The Shapley Value in Totally Convex Multichoice Games

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Abstract—In this paper, we introduce a class of totally convex multichoice cooperative games and prove that the Shapley value of such games is always in the core. © 2000 Elsevier Science Ltd.

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1. INTRODUCTION

Hsiao and Raghavan [1] introduced a class of multichoice cooperative games and found its Shapley value using an axiomatic approach. Later, Nouweland et al. [2] determined the Shapley value for multichoice cooperative games following its probabilistic interpretation. However, the values obtained by these two methods are quite different. In our paper, while avoiding the problem of inconsistency of the Shapley value between Hsiao-Raghavan and Nouweland, we consider a necessary and sufficient condition for the Shapley value by Nouweland to be in the core of a multichoice cooperative game.

It is well known that in the class of cooperative games in the characteristic function form, the Shapley value is in the core if the characteristic function is either convex [3], average convex [4], or totally convex [5]. The latter paper shows that the class of totally convex games includes that of average convex games. We discuss conditions for the Shapley value to be in the core for the class of multichoice cooperative games.

2. MULTICHOICE COOPERATIVE GAME

First of all, we describe the multichoice cooperative game (MCG) introduced in [1]. Let \( N = \{1, 2, \ldots, n\} \) be the set of players, \( M_i = \{0, 1, 2, \ldots, m_i\} \) the set of activity levels of player \( i \in N \). We assume that \( m_i = L, L \in R^+ \), for all \( i \in N \) as in [1]. A coalition in MCG is denoted by a vector \( s = (s_1, \ldots, s_n) \), where \( s_i \in M_i, i \in N \), shows activity level of player \( i \) in the coalition \( s \). If a player does not cooperate, his level of activity is set at zero. Hence,
the coalition in which no player participates is specified by the zero vector \( \theta = (0, \ldots, 0) \). We denote the set of all coalitions by \( M = M_1 \times \cdots \times M_n \). Throughout this paper, a coalition \( s \wedge t = (\min \{s_1, t_1\}, \min \{s_2, t_2\}, \ldots, \min \{s_n, t_n\}) \) is considered as the intersection of coalitions \( s \) and \( t \), and a coalition \( s \vee t = (\max \{s_1, t_1\}, \max \{s_2, t_2\}, \ldots, \max \{s_n, t_n\}) \) is admitted as the union of \( s \) and \( t \). A superadditive function \( v: M \to \mathbb{R}^1 \) with \( v(\emptyset) = 0 \) is called a characteristic function of an MCG. It is easily seen that \( v(m), m = (m_1, \ldots, m_n) \), is the maximal value of the characteristic function. We denote MCG by \( G(v, N) \).

Consider an \((m + 1) \times n\)-dimensional payoff matrix \( \xi = (\xi_{ij}) \) distributing \( v(m) \) among all players and their activity levels. A component \( \xi_{ij} \) shows the increase in payoff to player \( i \) when he changes his activity from level \( j - 1 \) to level \( j \). It is said that the payoff matrix \( \xi \) is **efficient** if \( \sum_{i=1}^n \sum_{j=0}^{m_i} \xi_{ij} = v(m) \) and it is **level increase rational** if \( \sum_{i=1}^n \sum_{j=0}^{m_i} \xi_{ij} \geq v((0, \ldots, 0, s_i, 0, \ldots, 0)) \), where \( i \in N, s_i \in M_i \). An efficient and level increase rational payoff matrix is called **imputation** and considered as a solution of \( G(v, N) \). Let \( I(v, N) \) be the set of all imputations in \( G(v, N) \). We shall say that the set \( G(v, N) = \{ \xi \in I(v, N) | \sum_{i: s_i \neq 0} \sum_{j=0}^{m_i} \xi_{ij} \geq v(s) \text{ for all } s \in M \} \) is the **core** of \( G(v, N) \).

### 3. TOTAL CONVEXITY

In [2], the following procedure of construction was proposed for the Shapley value. Suppose that a given coalition \( s \in M \) is formed step-by-step, starting from the zero coalition \( \theta = (0, \ldots, 0) \). On each stage of the procedure, one of the players has to increase his activity level by 1. Thus, the coalition \( s \in M \) will be created after \( k(s) = \sum_{i: s_i \neq 0} s_i \) steps, i.e., each player \( i \in N \) will reach his level of activity \( s_i \) in \( s \). Define \( M^+ = \{(i, j) | i \in N, j \in M_i \setminus \{0\} \} \). An **admissible order** is a bijection \( w: M^+ \to \{1, 2, \ldots, \sum_{i \in N} m_i \} \) satisfying \( w((i, j)) < w((i, j + 1)) \) for all \( i \in N \) and \( j \in \{1, 2, \ldots, m_i - 1\} \). The number of the admissible orders for \( G(v, N) \) is

\[
\Omega(m) = \frac{\prod_{i \in N} (m_i)!}{\prod_{i \in N} (m_i)!} = \frac{(L n)!}{(L!)^n}.
\]

Take an arbitrary coalition \( s \in M \) and then fix a player \( l \in N, s_l \neq 0 \). Suppose that by an admissible order \( w \), the given coalition \( s \) is created after the first \( k(s) \) steps, with the player \( l \) completing the formation of \( s \). The number of such orders is

\[
\Omega_l(s) = \frac{\prod_{i: (s_i | s_i - 1) \neq 0} (s_i)! \prod_{i: (s_i | s_i - 1) \neq 0} (L - s_i)!}{\prod_{i: (s_i | s_i - 1) \neq 0} (s_i)! \prod_{i \in N} (L - s_i)!},
\]

where \( s | s_l - 1 = (s_1, \ldots, s_l-1, s_l - 1, s_l+1, \ldots, s_n) \). One can have that \( \Omega_l(s) = 0 \) if \( s_l = 0 \). Nouweland et al. [2] showed that \( \phi = \{\phi_{ij}\} \), where \( i = 1, \ldots, n, j = 0, \ldots, m_i \), and

\[
\phi_{ij} = \sum_{s: s_i = j} \frac{\Omega_l(s)}{\Omega(m)} [v(s) - v(s | s_i - 1)]
\]

is the Shapley value of \( G(v, N) \).

Let \( G^s, s \in M \), be a subgame of the game \( G(v, N) \). Suppose that the characteristic function \( v^s \) of \( G^s \) is the restriction of \( v \) to the set \( M^s = \{ t \in M | 0 \leq t_i \leq s_i \text{ for each } i \in N \} \). Denote the Shapley value of \( G^s \) by \( \phi^s = \{\phi^s_{ij}\} \), \( i = 1, \ldots, N, j = 0, \ldots, s_i \). We shall say that according to the Shapley value, a coalition \( r \in M^s, s \in M \) obtains \( \phi(r) = \sum_{i\in r, s_i \neq 0} \sum_{j=0}^{m_i} \phi_{ji} \) in the game \( G(v, N) \).
and \( \phi^s(r) = \sum_{i:r_i \neq 0} \sum_{j=0}^r \phi_{ji}^s \) in the subgame \( G^s \). Now we will find a condition for \( \phi \) to be in \( C(v, N) \).

Introduce functions \( \delta_i(s) = v(s) - v(s|s_i - 1), s \in M, s_i - 1 \geq 0, i \in N \). Let \( t \) be a particular coalition in \( M \). From (3.1), we have

\[
\phi(t) = \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \phi_{ji}^t
\]

\[
= \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \sum_{s,s_s=s} \Omega_i(s) \Omega(m) \delta_i(s)
\]

\[
= \sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t} \Omega_i(s) \Omega(m) \delta_i(s).
\]

Note that \( t \leq m \), and hence,

\[
\sum_{s:(s \wedge t)_i \leq t} \Omega_i(s) \Omega(m) \geq \sum_{r:r \leq t} \Omega_i(r) \Omega(t).
\]

Thus, if

\[
\sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t} \Omega_i(s) \Omega(m) (\delta_i(s) - \delta_i(s \wedge t)) \geq 0,
\]

then expression (3.2) is greater than or equal to

\[
\sum_{i:t_i \neq 0} \sum_{r:r \leq t} \Omega_i(r) \Omega(t) \delta_i(r) = \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \Omega_i(r) \Omega(t) \delta_i(r) = \sum_{i:t_i \neq 0} \sum_{j=0}^{t_i} \phi_{ji}^t(t) = v^t(t) = v(t).
\]

For inequality (3.3), it is easily seen that

\[
\sum_{i:t_i \neq 0} \sum_{s:(s \wedge t)_i \leq t} = \sum_{s \in M} \sum_{i:(s \wedge t)_i \leq t},
\]

with the last summation being zero if \( s \wedge t = 0 \). Hence, we can conclude that the Shapley value is in the core if inequality (3.3) is satisfied.

**Definition.** \( G(v, N) \) is called a totally convex multichoice game if for any coalition \( t \in M \),

\[
\sum_{s \in M} \sum_{i:(s \wedge t)_i \leq t} \frac{\Omega_i(s)}{\Omega(m)} (\delta_i(s) - \delta_i(s \wedge t)) \geq 0.
\]

Moving backwards, from (3.4) to (3.2), we arrive at the fact that, if the Shapley value of \( G(v, N) \) lies in the core, then \( G(v, N) \) is totally convex. Thus, we have proved the following theorem.

**Theorem.** A necessary and sufficient condition for the Shapley value \( \phi \) of MCG \( G(v, N) \) to be in the core \( C(v, N) \) is total convexity of \( G(v, N) \).

The proof of the theorem is also valid for the games where players may have different numbers of activity levels: \( m_i \neq m_j \) for \( i \neq j \), where \( i, j \in N \).

Finally, we consider that the definition of a totally convex multichoice game coincides with another definition given by Izawa and Takahashi [5] for the class of \( n \)-person cooperative games in characteristic form. For the sake of simplicity, we draw on the following definition of total convexity proposed by Izawa and Takahashi [5].

**Definition.** A cooperative game \( (v, N) \) with the set of players \( N = \{1, \ldots, n\} \) and characteristic function \( v \) is totally convex if for any subset \( T \) of \( N \),

\[
\sum_{S \subseteq N} \sum_{i \in S \cap T} \frac{(|S| - 1)(n - |S|)!}{n!} [v(S) - v(S \setminus \{i\})] \geq 0.
\]

where the summation \( \sum_{S \subseteq N} \) is taken over all nonempty subsets \( S \) of \( N \).
Note that game \((v,N)\) is equivalent to the MCG \(G'(v,N)\), where \(M_i = \{0,1\}, i \in N\) and a one-to-one correspondence between \(M\) and \(2^N\) is constructed as follows: \(s_i = 0 \Leftrightarrow i \notin S\) and \(s_i = 1 \Leftrightarrow i \in S\) for each \(i \in N\). Then \((\Omega_i(s)/\Omega(m)) = ((|S| - 1)!(n - |S|)!/n!))\), and \(\delta_i(s) - \delta_i(s \cap t) = v(S) - v(S \setminus \{i\}) - v(S \cap T) + v(S \cap T \setminus \{i\})\), where \(s \in M\) is corresponded to \(S \subseteq N\). Thus, (3.5) coincides with (3.6) on the class of cooperative games.

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