu_r^+(t; \tau^+) \) for map \( f \) is same as \( \mu_f(t, t, t + \tau^+) \) for map \( g : Y \to \mathbb{R} \) where \( Y = X_{t,\infty} \) and \( g = f|_{X_{t,\infty}}. \)
2. \( \nu_r^-(t; \tau^-) \) for map \( f \) is same as \( \mu_f(t-\tau^-, t, t) \) for map \( g : Y \to \mathbb{R} \) where \( Y = X_{-\infty,t} \) and \( g = f|_{X_{-\infty,t}}. \)
3. \( e_r(t; \tau^-, \tau^+) \) for map \( f \) can be derived from the standard persistence numbers associated with map \( g : Y \to \mathbb{R} \) where \( Y = X \) and \( g : Y \to [0,1] \) with

\[ g^{-1}(0) = X_t \text{ and } g^{-1}(1) = X_{t+t^+} \sqcup X_{t-t^-} \]
\[ g|_{X_{t,t+t^+}} = \frac{1}{\tau^+}f + \frac{-t}{\tau^+} \text{ and } g|_{X_{t-t^-}}} = \frac{-1}{\tau^-}f + \frac{t}{\tau^-}. \]

### 3.2 Persistence for circle valued maps

The standard (sub level) persistence can not be extended to circle valued maps because the notion of sub level is not well defined for circle valued maps. However, we can extend the level persistence to such maps.

Let \( f : X \to S^1 = \mathbb{R}/\alpha \mathbb{Z} \) be a continuous circle valued map. Consider

1. \( \pi = \pi_f : \tilde{X} \to X \), the \( \mathbb{Z} \)-principal covering associated to \( f \), i.e., the pull back of the principal covering \( \mathbb{R} \to S^1 \) by the map \( f \). Equivalently, it is the principal covering associated to the 1-cocycle defined by \( f \) as described in section 2.1
2. \( p : \mathbb{R} \to S^1 \), defined by \( p(t) = t(\mod \alpha) \).
3. \( \tilde{f} : \tilde{X} \to \mathbb{R} \) is a map which satisfies \( p \cdot \tilde{f} = f \cdot \pi \). We make it unique by setting \( \tilde{f}(\tilde{x}) = 0 \) for a chosen base point \( \tilde{x} \) of \( \tilde{X} \).
The map $\tilde{f}$ is the infinite cyclic covering of $f$ which would be same as the one constructed (section 2.1) from the 1-cocycle associated to $f$. Observe that $\pi(\tilde{X}_t) = X_{p(t)} = f^{-1}(p(t))$.

**Definition 3.4** Define $L^\pm_r(\theta; \tau)$ for $f$ to be $L^\pm_r(t; \tau)$ for the map $\tilde{f}$ where $p(t) = \theta$. The definition involves the choice of $t$. It is easy to verify the independence of this choice.

Now, we are ready to define persistence for circle valued maps. Let $\theta \in S^1$ denote an angle and $X_\theta = f^{-1}(\theta)$. The level persistent homology with coefficients in a field for $f$ is the collection of vector spaces $L_r(\theta) := H_r(X_\theta)$ equipped with two filtrations:

- $L^+_r(\theta; \tau) \subseteq L^+_{r'}(\theta; \tau') \subseteq L_r(\theta), 0 \leq \tau \leq \tau' \leq \infty$,
- $L^-_r(\theta; \tau) \subseteq L^-_{r'}(\theta; \tau') \subseteq L_r(\theta), 0 \leq \tau \leq \tau' \leq \infty$.

Consequently we have the relevant persistence numbers:

- $l_r(\theta) := \dim L_r(t)$,
- $l^\pm_r(\theta; \tau) := \dim L^\pm_r(\theta; \tau')$, and
- $e_r(\theta; \tau', \tau'') := \dim (L^-_r(\theta; \tau') \cap L^+_r(\theta, \tau''))$.

By the above definition, the numbers $l^\pm_r(\theta; \tau)$ and $e(\theta; \tau', \tau'')$ for $f$ are the numbers $l^\pm(t; \tau)$ and $e(t; \tau', \tau'')$ for $\tilde{f}$. Note that the numbers $l^\pm(\theta; \tau)$ and $e(\theta; \tau', \tau'')$ can be computed from the level persistence numbers of $\tilde{f}$ and its restriction to $\tilde{X}_{t, t \pm 2\pi k}$ and $\tilde{X}_{t - 2\pi k', t + 2\pi k''}$ where $t$ is so that $p(t) = \theta$ and $k$, $k'$, and $k''$ are so that $2\pi k > \tau, 2\pi k' > \tau', 2\pi k'' > \tau''$, which in turn can be derived from persistence numbers for the maps $g$ as described in subsection 3.1.

### 3.3 Persistence for almost integral 1-cocycles

Given an almost integral 1-cocycle $(f, \alpha)$, we associate the circle ($S^1 = \mathbb{R}/\alpha\mathbb{Z}$) valued map $f : X \to S^1$ as described in section 2.1 and define the persistence of $f$ as the level persistence of the circle valued map $\tilde{f}$. It is possible to shortcut the involvement of the map $\tilde{f}$ and go directly to $\tilde{f}$ to define persistence of $f$, but this might obscure the topological meaning of the definition.

### 3.4 Tame maps

For finite calculations, one cannot allow the level sets change topology continuously. This is why we introduce the following restrictions of tameness on the maps. Similar conditions for standard persistence have been proposed before [3]. In most practical situations, the tameness condition holds for the maps of interest.

**Definition 3.5**. A continuous map $f : X \to Y$, $Y = \mathbb{R}$ or $S^1$ is called tame if there exists finitely many $\{t_1, t_2, \cdots t_N\}$ so that:

(i) for any $t \neq t_1, t_2, \cdots t_N$ there exists $\epsilon > 0$ so that $f : X_{t - \epsilon, t + \epsilon} \to [t - \epsilon, t + \epsilon]$ and the second factor projection $X_t \times [t - \epsilon, t + \epsilon] \to [t - \epsilon, t + \epsilon]$ are fiberwise homotopy equivalent,

(ii) for any $t_i$ there exists $\epsilon > 0$ so that the canonical inclusion $X_{t_i} \subset X_{t_i - \epsilon, t_i + \epsilon}$ is a deformation retraction.
The above definition can be considerably weakened by considering homotopy equivalence in place of homology equivalence, but in view of the fact that all our examples satisfy the above definition, we proceed with it. Generic smooth maps from a closed smooth manifold to $\mathbb{R}$ or $\mathbb{S}^1$ are tame and so are piecewise linear maps from a simplicial complex to $\mathbb{R}$ or $\mathbb{S}^1$. If $f : X \to \mathbb{S}^1$ is tame, then $\tilde{f} : \tilde{X} \to \mathbb{R}$ restricted to any $X_{t_1, t_2}$ is tame.

Observe that, for a tame map $f : X \to \mathbb{R}$, the vector spaces $B_r(t, t')$ are completely determined by the vector spaces $B_r(i, j) := B_r(t_i, t_j)$ and therefore $\beta_r(t, t')$ and $\mu_r(t', t, t'')$ by the numbers $\beta_r(i, j) := \beta_r(t_i, t_j)$ and $\mu_r(i, j, k) := \mu_r(t_i, t_j, t_k)$. The following relations between these numbers are known\cite{3, 8}.

$$\mu_r(i, k) = \mu_r(i, j, k) = \beta_r(i, k - 1) - \beta_r(i, k) - \beta_r(i - 1, k - 1) + \beta_r(i - 1, k),$$

$$\beta_r(i, j) = \sum_{j' > j, j' \leq i} \mu_r(i', j')$$

with $\dim H_r(X_{i, k}) = \beta_r(i, i)$.

One may observe similar properties for level persistence. Let $s_{2i} = t_i$ and $s_{2i-1}$ be any number between $t_{i-1}$ and $t_i$. Clearly, the vector spaces $L_r(\mathbb{t})$ and $L^\pm_r(\mathbb{t}, \tau)$ are completely determined by the vector spaces $L_r(i) := L_r(s_i), L^\pm_r(i, j) := L_r(s_i; \pm(s_j - s_i))$, and therefore $L^\pm_r(\mathbb{t}, \tau), \nu^\pm_r(\mathbb{t}, \tau)$, and $r_r(t, t', t'')$ by the numbers $l^\pm_r(i, j) := l^\pm_r(s_i; \pm(s_j - s_i)), \nu^\pm_r(i, j) := \nu^\pm_r(s_i; \pm(s_j - s_i))$, and $e_r(i, j, k) := e_r(s_i; s_i - s_j, s_k - s_i)$. Observe that

$$l^+(i, 0) = l^-(i, 0) = 0,$$

$$l^+(2i, 1) = l^-(2i - 1, 1) = 0,$$

$$e(i, 0, 0) = e(i, 1), e(i, 0, 1) = e(i, 1, 1) = 0$$

and

$$l^+_r(i, j) = \sum_{0 \leq k \leq j} \nu^+_r(i, k), l^-_r(i, j) = \sum_{0 \leq k \leq j} \nu^-_r(i, k).$$

This shows level persistence numbers can be computed from $\nu^\pm_r(\cdot, \cdot)$ which in turn can be computed from $\mu_r(\cdot, \cdot)$ (section\cite{3, 4}).

### 4 Algorithm

The algorithm to compute persistence for an almost integral 1-cocycle $f$ follows the logical sequence that defines its relevant persistence numbers in the previous section. It considers the associated circle valued map $f$, and computes the level persistence for the cyclic covering $\tilde{f}$. We assume $\mathbb{Z}_2$-homology and describe how one can adapt a matrix version\cite{3, 8} of the standard persistence algorithm\cite{7} to compute level persistence for a $\mathbb{R}$-valued map, then how to extend it to circle valued maps. In this development, we encounter subspaces of simplicial complexes which are cell complexes. We begin with its definition.

**Definition 4.1** A geometric cell complex $X$ is a union of a collection $\mathcal{X}$ of non degenerate convex cells with disjoint interiors which satisfy the property that, for any cell $\sigma \in \mathcal{X}$, all its faces belong to $\mathcal{X}$.

We consider finite cell complexes $X$ equipped with a filtration $\mathcal{F} : \equiv X_0 \subseteq \cdots \subseteq X_k \subseteq X_{k+1} \cdots \subseteq X_m = X$ with $X_i$ being sub complexes of $X$.

**Definition 4.2** A total order on $\mathcal{X} = \{\sigma_1, \ldots, \sigma_n\}$ is called topologically consistent if the condition A below is satisfied and filtration compatible if the condition B below is satisfied.

- **Condition A.** $\sigma_i$ is a face of $\sigma_j$ implies $i < j$.

- **Condition B.** $\sigma_i \in X_k$ and $\sigma_j \in X_{k'} \setminus X_k$ with $k < k'$ implies $i < j$. 


Given a filtration $\mathcal{F}$, one can canonically modify any total order which satisfies Condition A into one which satisfies both conditions A and B. This is done in the following way:

Find the first simplex $\sigma$ in the order which violates Condition B. Let $\tau$ be the simplex immediately preceding $\sigma$ in the total order. The violation of condition B by $\sigma$ implies that $\tau \in X_j$ and $\sigma \in X_i$ where $i < j$. Permute $\sigma$ with $\tau$. Observe that condition A continues to hold; indeed the only possible violation of condition A is if $\tau$ is a face of $\sigma$ which would imply that $j \leq i$. If $\sigma$ still violates Condition B in the new position, then continue moving $\sigma$ to the left until it does not violate Condition B anymore. All other simplexes which were initially preceding $\sigma$ do satisfy condition B. So, we have one less simplex which violates Condition B.

**Persistence algorithm** [5, 7]. The input to the algorithm is a matrix $M$ that represents incidence structure of a cell complex $X$ equipped with a filtration. The algorithm derives the numbers $\mu_r(i, j)$ and the Betti numbers of $X$ in case of $\mathbb{Z}_2$-homology. The numbers $\mu_r(i, j)$ determine the numbers $\beta_r(i, j)$ for this filtration. Let $X = \{\sigma_1, \ldots, \sigma_n\}$ where ordering of the cells is topologically consistent and filtration compatible. An entry $M[i, j]$ in the incidence matrix $M$ is set to the incidence number of $\sigma_i$ and $\sigma_j$, that is, $M[i, j] := I(\sigma_i, \sigma_j)$ where $I(\sigma_i, \sigma_j)$ is 1 if $\sigma_i$ is a face of codimension 1 of $\sigma_j$ and 0 otherwise.

Let $U$ be the class of $n \times n$ upper triangular matrices with zero on diagonal and all entries zero or 1. The matrix $M$ is in this class. Given a matrix $M \in U$, for any column $j$, denote by $\text{low}(j)$ the largest $i$ so that $M[i, j] \neq 0$. If the column $j$ has all entries zero $\text{low}(j)$ is not defined.

The persistence algorithm uses column additions to transform the matrix $M$ within the class $U$ with the additional property: no two columns $j$ and $j'$ have $\text{low}(j) = \text{low}(j')$ if they are defined. A matrix with this property is called in “reduced form”. The algorithm works by adding columns from left to right. Finally, the algorithm ends up with a matrix in the reduced form. The matrix in the reduced form obtained from $M$ carries explicit homological information about $X$ and the filtration. For example, the dimension of $H_r(X; \mathbb{Z}_2)$ is the cardinality of the set of zero columns $j$ in the reduced form where $\sigma_j$ has dimension $r$. To describe the number $\mu_r(i, j)$ we consider all pairs $(\sigma_k, \sigma_{k'})$ with $\text{low}(k') = k$, and observe that $\mu_r(i, j)$ is the number of such pairs which in addition satisfy

1. $k$ is an index corresponding to a cell in $X_i \setminus X_{i-1}$,
2. $k'$ is an index corresponding to a cell in $X_j \setminus X_{j-1}$,
3. dimension of $\sigma_k$ is exactly $r$.

One can also read directly the numbers $\beta_r(i, j)$.

If $X$ is a simplicial complex, $f : X \to \mathbb{R}$ a generic linear map, and $t_0 < t_1, \ldots, t_i < \cdots$ are the values of $f$ on vertices of $X$, then the filtration of $X$ by topological spaces $f^{-1}((0, t])$ is not a filtration by simplicial sub complexes. Apparently the above algorithm can not be applied. However it is possible to show that this topological filtration is homotopically the same (hence has the same persistence) as the filtration provided by the sub complexes $X_i$ consisting of the union of simplexes $\sigma$ so that $f(\sigma) \subset (0, t]$. 

**Computing level persistence for $\mathbb{R}$-valued maps.** In the case of level persistence, the calculation of the numbers $\mu_r(i, j)$ for a tame map is reduced to the calculation of the numbers $\mu_r(\cdot, \cdot)$ for subspaces of $X$, precisely $X_t$, $X_{t, \infty}$, $X_{-\infty, t}$, and $X_{t_1, t_2}$ as indicated at the end of section 3.1. In what follows we describe how to derive the matrix $M$ for $X_t$, $X_{t, \infty}$, $X_{-\infty, t}$, $X_{t_1, t_2}$ from the matrix $M$ for $X$.

Let $X$ be a simplicial complex and $f : X \to \mathbb{R}$ be a generic linear map. Observe that $X_t$, $X_{t, \infty}$, $X_{-\infty, t}$, $X_{t_1, t_2}$ are cell complexes as indicated in the previous section and not simplicial complexes. However, the incidence structure of their cells can be described in terms of the incidences of simplexes in $X$. This key observation allows us to construct the incidence matrix of the cell complexes $X_t$, $X_{t, \infty}$, $X_{-\infty, t}$, $X_{t_1, t_2}$ from
the matrix $M(X)$ as detailed below. For any simplex $\sigma$, let $[\sigma]$ denote the closed interval which is the convex hull of the numbers \{ $f(v)$, $v$ a vertex of $\sigma$ \}.

**Cell complex $X_t$.** For any $t$ introduce the matrix $\hat{M}(t)$ as the minor of $M(X)$ consisting of the rows and columns $i$ with $t \in \text{int} [\sigma_i]$ and regard the simplex $\sigma_i$ as the cell $\hat{\sigma}_i$ with $\dim (\hat{\sigma}_i) = \dim (\sigma_i) - 1$. If there are $\ell < n$ such simplexes, $\hat{M}(t)$ is an $\ell \times \ell$ matrix. Clearly, $\hat{M}(t)$ respects the order of the cells induced by the order of simplexes used for $M(X)$.

- If no vertex takes the value $t$, then $M(X_t) = \hat{M}(t)$ is the incidence matrix of the cell complex $X_t$.
- If there exists a vertex $v$ (unique since $f$ is generic) so that $f(v) = t$, then the incidence matrix $M(X_t)$ is an $(\ell + 1) \times (\ell + 1)$ matrix with one additional row and column corresponding to the vertex $v$ viewed as a cell of dimension 0 which should be indexed before all other cells. The entry for the pair $(v, \hat{\sigma}_j)$ in $M(X_t) = 1$ if $\sigma_j$ is a 2-simplex which has $v$ as a vertex and 0 otherwise.

**Cell complexes $X_{t,\infty}$, $X_{-\infty,t}$, $X_{t_1,t_2}$.** To describe the incidence matrices which correspond to the cell complexes $X_{t,\infty}$, $X_{-\infty,t}$, and $X_{t_1,t_2}$ with $t_1 < t_2$, consider the collection of simplexes:

\[
\mathcal{X}(t, \infty) := \{ \sigma, [\sigma] \cap (t, \infty) \neq \emptyset \}
\]
\[
\mathcal{X}(-\infty, t) := \{ \sigma, [\sigma] \cap (-\infty, t) \neq \emptyset \}
\]
\[
\mathcal{X}(t_1, t_2) := \{ \sigma, [\sigma] \cap (t_1, t_2) \neq \emptyset \}.
\]

Let the matrices $\hat{M}(t, \infty)$, $\hat{M}(-\infty, t)$, and $\hat{M}(t_1, t_2)$ be the minors of $M(X)$ whose rows /columns are the simplexes in the sets $\mathcal{X}(t, \infty)$, $\mathcal{X}(-\infty, t)$, and $\mathcal{X}(t_1, t_2)$ respectively. We keep the dimension of the cell same as that in the matrix $M(X)$. The matrices $M(X_{t,\infty})$ and $M(X_{-\infty,t})$ are matrices of the form

\[
\begin{bmatrix}
A & B^+ \\
0 & D^+
\end{bmatrix}
\] and \[
\begin{bmatrix}
D^- & B^- \\
0 & A
\end{bmatrix}
\]

respectively,

where $A = M(X_t)$, $D^+ = \hat{M}(t, \infty)$, and $D^- = \hat{M}(-\infty, t)$. The matrix $B^+$ is a $s \times m$ matrix where $X_t$ has $s$ cells and $\mathcal{X}(t, \infty)$ has $m$ simplexes. We set $B^+ [i,j] = 1$ if the $i$-th cell in $X_t$ and the $j$-th simplex in $\mathcal{X}(t, \infty)$ are either $\hat{\sigma}$ and $\sigma$ respectively, or a vertex of $\hat{X}$ and a 1-simplex incident to it respectively. All other entries in $B^+$ are 0. Define $B^-$ analogously by considering simplexes in $\mathcal{X}(-\infty, t)$. The incidences among the cells in $X_{t,\infty}$ are of three types, the ones among the cells in $X_t$, the ones among the cells in $X_{t,\infty}$ that are not in $X_t$, and the ones among the cells one of which is in $X_t$ and the other is in $X_{t,\infty}$ but not in $X_t$. The matrices $A$, $D^+$, and $B^+$ capture these three types of incidences respectively. Clearly, similar observations can be made about the incidences among the cells in $X_{-\infty,t}$.

Let $\hat{M}(t_1, t_2)$ denote the minor of $M(X)$ whose rows and columns correspond to the simplexes in $\mathcal{X}(t_1, t_2)$. The matrix $M(X_{t_1,t_2})$ is

\[
\begin{bmatrix}
A & B^+ & 0 \\
0 & D & B^- \\
0 & 0 & C
\end{bmatrix}
\]

where $A = M(X_{t_1})$, $C = M(X_{t_2})$, $D = \hat{M}(t_1, t_2)$. The matrix $B^+$ has the entries as defined in the previous case with $X_{t_1}$ and $\mathcal{X}(t_1, t_2)$ playing the roles of $X_t$ and $\mathcal{X}(t, \infty)$ respectively. The matrix $B^-$ is analogous.

The order of the cells as suggested by the matrices $M(X_t)$, $M(X_{-\infty,t})$, $M(X_{t,\infty})$ and $M(X_{t_1,t_2})$ satisfies Condition A but not necessarily Condition B w.r.t. the filtration induced from the map $g$ (end of section 3.1). If necessary it can be canonically changed as indicated at the beginning of this section.
Computing level persistence for a circled value map. Let \( f : X \to S^1 \) be a generic linear map. For simplicity in writing we suppose that \( f(\sigma) \) is an angular interval smaller than \( \pi \). We need the incidence structure of the simplexes in the covering space \( \widetilde{X} \) and the linear \( \mathbb{R} \)-valued map \( \tilde{f} \). However, the space \( \widetilde{X} \) is infinite. So, we compute only a finite subspace of \( X \) which is sufficient for computing relevant persistence numbers and the restriction of \( \tilde{f} \) to these subspaces.

We show how to construct a simplicial complex from \( X \) that contains \( \widetilde{X}_{t,t+2\pi k} \). The constructions of \( \widetilde{X}_{t-2\pi k,t}, \widetilde{X}_{t-2\pi k_1,t+2\pi k_2} \) are analogous.

Let \( \mathcal{X} \) denote the set of simplexes in \( X \). For any \( \theta \in S^1 \), decompose \( \mathcal{X} \) as a disjoint union \( \mathcal{X} = T^\theta \sqcup L^\theta \sqcup \partial_- L^\theta \sqcup \partial_+ L^\theta \) where (see Figure 1)

- \( L^\theta \) consists of the set of all simplexes whose closure do intersect the level \( X_\theta \). Let \( L^\theta \) be the simplicial complex generated by simplexes in \( L^\theta \),
- \( T^\theta \) is the set of simplexes which do not belong to \( L^\theta \). Let \( T^\theta \) denote the simplicial complex generated by the simplexes in \( T^\theta \) and consider \( T^\theta \cap L^\theta \). This simplicial complex is the disjoint union of two simplicial complexes \( \partial_- L^\theta \) and \( \partial_+ L^\theta \) characterized by \( f(\sigma) < \theta \) for \( \sigma \in \partial_- L^\theta \) and \( f(\sigma) > \theta \) for \( \sigma \in \partial_+ L^\theta \).
- \( \partial_{\pm} L^\theta \) represent the simplexes in \( \partial_{\pm} L^\theta \).

Our purpose is to build a collection of simplicial complexes which are equivalent with the space \( \widetilde{X}_{t,t+2\pi k} \) where \( p(t) = \theta \).

Introduce a nested sequence of simplexes \( \tilde{X}^\theta(0) \subseteq \tilde{X}^\theta(1) \subseteq \cdots \tilde{X}^\theta(k) \) as follows. Since we will repeat copies of each of the sets \( T^\theta, L^\theta, \partial_- L^\theta, \partial_+ L^\theta \), let \( T^\theta(n), L^\theta(n), \partial_- L^\theta(n), \partial_+ L^\theta(n) \) denote their \( n \)-th copies respectively. Taking \( L^\theta(0) = L^\theta, \partial^\theta(0) = T^\theta, \partial_+ L^\theta(0) = \partial_+ L^\theta \), define inductively,

\[
\tilde{X}^\theta(0) = \partial_- L^\theta \\
\tilde{X}^\theta(n+1) = \tilde{X}^\theta(n) \sqcup L^\theta(n) \sqcup \partial_+ L^\theta(n) \sqcup T^\theta(n) \sqcup \partial_- L^\theta(n+1)
\]

Taking \( I_0(\sigma, \tau) = I(\sigma, \tau) \), the incidences among the simplexes are described by

\[
I_{n+1}(\sigma, \tau) = I_n(\sigma, \tau) \quad \text{if } \sigma \in \tilde{X}^\theta(n) \text{ and } \tau \text{ is a face of } \sigma \\
I_{n+1}(\sigma, \tau) = I(\sigma, \tau) \quad \text{if } \sigma \in L^\theta(n) \sqcup \partial_+ L^\theta(n) \subset L^\theta \text{ and } \tau \text{ is a face of } \sigma \\
I_{n+1}(\sigma, \tau) = I(\sigma, \tau) \quad \text{if } \sigma \in T^\theta(n) \sqcup \partial_- L^\theta(n+1) \subset T^\theta \text{ and } \tau \text{ is a face of } \sigma.
\]

In all other cases \( I_{n+1}(\sigma, \tau) = 0 \).

Notice that each \( \tilde{X}^\theta(i) \) forms a simplicial complex \( \tilde{X}^\theta(i) \) (Figure 1). To describe \( \tilde{f} \) it suffices to provide its values on vertices. We write

\[
P = \partial_- L^\theta \sqcup L^\theta \sqcup \partial_+ L^\theta \sqcup T^\theta
\]
and $P_0$ for the subset of vertices in $P$. We write $P_0(n)$ for the $n$-th copy of $P_0$ and define $\tilde{f}(n) : P_0(n) \to \mathbb{R}$ by $\tilde{f}(n) := \tilde{f} + 2\pi n$ where $\tilde{f} = p^{-1} \cdot f$ with $p : [t - \pi, t + \pi] \to S^1$ which sends $t$ to $\theta$. Once defined on vertices, $\tilde{f}$ is extended by linearity to each simplex of the simplicial complex $\tilde{X}^\theta(n)$.

Note that an order of the simplexes of $X$ satisfying condition A induces an order on the simplexes of $T^\theta(n)$, $\partial \pm L^\theta(n)$ and $L^\theta(n)$ and by juxtaposition an order on the simplexes of $\tilde{X}^\theta(n)$ which continue to satisfy condition A. It implies that one can build a matrix $M(\tilde{X}^\theta(n))$ which satisfies condition A by juxtaposing the minors of $M(X)$ that represent $L^\theta$, $\partial \pm L^\theta$, $T^\theta$, and their copies in an appropriate order. Note also that $\tilde{X}^\theta(n)$ is a sub complex of $\tilde{X}$ and therefore the restriction of $\tilde{f}$ provides tame maps on each of these spaces. The columns and rows of $M(\tilde{X}^\theta(n))$ can be reordered so that they become filtration compatible with $\tilde{f}$ or $g$ by the method indicated at the beginning of this section.

**Algorithm for almost integral 1-cocycles.** If $(f, \alpha)$ is an almost integral 1-cocycle, hence given by $M(X)$ and a real number $f(e)$ for each edge, we have to provide the map $f : X \to S^1$ hence a matrix $M(X)$ and an angle valued map on vertices and calculate the level persistence. For this purpose we choose a base point vertex $x$ and assign to it the angle value $0$. For any other vertex $y$ choose a sequence of consecutive vertices $\{x = y_0, y_1, \cdots y_{L-1}, y_L = y\}$ and assign to $y$ the angle $(\sum_{0 \leq i < L-1} f(y_i, y_{i+1})) \mod \alpha)2\pi/\alpha$. The result is independent of the choice of $x$. We continue then as described above.

---

4 $p$ is bijective and continuous, but $p^{-1}$ is not continuous
References


Appendix

In this Appendix we show that, for a tame map \( f : X \to \mathbb{R} \), the relevant level persistence numbers determine the persistence numbers for both \( f \) and \(-f\). We suppose \( X \) is compact so \( f \) is bounded from above and below.

Let \( \{t_1, t_2, \ldots, t_k\} \) be the critical values of \( f \) and let \( \{t'_1, \ldots, t'_{k+1}\} \) regular values so that \( t'_1 < t_1 < t'_2 < t_2 < \cdots < t'_k < \cdots < t_k < t'_{k+1} \). Consider \( s_{2i} = t_i \) and \( s_{2i-1} = t'_i \). Clearly \( X_{s_1} = \emptyset \) and \( X_{s_{2k+1}} = X \), therefore \( X_{s_1, s_i} = X_{\infty, s_i} \) and \( X_{s_i, s_{2k+1}} = X_{s_i, \infty} \). Note that the tameness of \( f \) implies that the inclusions \( X_{s_i} \subset X_{s_i, s_{i+1}} \) for \( i \) even and \( X_{s_{i+1}} \subset X_{s_i, s_{i+1}} \) for \( i \) odd induce isomorphism in homology.

The relevant level persistence numbers are \( l_r(s_i), l^r_t(s_i) \) and \( e_r(s_i) \) while the relevant persistence numbers for \( f \) are \( \kappa_r(i) = \dim H_r(X_{\infty, s_i}) \) and \( \kappa_r(s_i, s_{i+k}) = \dim \ker(H_r(X_{-\infty, s_i}) \to H_r(X_{-\infty, s_{i+k}})) \) for \(-f\) are \( \dim H_r(X_{s_i, \infty}) \) and \( \dim \ker(H_r(X_{s_i, \infty}) \to H_r(X_{s_{i+k}, \infty})) \).

To show that the numbers \( l_r, l^r_t, e_r \) determine the numbers \( \kappa_r \) we will use the algebraic proposition below:

**Proposition 4.3** Let

\[
\begin{align*}
0 \to A_N &\to B_N &\to \cdots &\to A_n &\to B_n &\to C_n &\to A_{n-1} &\to \cdots &\to A_0 &\to B_0 &\to C_0 &\to 0 \\
0 \leq n \leq N &\text{ be an exact sequence of vector spaces. Then any three independent collections of numbers } &\{\dim A_n\}, &\{\dim B_n\}, &\{\dim C_n\}, &\{\dim(\ker \alpha_n)\}, &\{\dim(\ker \beta_n)\}, &\{\dim(\ker \delta_n)\} &\text{ determine the other three. (Here independent means not constrained by the obvious equalities (1), (2), (3) below.)}
\end{align*}
\]

We have the following two long exact sequences:

1. The Mayer-Vietoris sequence associated to \( X_{t, t''} = X_{t, t'} \cup X_{t, t''}, X_{t, t'} \cap X_{t', t''} = X_t, t \leq t' \leq t'' \).

\[
\begin{align*}
&\quad A_n \to B_n \to \cdots \to A_1 \to B_1 \to C_1 \to A_0 \to B_0 \to C_0 \\
&\quad \alpha_n \to \beta_n \to \gamma_n \to \alpha_{n-1} \to \beta_{n-1} \to \gamma_{n-1} \to \cdots.
\end{align*}
\]

2. The long exact sequences of the pairs \( X_{t, t'} \subset X_{t, t''} \) and \( X_{t', t''} \subset X_{t, t''} \), \( t \leq t' \leq t'' \).

\[
\begin{align*}
(I) \quad &\quad \ldots \to H_n(X_{t, t'}) \to H_n(X_{t, t'}) \to H_n(X_{t, t'}) \to H_n(X_{t, t'}) \to H_n(X_{t, t'}) \to \cdots. \\
(II) \quad &\quad \ldots \to H_n(X_{t', t''}) \to H_n(X_{t', t''}) \to H_n(X_{t', t''}) \to H_n(X_{t', t''}) \to H_n(X_{t', t''}) \to \cdots.
\end{align*}
\]

For \( t \leq t' \leq t'' \), we also have by excision theorem

\[
(III) \quad H_n(X_{t, t''}, X_{t, t'}) = H_n(X_{t', t''}, X_{t', t'}) \quad H_n(X_{t, t''}, X_{t', t''}) = H_n(X_{t, t'}, X_{t', t''}).
\]

**Proof:**[Proposition 4.3]: Note first that the exactness (\( * \)) implies the following equalities

\[
\begin{align*}
(1) \quad &\quad \dim A_n = \dim \ker \alpha_n + \dim \ker \beta_n \\
(2) \quad &\quad \dim B_n = \dim \ker \beta_n + \dim \ker \delta_n \\
(3) \quad &\quad \dim C_n = \dim \ker \delta_n + \dim \ker \alpha_{n-1}
\end{align*}
\]

Then we obtain
dim $A_n$, dim $B_n$, dim ker $\alpha_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine dim ker $\beta_n$, dim ker $\delta_n$, and dim $C_n$, 0 $\leq n \leq N$.

• dim $A_n$, dim $B_n$, dim ker $\beta_n$, 0 $\leq n \leq N$ in view of (2), (1), (3) determine dim ker $\delta_n$, dim ker $\alpha_n$ and dim $C_n$, 0 $\leq n \leq N$.

• dim $A_n$, dim $C_n$, dim ker $\alpha_n$, 0 $\leq n \leq N$ in view of (1), (3), (2) determine dim ker $\beta_n$, dim ker $\delta_n$, and dim $B_n$, 0 $\leq n \leq N$.

• dim $A_n$, dim $C_n$, dim ker $\beta_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine dim $A_n$, dim ker $\delta_n$ and dim $C_n$, 0 $\leq n \leq N$.

• dim ker $\alpha_n$, dim ker $\beta_n$, dim $B_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine dim ker $\delta_n$, dim $B_n$ and dim $A_n$, 0 $\leq n \leq N$.

• dim ker $\alpha_n$, dim ker $\beta_n$, dim $A_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine ker dim $\beta_n$, dim $B_n$ and dim $C_n$, 0 $\leq n \leq N$.

• dim ker $\alpha_n$, dim ker $\beta_n$, dim ker $\delta_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine dim $A_n$, $B_n$, and dim $C_n$, 0 $\leq n \leq N$.

• dim $A_n$, dim ker $\alpha_n$, dim ker $\delta_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) determine dim ker $\beta_n$, dim $B_n$ and dim $C_n$, 0 $\leq n \leq N$.

• dim $A_n$, dim $B_n$, dim $C_n$, 0 $\leq n \leq N$ in view of (1), (2), (3) and the fact that dim ker $\alpha_N = 0$ determine dim ker $\alpha_n$, dim ker $\beta_n$, dim ker $\delta_n$, 0 $\leq n \leq N$.

All other possible situations can be recovered from these cases.

To conclude that the level persistence numbers determine the persistence numbers we proceed as follows.

Use the equality dim $H_r(X_{s_i,s_i+1}) = \dim H_r(X_{s_i})$ if $i$ even and dim $H_r(X_{s_i,s_i+1}) = \dim H_r(X_{s_i+1})$ if $i$ odd which follow from tameness, and apply Proposition 4.3 to the exact sequence (1). In this way we derive from the numbers $l_r(s_i)$ and $e_r(s_i; s_i - s_i - k, s_i + k' - s_i)$ the numbers dim $H_r(X_{s_i,s_i+k})$ for all $i, k$.

Apply Proposition 4.3 to the long exact sequence 2(I) for $t = t'$ (resp. 2(II) for $t' = t''$), and in view of (III) derive from the numbers $l_r(s_i)$ and $l^+_r(s_i; s_i + k - s_i)$ (resp. $l^-_r(s_i; s_i - s_i - k)$) the numbers dim $H_r(X_{s_i,s_i+k}, X_{s_i+s_i+k})$ (resp. dim $H_r(X_{s_i,s_i+k+s_i+k})$).

Apply Proposition 4.3 to the long exact sequence 2(II) (resp. 2(II)) and derive from dim $H_r(X_{s_i,s_i+k})$ dim $H_r(X_{s_i,s_i+k}, X_{s_i+k})$ (resp. dim $H_r(X_{s_i,s_i+k}, X_{s_i+s_i+k+k})$) the numbers dim ker $H_r(X_{s_i+s_i+k})$ (resp. dim ker $H_r(X_{s_i+s_i+k+k})$) (resp. dim ker $H_r(X_{s_i+s_i+k+k})$). Taking $s_i = s_1$ (resp. $s_i+k+r = s_{2k+1}$) one obtains the persistence numbers for $f$ and $-f$. 

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