THE CROUZEIX-RAVIART FE ON NONMATCHING GRIDS WITH AN APPROXIMATE MORTAR CONDITION

TALAL RAHMAN *, PETTER BJØRSTAD †, AND XUEJUN XU ‡

Abstract. A new approximate mortar condition is proposed for the lowest order Crouzeix-Raviart finite element on nonmatching grids, which uses only the nodal values on the interface for the calculation of the mortar projection. This approach allows for improved and more flexible algorithms compared to those for the standard mortar condition where nodal values in the interior of a subdomain, those closest to a mortar side of the subdomain, are also required in the calculation.

Key words. Crouzeix-Raviart FE, nonmatching grids, mortar condition, additive Schwarz

AMS subject classifications. 65F10, 65N30

1. Introduction. The general concept of the mortar technique was originally introduced by Bernardi, Maday and Patera in [3], which provides a useful tool for coupling different discretization schemes. In recent years, the approach has largely been applied to nonmatching grids for the design of algorithms which are very well suited for parallel implementation, and which can easily handle complicated geometries and heterogeneous materials. In order to ensure that the overall discretization makes sense, an optimal coupling between the meshes is required. In a standard mortar technique, this condition is realized by applying the condition of weak continuity on the solution, called the mortar condition, saying that the jump of the solution along the interface between two meshes is orthogonal to some suitable test space. The mortar technique has been extensively studied by many authors. A saddle point formulation for the mortar technique was studied in [2]. Later, an extension to three dimensions was introduced in [1]. Further analysis and extensions of the mortar technique can be found in [5, 9, 15, 22, 23, 24, 10], and the references therein.

In this paper, we are interested in the application of the mortar technique on nonmatching grids, where in each subgrid, the nonconforming P1 or the lowest order Crouzeix-Raviart (CR) finite element [13] is used for the discretization. The first analysis of the mortar technique for such an element was given by Marcinkowski in [15]. In the event of applying the mortar condition, it is necessary to know the function on the interface. For the conforming P1 finite element, it is enough to know the nodal values along the interface. However, for the nonconforming P1 finite element (the lowest order Crouzeix-Raviart finite element), where the degrees of freedom are associated with the edge midpoints, the function on the interface depends on the nodal values corresponding to interface nodes and some subdomain interior nodes lying closest to the interface, see [15] and Fig. 1.1 for illustration.

A variant of the mortar technique has recently been proposed in [18], where the standard mortar condition [15] has been modified so that the method will use only the nodal values on the interface. However, the error estimate as given in [18], requires

---

*Department of Mathematics, University of Bergen, c/o Center for Integrated Petroleum Research, Allégt. 41, 5007 Bergen, Norway, Phone: + 47 55583690, Fax: + 47 55588265 (Email: talal.rahman@math.uib.no)
†Department of Mathematics, University of Bergen, Johannes Brunsgt. 12, 5008 Bergen, Norway, Phone: + 47 55584867, Fax: + 47 55589672 (Email: petter.bjorstad@math.uib.no)
‡LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O.Box 2719, 100080 Beijing, Peoples Republic of China, Phone: + 86 10 62627487, Fax: + 86 10 62542285 (Email: xxj@lsec.cc.ac.cn)
the mesh sizes to be of the same order. In this paper, we improve this situation by modifying the mortar condition a step further, giving us an error estimate which is free from any such requirement on the mesh sizes.

The motivation to replace the standard mortar condition with an approximate mortar condition which uses nodal values only on the interface, lies in the fact that the new approach will facilitate the design of improved and less intricate algorithms for the CR finite element on nonmatching grids. As we know, there already exists a number of algorithms for the conforming P1 mortar finite element, many of which, in particular the domain decomposition algorithms, can be extended to the CR mortar finite element, see for instance in [15, 19]. In order to be able to extend readily, it is desirable that the latter uses similar geometrical information as the former one, for instance using nodal values only on the interface for the mortar condition. An important application of the approximate mortar condition is the solution of elliptic problems with discontinuous coefficients using, for instance, parallel domain decomposition algorithms of the nonoverlapping type. As, in the CR standard mortar case, basis functions corresponding to some interior nodes of a subdomain may have non zero support on the neighboring subdomain, special care with respect to those nodes is required in defining the nonoverlapping subspaces in the algorithm, cf. [16, 19]. This results in a more intricate code as compared to the code in the approximate mortar case. In fact, the algorithm gets affected even further, the stiffness matrix corresponding to subspaces living on the interfaces loose their block diagonal structure which is natural in the approximate mortar approach, cf. [19]. It is also important to note that, in the application of the mortar condition for the CR finite element, the matrix on the nonmortar side, called the slave matrix, is a diagonal matrix. An application of the standard mortar condition simply destroys the advantage of having a diagonal slave matrix in the design of any effective algorithms. Another important application of the approximate mortar condition, which seems now obvious, is the construction of the intergrid transfer operator which is a vital and already an intricate part of a multilevel or multigrid algorithm, cf. [25, 17]. Although, in the analysis of all such algorithms, those mentioned in here for 2D problems, one may hardly notice the difference between using the standard and the approximate mortar conditions, the difference is significant in the implementation, specially for 3D problems where the interfaces are no longer straight lines but planes. An analysis of a 3D extension of the CR approximate mortar finite element method is already in progress.

The concept of using an approximate mortar condition has started to receive more
and more attention. Recently, an additive Schwarz preconditioner for an approximate CR mortar finite element has been used to precondition the standard CR mortar finite element, see [16]. Approximate constraint has been used also in other context, see [4] for instance, where approximate constraint was indeed necessary for coupling the wavelet and the finite element.

The paper is organized as follows. In Section 2, a detailed description of the new approximate mortar condition is given, and the corresponding discrete problem is then formulated. The analysis of the new approach is found in Section 3, and a brief discussion on the implementation is given in Section 5. For the purpose of experimenting, in Section 4, we also propose an additive Schwarz preconditioner for the new approximate CR mortar finite element. Numerical results supporting our theory are presented in the final section.

2. The discrete problem. Let Ω ⊂ R^2 be a bounded, simply connected polygonal domain, partitioned into a collection of nonoverlapping polygonal subdomains, Ω_i, i = 1, . . . , N. We consider the following problem: Find u* ∈ H^1 0(Ω) such that

\begin{equation}
\tag{2.1}
a(u^*, v) = f(v), \quad v \in H^1_0(Ω),
\end{equation}

where

\[ a(u, v) = \sum_{i=1}^{N} \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \sum_{i=1}^{N} \int_{\Omega_i} f v \, dx. \]

We consider the subdomains to be geometrically conforming. With each subdomain Ω_i, we associate a quasi-uniform triangulation T_h(Ω_i) of mesh size h_i. The resulting triangulation can be nonmatching across subdomain interfaces.

Let X_h(Ω_i) (or simply X_h(Ω_i)) be the nonconforming P1 (Crouzeix-Raviart) finite element space defined on a shape regular triangulation T_h(Ω_i) (or simply T_h(Ω_i)) of Ω_i, consisting of functions which are piecewise linear in each triangle τ ∈ T_h(Ω_i), continuous at the interior edge midpoints of Ω^CR_h, and vanishing at the edge midpoints of ∂Ω^CR_h ∩ ∂Ω lying on the boundary ∂Ω. We use a subscript for h only when we need to differentiate the discretization of one domain from the other. Let X_h^0(Ω_i) ⊂ X_h(Ω_i) be the space of functions whose values at ∂Ω^CR_h are equal to zero. Here, Ω^CR_h and ∂Ω^CR_h represent the sets of edge midpoints, i.e., the nonconforming P1 (Crouzeix-Raviart) nodal points, of Ω_i and ∂Ω_i, respectively. Note that Ω^CR_h = Ω^CR_i ∪ ∂Ω^CR_h and ∂Ω^CR_h ∩ ∂Ω^CR_i is an empty set. Without the superscript CR, the sets represent the corresponding sets of triangle vertices, i.e., the P1 conforming nodal points. Using X_h(Ω_i), we define the product space X_h on the whole domain as

\[ X_h(Ω) = X_h(Ω_1) \times X_h(Ω_2) \ldots \times X_h(Ω_N). \]

Let Γ_{ij} be an open edge common to Ω_i and Ω_j, i.e., Γ_{ij} = ∂Ω_i ∩ ∂Ω_j. Note that each interface Γ_{ij} inherits two different discretization from its two sides, T_h(γ_{ij}) and T_h(γ_{ji}). We select one side of Γ_{ij} as the master side, called the mortar, and the other side as the slave side, called the nonmortar. We define the skeleton S = (∂Ω_i) \ ∂Ω of the decomposition as follows:

\[ S = \bigcup_m \tau_m, \quad \text{with} \quad \gamma_m \cap \gamma_n = \emptyset \quad \text{if} \quad m \neq n, \]

where each γ_m denotes an open mortar edge. We write γ_m as γ_m(i), if it is an edge of Ω_i, i.e., γ_m(i) ⊂ ∂Ω_i. Let δ_m = δ_m(j) ⊂ ∂Ω_j be the corresponding open nonmortar
edge of $\Omega_j$ that occupies the same geometrical space as $\gamma_{m(i)}$, i.e., $\gamma_{m(i)} = \Gamma_{ij} = \delta_{m(j)}$ (See Figure 2.1 for an illustration).

![Figure 2.1. Illustrating a mortar- ($\gamma_{m(i)}$) and the corresponding nonmortar- ($\delta_{m(j)}$) side of a subdomain interface ($\Gamma_{ij}$) with nonmatching meshes on both sides.](image)

Since the triangulations on $\Omega_i$ and $\Omega_j$ do not match on their common interface $\Gamma_{ij}$, the functions in $X_h(\Omega)$ are discontinuous on the set $\gamma_{m(i)}$ or $\delta_{m(j)}$ of edge midpoints on $\Gamma_{ij}$. In other words, $X_h(\gamma_{m(i)}) = X_h(\Omega_i)|_{\gamma_{m(i)}}$ differ from $X_h(\delta_{m(j)}) = X_h(\Omega_j)|_{\delta_{m(j)}}$ at those points. A weak continuity condition, called the mortar condition, is therefore imposed. Let $u_h \in X_h$, where $u_h = \{u_i\}_{i=1}^N$. A function $u_h \in X_h$ satisfies the mortar condition on $\delta_{m(j)} = \Gamma_{ij} = \gamma_{m(i)}$, if

$$Q_m J_m u_i = Q_m u_j,$$

where $J_m$ is an interpolation operator to be defined below, and $Q_m$ is the $L^2$-projection operator: $Q_m : L^2(\Gamma_{ij}) \rightarrow M^{h_j}(\delta_{m(j)})$ defined as

$$\langle Q_m u, \psi \rangle_{L^2(\delta_{m(j)})} = \langle u, \psi \rangle_{L^2(\delta_{m(j)})}, \quad \forall \psi \in M^{h_j}(\delta_{m(j)}),$$

where $M^{h_j}(\delta_{m(j)}) \subset L^2(\Gamma_{ij})$ is the test space of functions which are piecewise constant on the triangulation of $\delta_{m(j)}$, and $(\cdot, \cdot)_{L^2(\delta_{m(j)})}$ denotes the $L^2$ inner product on $L^2(\delta_{m(j)})$. In the original mortar method (cf. [15]), we remark that the operator $J_m$ is simply the identity operator.

Let $P^1(\tau)$ be the space of linear functions on a triangle $\tau \in T_h(\Omega_i)$, uniquely determined by their values at the vertices. Denote by $Z_h(\Omega_i) = \Pi_{\tau \in T_h(\Omega_i)} P^1(\tau)$ the space of piecewise linear functions defined on the triangulation $T_h(\Omega_i)$. Let $T^\gamma_h(\Omega_i)$ be the triangulation associated with the $\Omega_i$, which is obtained as a result of dividing the edges of $T_h(\Omega_i)$. Denote by $Y^\gamma_h(\Omega_i)$, the conforming space of piecewise linear and continuous functions on the triangulation $T^\gamma_h(\Omega_i)$. The functions of this space are defined by their values at the set $\Omega^\gamma_h(i)$ of all edge endpoints of $T^\gamma_h(\Omega_i)$. Let $\bar{Y}^\gamma_h(\Omega_i) \subset Y^\gamma_h(\Omega_i)$ be the space of functions whose values at $x \in \partial \Omega^\gamma_h(i)$ are equal to zero. It is easy to see that $\Omega^\gamma_h(i) = \Omega^CR_{ih} \cup \Omega_{ih}$. Let $E$ denote an edge (a triangle edge) or an edge segment. The midpoint, and the left and right end points of each edge (or edge segment) $E$, is denoted by $x_m^E$, $x_l^E$ and $x_r^E$, respectively. The length of $E$ is denoted by $h_E$. 
We now define the operator $J_m : X_h(\gamma_m) \rightarrow Z_h(\gamma_m)$. It is based on the definition of another interpolation operator $I_m : X_h(\gamma_m) \rightarrow Y^1_h(\gamma_m)$, see Definition 2.1.

**Definition 2.1.** For $u \in X_h(\gamma_m)$, $I_m u \in Y^1_h(\gamma_m)$ is defined by the nodal values as

$$I_m u(x) = \begin{cases} 
    u(x), & x \in \overline{\gamma}_m, \\
    \frac{h_x}{h_x + h_e} u(x_m^e) + \frac{h_e}{h_x + h_e} u(x_m^l), & x \in \gamma_{mb}, \\
    u(x_m^e) + \frac{h_x}{h_x + h_e} (u(x_m^l) - u(x_m^e)), & x \in \partial \gamma_{mb}.
\end{cases}$$

Here, $\gamma_l$ and $\gamma_r$ are the left- and the right neighboring edges of $x \in \gamma_{mb}$, respectively. $\gamma_m$ represents a triangle edge of $\mathcal{T}_h(\gamma_m)$, touching $\partial \gamma_m$, and $\gamma_m'$ is the corresponding neighboring edge.

The interpolation is done basically by first joining the neighboring edge midpoints using straight lines, and then simply extending the two end straight lines towards the end of the mortar $\gamma_m$, see Fig. 2.2. The operator $J_m$ can now be defined using $I_m$.

**Definition 2.2.** For $u \in X_h(\gamma_m)$, $J_m u \in Z_h(\gamma_m)$ is a piecewise linear function on the edges, $\{\gamma_m\}$, defined by its values at the two end points, $x_m^l \in \overline{\gamma}_{mb}$ and $x_m^r \in \overline{\gamma}_{mb}$, of each such edge $\gamma_m$. If $\gamma_m$ is an interior edge of $\gamma_m$, then

$$J_m u(x) = \begin{cases} 
    u(x_m^l) + \frac{1}{2} \left\{ I_m u(x_m^r) - I_m u(x_m^l) \right\}, & x = x_m^l, \\
    u(x_m^r) + \frac{1}{2} \left\{ I_m u(x_m^l) - I_m u(x_m^r) \right\}, & x = x_m^r.
\end{cases}$$

It is easy to see that, if $\gamma_m$ is a boundary edge of $\gamma_m$, then $J_m u(x) = I_m u(x)$ for $x = x_m^l, x_m^r$.

As seen from Fig. 2.2, for each edge $\gamma_m$ with edge midpoint $x_m^e$, the straight line $J_m u$ passes through $u(x_m^e)$, in other words $J_m u(x_m^e) = I_m u(x_m^e) = u(x_m^e)$. It is not difficult to see that the operator $J_m$ preserves all linear functions on the mortar.

---

![Illustrating the interpolation operator $J_m$.](image-url)

As seen from Fig. 2.2, for each edge $\gamma_m$ with edge midpoint $x_m^e$, the straight line $J_m u$ passes through $u(x_m^e)$, in other words $J_m u(x_m^e) = I_m u(x_m^e) = u(x_m^e)$. It is not difficult to see that the operator $J_m$ preserves all linear functions on the mortar.
Let $V_h \subset X_h$ be the subspace of functions which satisfy the mortar condition for all $\delta_m \subset S$. Since functions of $V_h$ are not continuous, for $u, v \in X_h$, we use the broken bilinear form $a_h(\cdot, \cdot)$ defined by

$$a_h(u, v) = \sum_{i=1}^{N} a_i(u, v) = \sum_{\tau \in T_h(\Omega_i)} \int_{\tau} \nabla u \cdot \nabla v \, dx,$$

and the corresponding broken $H^1$ seminorm $|u|_{H_h^1(\Omega_i)} = \sum_{\tau \in T_h(\Omega_i)} |u|_{H^1(\tau)}^2$. The discrete problem takes the following form: Find $u_h^* = (u_i^*)_i \in V_h$ such that

$$a_h(u_h^*, v) = f(v), \quad \forall v \in V_h.$$  

In our analysis, we will be using the subspace $H^1_2(\delta_m)$ which is defined as consisting of functions $v \in H^1(\Omega_i)$ such that their trivial extensions by zero to all of $\partial \Omega_j$ belong to the space $H^1(\partial \Omega_j)$, cf. [14]. The corresponding norm $\| \cdot \|_{H^1_2(\delta_m)}$ is defined as

$$\| v \|_{H^1_2(\delta_m)} = \inf_{u \in H^1_{0,\partial \Omega_j}(\Omega_j)} \| u \|_{H^1(\Omega_j)},$$

where $H^1_{0,\delta}$ is the closure in $H^1$ of all $C^\infty$ functions vanishing on $\delta$.

3. **Analysis.** Since, by construction, both $u \in X_h(\gamma_m)$ and $J_m u \in Z_h(\gamma_m)$ are linear on any edge $E$ belonging to an interface $\gamma_m$, and they have the same value at the edge midpoint $x^E_m$, their integral averages over the edge are the same, in other words, $\frac{1}{|E|} \int_E J_m u \, dx = \frac{1}{|E|} \int_E u \, dx$. This will see is essential for our analysis.

**Definition 3.1.** For any interior edge $E \subset \gamma_m$, where $\gamma_m$ is a subdomain interface, such that $E$ is not touching the boundary $\partial \gamma_m$, then we call $B_E$ the set of all triangles $\tau$ such that $\partial \tau \cap E \neq \emptyset$, that is, all triangles touching $E$, see Figure 3.1 (middle). If $E$ is a boundary edge of $\gamma_m$, see Figure 3.1 (left), then $B_E$ is the set of all triangles touching $E$ excluding the ones touching only at the endpoint $\partial \varepsilon \cap \partial \gamma_m$ of $E$. We call $B_{E,E}$ the set of triangles $\tau$ connecting the two triangle edges $E_i$ and $E_j$, see Figure 3.1 (right), in such a way that $\partial \tau \cap E \neq \emptyset$, where $E = (x^E_m, x^E_n)$ is the straight line joining the two edge midpoints $x^E_m$ and $x^E_n$.

Using the continuity of the function $u$ at edge midpoints, and the discrete equivalence of the $H^1$-seminorm, it is easy to see for any two neighboring edges $E_i$ and $E_j$ with $\partial E_i \cap \partial E_j \neq \emptyset$, that the following holds,

$$\left( u(x^E_m) - u(x^E_n) \right)^2 \leq c \sum_{\tau \in B_{E,E} \cap \partial E} |u|_{H^1(\tau)}^2,$$

where the sum is taken over the elements of the set $B_{E,E}$.

Now, we follow the definition of $I_m$, cf. Definition 2.1. If $E$ is an interior edge of $\gamma_m$, then a simple calculation yields

$$I_m u(x^E_m) - I_m u(x^E_n) = \left[ \frac{h_E}{h_E + h_{E_i}} \left( u(x^E_m) - u(x^E_n) \right) + \frac{h_{E_i}}{h_E + h_{E_i}} \left( u(x^E_n) - u(x^E_m) \right) \right].$$
Fig. 3.1. Shaded areas representing $B_E$ (left and middle), and $B_{E_iE_j}$ (right).

where $E_i$ and $E_r$ are respectively the left- and the right neighboring edges of $E$. If $E$ is the right boundary edge of $\gamma_m$, then we have the special case

$$I_m u(x^E_r) - I_m u(x^E_l) = \frac{2h_E}{h_E + h_{E_i}} \left\{ u(x^E_{m_r}) - u(x^E_{m_l}) \right\},$$

and, similarly, if $E$ is the left boundary edge of $\gamma_m$, then

$$I_m u(x^E_r) - I_m u(x^E_l) = \frac{2h_E}{h_E + h_{E_r}} \left\{ u(x^E_{m_r}) - u(x^E_{m_l}) \right\}.$$

Hence, by applying (3.1), we get

$$(I_m u(x^E_r) - I_m u(x^E_l))^2 \leq c \sum_{\tau \in B_E} |u|^2_{H^1(\tau)},$$

for each edge $E \subset \gamma_m$, where the sum is taken over the elements of the set $B_E$. Note that $B_E$ equals to either $B_{EiE}$ or $B_{EEr}$ if $E$ is a boundary edge, and equals to $B_{EEi} \cup B_{EEr}$ if $E$ is an interior edge.

Now, we follow the definition of $J_m$, cf. Definition 2.2. For any edge $E \subset \gamma_m$, the following is true.

$$J_m u(x^E_r) - u(x^E_m) = -(J_m u(x^E_l) - u(x^E_m)) = \frac{1}{2} \left[ I_m u(x^E_r) - I_m u(x^E_l) \right].$$

If $x \in E$ is a point lying on the right of $x^E_m$, then

$$J_m u(x) - u(x^E_m) = \frac{\text{dist}(x, x^E_m)}{h_E} \left[ I_m u(x^E_r) - I_m u(x^E_l) \right],$$

or lying on the left of $x^E_m$, then

$$J_m u(x) - u(x^E_m) = -\frac{\text{dist}(x, x^E_m)}{h_E} \left[ I_m u(x^E_r) - I_m u(x^E_l) \right],$$

where $\text{dist}(x, x^E_m)$ is the distance between the points $x$ and $x^E_m$. If $x^E$ is a point anywhere on the edge $E$, then by applying (3.2), we get

$$(J_m u(x^E_r) - u(x^E_m))^2 \leq c \sum_{\tau \in B_E} |u|^2_{H^1(\tau)},$$
for edge $E \subset \gamma_m$, where the sum is taken over the elements of the set $B_E$.

The properties of $J_m$ can be stated in the following lemmas.

**Lemma 3.2.** For $u_i \in X_h(\Omega_i)$, and each $E \subset \gamma_m$, we have

$$\| u_i - J_m u_i \|^2_{L^2(E)} \leq c h_i \sum_{\tau \in B_E} |u_i|^2_{H^1(\tau)},$$

and

$$\| u_i - J_m u_i \|^2_{L^2(\gamma_m(i))} \leq c h_i |u_i|^2_{H^1_h(\Omega_i)}.$$

**Proof.** Let $\overline{u_i}_E$ denote the average of $u_i$ over the edge $E$, that is $\overline{u_i}_E = \frac{1}{|E|} \int_E u_i \, dx$. From the trace- and the Poincaré inequality, and (3.3), we get

$$\| u_i - J_m u_i \|^2_{L^2(E)} \leq 2 \| u_i - \overline{u_i}_E \|^2_{L^2(E)} + 2 \| J_m u_i - \overline{u_i}_E \|^2_{L^2(E)} \leq c h_i |u_i|^2_{H^1(\tau)} + c h_i \sum_{x = x_i^E, x_i^R} (J_m u_i(x) - u_i(x^E))^2 \leq c h_i |u_i|^2_{H^1(\tau)} + c h_i \sum_{\tau \in B_E} |u_i|^2_{H^1(\tau)} \leq c h_i \sum_{\tau \in B_E} |u_i|^2_{H^1(\tau)}.$$ 

The second inequality follows immediately from the first one by taking the sum over the triangle edges $E$ of $\gamma_m(i)$. \[\square\]

The next lemma is crucial for the analysis. The factor $h_i^{1/2}$ in its estimate is essential for our estimate of the consistency error. In the first part of its proof, we will need the following operator $M_i : X_h(\Omega_i) \to Y_h(\Omega_i)$, cf. [21]: for $u \in X_h(\Omega_i)$,

$$M_i u(x) = \begin{cases} u(x), & x \in \Omega_i, \\ \frac{1}{n_x} \sum_{\tau \in B_E} u_{\tau}(x), & x \in \Omega_{ih}, \end{cases}$$

for $\Omega_{ih}$ the set of all triangles of $T_h(\Omega_i)$ sharing the vertex $x$, and $n_x$ is the number of such elements. Following are the properties of $M_i$, which we will use:

$$\| u - M_i u \|^2_{L^2(E)} \leq c h_i \sum_{\tau \in B_E} |u_{\tau}|^2_{H^1(\tau)} \quad \text{and} \quad |M_i u_{\tau}|^2_{H^1_h(\Omega_i)} \leq c |u_{\tau}|^2_{H^1_h(\Omega_i)}.$$ 

In the first inequality above, $E$ is an edge of $\partial \Omega_i$, where $\hat{B}_E$ is the set of all triangles $\tau$ touching $E$. We remark here that the number of triangles touching any vertex in a subdomain is bounded independently of the mesh size. The first estimate of (3.4) can be proved in a similar way as the proof of the first estimate in Lemma 3.2, and so we omit the proof here. For the proof of the second estimate we refer to [21].

**Lemma 3.3.** For $u_i \in X_h(\Omega_i)$, we have

$$\| J_m u_i - Q_m J_m u_i \|^2_{L^2(\gamma_m(i))} \leq c h_i^{1/2} |u_i|_{H^1_h(\Omega_i)}.$$
Proof. We partition $\gamma_m$ into a collection of nonoverlapping subintervals or edge segments, $\mathcal{E}_o$, which are intersections between the edges from the mortar side and the edges from the nonmortar side. For simplicity we assume that each $\mathcal{E}_o$, shown as a thick line segment in figures 3.2 - 3.3 (left pictures), corresponds to a complete edge.

For the case where $\mathcal{E}_o$ may not be a complete edge but rather an edge segment shown as a thick line segment in figures 3.2 - 3.3 (right pictures), the analysis are similar and will only contribute to the constant $c$. For the proof of the present lemma, we consider separately the two different cases of the mesh sizes: the mesh on the mortar side is finer than the mesh on the nonmortar side (Case-A), and vice versa (Case-B).

The general case where the mesh on the mortar side may be finer than the mesh on the nonmortar side at some region along the interface and coarser in the other region, can be formed as a combination of the two cases.

\[ \gamma_m(i) \quad \delta_m(j) \]
\[ \mathcal{E} \quad \mathcal{E}_o \]
\[ \frac{1}{\mathcal{E}_o} \quad \frac{1}{\mathcal{E}} \]

Fig. 3.2. Illustrating $h_i < h_j$.

Case-A: Let $\mathcal{E} \subset \delta_m(j)$ and $\mathcal{E}_o \subset \gamma_m(i)$, where $\mathcal{E}_o \subset \mathcal{E}$ such that $h_{\mathcal{E}_o} \leq h_\mathcal{E}$, cf. Fig. 3.2. Now, by the triangle inequality and the $L^2$ stability of the projection operator $Q_m$, we can write

\[
\| J_m u_i - Q_m J_m u_i \|_{L^2(\gamma_m(i))}^2 \\
\leq c \left\{ \| J_m u_i - u_i \|_{L^2(\gamma_m(i))}^2 + \| u_i - Q_m u_i \|_{L^2(\gamma_m(i))}^2 \\
+ \| Q_m (u_i - J_m u_i) \|_{L^2(\gamma_m(i))}^2 \right\}
\leq c \left\{ \sum_{\mathcal{E} \subset \delta_m(j)} \| u_i - J_m u_i \|_{L^2(\mathcal{E})}^2 + \| u_i - Q_m u_i \|_{L^2(\gamma_m(i))}^2 \right\}.
\]

Owing to the same $L^2$ stability, the second term above can further be estimated as the following.

\[
\| u_i - Q_m u_i \|_{L^2(\gamma_m(i))}^2 \\
\leq c \left\{ \sum_{\mathcal{E} \subset \delta_m(j)} \| u_i - M_i u_i \|_{L^2(\mathcal{E})}^2 + \| M_i u_i - Q_m M_i u_i \|_{L^2(\gamma_m(i))}^2 \right\}.
\]

Using Lemma 3.2 and the first estimate of (3.4) for each edge $\mathcal{E}_o \subset \mathcal{E}$, adding their contributions, and finally noting that $h_{\mathcal{E}_o} \leq h_\mathcal{E}$, the terms $\| u_i - J_m u_i \|_{L^2(\mathcal{E})}^2$ and $\| u_i - M_i u_i \|_{L^2(\mathcal{E})}^2$ can be estimated as $c h_j \sum_{\tau \in \mathcal{B}_e} |u_i|_{H^1(\tau)}^2$ and $c h_j \sum_{\tau \in \mathcal{B}_e} |u_i|_{H^1(\tau)}^2$, respectively. Now summing over $\mathcal{E} \subset \delta_m(j)$, in each case, we get the bound equal to...
of $I^3$. We note that the function $\mathcal{M}_i u_i - Q_m \mathcal{M}_i u_i \|_{L^2(\gamma_{m(i)})}$. To do so, we first note that, cf. [1],

$$\| \mathcal{M}_i u_i - Q_m \mathcal{M}_i u_i \|_{L^2(E)}^2 \leq c h_j |u_i|^2_{H^1(E)}.$$ 

Now, by adding together the estimates over $E \subset \delta_{m(j)}$, using the trace inequality $|\mathcal{M}_i u_i|_{H^\frac{1}{2}(\gamma_{m(i)})} \leq c |\mathcal{M}_i u_i|_{H^1(\Omega)}$, and finally applying the second estimate of (3.4) we again get the bound $c h_j |u_i|^2_{H^1(\Omega)}$. Replacing the three estimates into (3.6) proves the lemma for the present case.

**Case - B:** Let $E \subset \gamma_{m(i)}$ and $\mathcal{E}_o \subset \delta_{m(j)}$, and $\mathcal{E}_o \subset E$ such that $h_{\mathcal{E}_o} \leq h_E$, cf. Fig. 3.3. We note that the function $J_m u_i - Q_m J_m u_i$ is linear over $\mathcal{E}_o$. Now,

$$\| J_m u_i - Q_m J_m u_i \|_{L^2(\delta_{m(j)})}^2 = \sum_{E \subset \delta_{m(j)}} \| J_m u_i - Q_m J_m u_i \|_{L^2(\mathcal{E}_o)}^2 \leq c \sum_{E \subset \delta_{m(j)}} h_{\mathcal{E}_o} \sum_{x = x_{m(i)}^E, x_{m(i)}^E} (J_m u_i(x) - Q_m J_m u_i(x))^2.$$ 

![Fig. 3.3. Illustrating $h_j < h_i$.](image)

On the edge $\mathcal{E}_o$, $Q_m J_m u_i$ is a constant equal to $J_m u_i(x_{m(i)}^E)$. The difference $(J_m u_i(x) - Q_m J_m u_i(x))$ is zero at $x = x_{m(i)}^E$, and at $x = x_{m(i)}^E$ and $x_{m(i)}^E$, it can be written as follows.

$$J_m u_i(x_{m(i)}^E) - Q_m J_m u_i(x_{m(i)}^E) = - \left( J_m u_i(x_{m(i)}^E) - Q_m J_m u_i(x_{m(i)}^E) \right)$$

$$= - \left( \frac{1}{h_E} \left( I_m u_i(x_{m(i)}^E) - I_m u_i(x_{m(i)}^E) \right) \right) \frac{h_{\mathcal{E}_o}}{2},$$

where the term inside the curly brackets represents the slope of $J_m u_i$, or equivalently of $I_m u_i$, along the mortar. Hence, for $x = x_{m(i)}^E$ and $x_{m(i)}^E$, it follows from (3.2) that

$$\left( J_m u_i(x) - Q_m J_m u_i(x) \right)^2 \leq c \left( \frac{h_{\mathcal{E}_o}}{h_E} \right)^2 \left( I_m u_i(x_{m(i)}^E) - I_m u_i(x_{m(i)}^E) \right)^2$$

$$\leq c \left( \frac{h_j}{h_i} \right)^2 \sum_{E \subset \delta_{m(j)}} |u_i|^2_{H^1(\Omega)},$$

where the sum is taken over the set $B_{\mathcal{E}}$ of triangles along the mortar side, those touching the edge $E$. The number of occurrence of the above sum in the final calculation
is \( \approx h_i/h_j \). This results into an \( h_j \) in the final calculation, since \( h_{E_i}(h_i)^2 h_i \leq h_j \) due to \( \frac{h_i}{h_j} \leq 1 \). Hence, we arrive at

\[
\| J_m u_i - Q_m J_m u_i \|_{L^2(\gamma_m(i))} \leq c h_j |u_i|_{H^1_0(\Omega_i)}^2.
\]

The lemma thus holds for both cases. For the general case where both Case A and B exist along the interface \( \gamma_m(i) = \delta_{m(j)} \), we simply need to take an appropriate combination of the estimates, that is the estimates on \( E \) from Case A and the estimates on \( E_i \) from Case B, to show that the lemma holds. \( \square \)

In the following, we briefly introduce some special operators from [15, 21], which we will need for our analysis. Let \( \Pi_m : L^2(\delta_m(j)) \rightarrow Y^0_h(\delta_m(j)) \), where \( Y^0_h(\delta_m(j)) \) is the set of continuous and piecewise linear functions on \( \delta_{m(j)} \), defined uniquely by their values at \( x \in \delta_{m(j)} \), and taking zero values at the boundary. \( \Pi_m u \) on \( \delta_{m(j)} \), is defined as follows, cf. [15],

\[
\Pi_m u(x) = Q_m u(x), \quad \forall x \in \delta^{CR}_{m(j)}h.
\]

From the definition of \( Q_m \), cf. (2.3), it follows that \( Q_m(Q_m u)(x) = (Q_m u)(x) \) for \( x \in \delta^{CR}_{m(j)}h \), giving \((\Pi_m(Q_m u))(x) = (Q_m(Q_m u))(x) = Q_m u(x) = \Pi_m u(x), \quad \forall x \in \delta^{CR}_{m(j)}h.
\)

Hence, on \( \delta_{m(j)} \),

\[(3.7) \quad \Pi_m u = \Pi_m(Q_m u).\]

The stability of \( \Pi_m \) is stated in the following lemma, see [15] for proof.

**Lemma 3.4.** For \( u \in H^1_0(\delta_{m(j)}) \),

\[
\| \Pi_m u \|_{H^1_0(\delta_{m(j)})} \leq c \| u \|_{H^1_0(\delta_{m(j)})},
\]

and for \( u \in L^2(\delta_{m(j)}) \),

\[
\| \Pi_m u \|_{L^2(\delta_{m(j)})} \leq c \| u \|_{L^2(\delta_{m(j)})}.
\]

In the following, we define the two discrete harmonic extension operators, \( H^{CR}_h : X_h(\partial \Omega_j) \rightarrow X_h(\Omega_j) \) and \( H^2_2 : Y^0_h(\partial \Omega_j) \rightarrow Y^0_h(\Omega_j) \), corresponding to the nonconforming P1 functions and the conforming P1 functions, respectively. For \( u \in X_h(\partial \Omega_j) \), \( H^{CR}_h u \in X_h(\Omega_j) \) is the solution of

\[(3.8) \quad a_h(H^{CR}_h u, v) = 0, \quad \forall v \in X^0_h(\Omega_j),\]

where \( H^{CR}_h u(x) = u(x), \) for \( x \in \partial \Omega^{CR}_j \). Similarly, for \( u \in Y^0_h(\partial \Omega_j) \), \( H^2_2 u \in X_h(\Omega_j) \) is the solution of

\[(3.9) \quad a_h(H^2_2 u, v) = 0, \quad \forall v \in Y^0_h(\Omega_j),\]

where \( H^2_2 u(x) = u(x), \) for \( x \in \partial \Omega^{CR}_j \).

We state another useful operator \( M^+_j : Y^0_2(\Omega_j) \rightarrow X_h(\Omega_j) \), cf. [21], which is defined as follows: for \( u \in Y^0_2(\Omega_j) \),

\[
M^+_j u(x) = u(x), \quad \forall x \in \Omega^{CR}_j.
\]
For any \( u \in Y_h(\Omega_j) \), the following holds, cf. [21],

\[
|M_j^+ u|_{H^1_h(\Omega_j)} \leq c |u|_{H^1_h(\Omega_j)}.
\]

(3.10)

In the next lemma we estimate the \( H^1_h \)-seminorm over a subdomain, by the \( H^1_\infty \)-norm on a non-mortar side of the subdomain.

**Lemma 3.5.** Let \( u \in X_h(\Omega_j) \) be discrete harmonic function in \( \Omega_j \), in the sense of (3.8), with \( u = 0 \) at \( x \in \partial \Omega^R_{j(h)} \setminus \delta_m(\Omega_j) \). Then

\[
|u|_{H^1_h(\Omega_j)} \leq c \| \Pi_m u \|_{H^1_\infty(\delta_m)}.
\]

(3.11)

**Proof.**

\[
|u|_{H^1_h(\Omega_j)} \leq c |M_j^+ \mathcal{H} \Pi_m u|_{H^1_h(\Omega_j)}
\]

\[
\leq c |\mathcal{H} E_m \Pi_m u|_{H^1_h(\Omega_j)}
\]

\[
\leq c \| \Pi_m u \|_{H^1_\infty(\delta_m)}
\]

The first inequality follows from the fact that \( u \) is itself discrete harmonic, and therefore has the minimal energy over all functions whose boundary values at \( x \in \partial \Omega^R_{j(h)} \), are the same as those of \( u \). The second inequality follows from (3.10), and the third inequality follows from the property of a conforming discrete harmonic functions, cf. [7].

\( E_m \) is the zero extension operator from \( L^2(\delta_m) \) to \( L^2(\partial \Omega_j) \).

**Theorem 3.6 (Error).** Let \( u \in H^2(\Omega) \) and \( u_h \in V_h \) be the exact solution of (2.1) and (2.4), respectively, then

\[
|u - u_h|_{H^1_h(\Omega)} \leq c \left( \sum_{k=1}^N h_k^2 |u|_{H^2(\Omega_k)}^2 \right)^{1/2}.
\]

(3.12)

The proof follows from the second Strang lemma, cf [12],

\[
|u - u_h|_{H^1_h(\Omega)} \leq c \inf_{v \in V_h} |u - v|_{H^1_h(\Omega)}
\]

(3.13)

\[
+ \sup_{w \in V_h} \sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} \int_{\partial \tau} \frac{\partial u}{\partial \eta} \frac{w}{|w|} \, ds,
\]

and Lemma 3.7-3.8.

**Lemma 3.7 (Consistency error).** Let \( u \in H^2(\Omega) \) be the exact solution of (2.1). Then for any \( w = \{w_k\} \in V_h \),

\[
\sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} \int_{\partial \tau} \frac{\partial u}{\partial \eta} \frac{w}{|w|} \, ds \leq c \left( \sum_{k=1}^N h_k^2 |u|_{H^2(\Omega_k)}^2 \right)^{1/2}
\]

(3.14)

**Proof.** Note that

\[
\sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} \int_{\partial \tau} \frac{\partial u}{\partial \eta} w_k \, ds = \sum_{\tau \in T_h(\Omega_k), \epsilon \subset (\partial \tau \setminus S)} \int_{\partial \tau} \frac{\partial u}{\partial \eta} |w| \, ds
\]

\[
+ \sum_{\Gamma_i \subset S} \int_{\Gamma_i} \frac{\partial u}{\partial \eta} |w| \, ds
\]
Using Lemma 8.3.7 and Lemma 8.3.9 of [12], for the interior and boundary edge, respectively, one can estimate the first term as follows

\[ \sum_{\tau \in \mathcal{T}_k(\Omega_k), \, \mathcal{E} \subset (\partial \tau \setminus S)} \int_{\mathcal{E}} \frac{\partial u}{\partial \eta}[w] \, ds \leq c \left( \sum_{k=1}^{N} h_k^2 |u|_{H^2(\Omega_k)}^{\mathcal{E}} \right)^{1/2} |w|_{H^1(\Omega)} \]

In the following, we estimate the second term.

Let \( [w] = w_i - w_j \) be the jump across \( \Gamma_{ij} \), where \( \gamma_{m(i)} = \Gamma_{ij} = \delta_{m(j)} \). Let \( \alpha = \{ \alpha_{\mathcal{E}} \}_{\mathcal{E} \subset \gamma_{m(i)}} \) be the piecewise constant function on \( \gamma_{m(i)} \), where \( \alpha_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \frac{\partial u}{\partial \eta} \, ds \) for each triangle edge \( \mathcal{E} \subset \gamma_{m(i)} \). Similarly, we define the piecewise constant function on \( \delta_{m(j)} \) as \( \beta = \{ \beta_{\mathcal{E}} \}_{\mathcal{E} \subset \delta_{m(j)}} \), where \( \beta_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \frac{\partial u}{\partial \eta} \, ds \) for each triangle edge \( \mathcal{E} \subset \delta_{m(j)} \).

We get immediately the following two relations,

\[ \int_{\gamma_{m(i)}} \alpha(w_i - J_m w_i) \, d\sigma = 0 \]

and

\[ \int_{\delta_{m(j)}} \beta(J_m w_i - w_j) \, d\sigma = 0. \]

The first one follows from the fact that \( \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} J_m w_i \, dx = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w_i \, dx \), by construction, for each \( \mathcal{E} \subset \gamma_{m(i)} \), and the second one is due to the mortar condition, giving, for each \( \mathcal{E} \subset \delta_{m(j)} \), that \( w_i(x_m) = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} J_m w_i \, dx \). Now, using these relations, we get

\[
\int_{\Gamma_{ij}} \frac{\partial u}{\partial \eta}[w] \, d\sigma \leq \int_{\gamma_{m(i)}} \left( \frac{\partial u}{\partial \eta} - \alpha \right) (w_i - J_m w_i) \, d\sigma + \int_{\delta_{m(j)}} \left( \frac{\partial u}{\partial \eta} - \beta \right) (J_m w_i - w_j) \, d\sigma
\]

\[
\leq \left\| \frac{\partial u}{\partial \eta} - \alpha \right\|_{L^2(\gamma_{m(i)})} \| w_i - J_m w_i \|_{L^2(\gamma_{m(i)})} + \left\| \frac{\partial u}{\partial \eta} - \beta \right\|_{L^2(\delta_{m(j)})} \| J_m w_i - w_j \|_{L^2(\delta_{m(j)})}
\]

We have

\[
\left\| \frac{\partial u}{\partial \eta} - \alpha \right\|_{L^2(\gamma_{m(i)})} \leq c \sum_{\mathcal{E} \subset \gamma_{m(i)}} h_{\mathcal{E}} \left\| \frac{\partial u}{\partial \eta} \right\|_{H^1(\mathcal{E})} \leq c \sum_{\mathcal{E} \subset \gamma_{m(i)}} h_{\mathcal{E}} \left\| \frac{\partial u}{\partial \eta} \right\|_{H^1(\mathcal{E})}
\]

\[
\leq c h_i |u|_{H^2(\Omega_i)}^2
\]

where the last sum is taken over the triangles \( \tau_{\mathcal{E}} \in \mathcal{T}_h(\Omega_i) \) those having \( \mathcal{E} \subset \gamma_{m(i)} \) as one of their edges. Similarly,

\[
\left\| \frac{\partial u}{\partial \eta} - \beta \right\|_{L^2(\delta_{m(j)})} \leq c h_j |u|_{H^2(\Omega_j)}^2
\]
Also, it follows from Lemma 3.2 that
\[ \| w_i - J_m w_i \|_{L^2(\gamma_{m(i)})}^2 \leq c h_i |w_i|_{H^1_k(\Omega_i)}^2. \]

Now, using the mortar condition \( Q_m J_m w_i = Q_m w_j \) and Lemma 3.3, we get
\[
\| J_m w_i - w_j \|_{L^2(\gamma_{m(j)})}^2 \leq 2 \| J_m w_i - Q_m J_m w_i \|_{L^2(\gamma_{m(j)})}^2 + 2 \| Q_m w_j - w_j \|_{L^2(\gamma_{m(j)})}^2 \\
\leq c \left( h_j |w_i|_{H^1_k(\Omega_i)}^2 + h_j |w_j|_{H^1_k(\Omega_i)}^2 \right).
\]

Finally,
\[
\int_{\Gamma_{ij}} \frac{\partial u}{\partial n} \, d\sigma \leq c \left( h_i |u|_{H^2(\Omega_i)} |w_i|_{H^1(\Omega_i)} + h_j |u|_{H^2(\Omega_j)} \left( |w_i|_{H^1_k(\Omega_i)} + |w_j|_{H^1_k(\Omega_i)} \right) \right).
\]

Now, summing over the interfaces \( \Gamma_{ij} \subset \mathcal{S} \), and using the Cauchy-Schwarz inequality,
\[
\left( \sum_i (a_i b_i) \right)^{1/2} \leq \left( \sum_i a_i^2 \right)^{1/2} \left( \sum_i b_i^2 \right)^{1/2},
\]
we get the proof. \( \square \)

**Lemma 3.8 (Approximation error).** For any function \( u \in H^1_0(\Omega) \), with \( u_{|\Omega_k} \in H^2(\Omega_k) \), we have
\[
(3.15) \quad \inf_{v \in V_h} |u - v|_{H^1_0(\Omega)} \leq c \left( \sum_{k=1}^N h_k^2 |u|_{H^2(\Omega_k)}^2 \right)^{1/2}
\]

**Proof.** Let \( I_h \) be the conforming P1-interpolant of \( u \) on \( T_h(\Omega_k) \), for \( k = 1, \ldots, N \), and define \( \bar{v} = I_h u = \{ \bar{v}_k \}_{k=1}^{N} \in X_h \). We note that \( \bar{v} \) does not satisfy the mortar condition, and so \( \bar{v} \notin V_h \). Let \( v = \bar{v} + w \), where \( w = \{ w_k \}_{k=1}^{N} \in X_h \), and \( w_k = \sum_{\delta_{m(k)} \subset \partial \Omega_k} w_{m(k)} \in X_h(\Omega_k) \). The function \( w_{m(k)} \in X_h(\Omega_k) \) is defined as follows.
\[
w_{m(k)}(x) = \begin{cases} 
Q_m(J_m \bar{v}_l - \bar{v}_k)(x), & x \in \delta_{m(k)}^{CR}, \\
0, & x \in \delta_{m(k)}^{CR} \setminus \delta_{m(k)}^{CR}, \\
\text{Discrete harmonic (cf. 3.8),} & x \in \Omega_{kh}^{CR}.
\end{cases}
\]
It is easy to see that \( v \in V_h \).
\[
|u - v|_{H^1_0(\Omega_k)}^2 \leq |u - \bar{v}_k|_{H^1(\Omega_k)}^2 + |w_k|_{H^2(\Omega_k)}^2 + c h_k^2 |w|_{H^2(\Omega_k)}^2 + c h_k^2 |w_k|_{H^2(\Omega_k)}^2
\]
Here we have used \( |u - I_h u|_{H^1(\Omega_k)} \leq c h_k |u|_{H^2(\Omega_k)} \) from the Bramble-Hilbert lemma.

Now using Lemma 3.5 and (3.7), we have
\[
|w_{m(k)}|_{H^1_0(\Omega_k)}^2 \leq \| \Pi_m(J_m \bar{v}_l - \bar{v}_k) \|_{H^2(\delta_{m(k)})}^2 + \| \Pi_m(J_m \bar{v}_l - \bar{v}_k) \|_{H^2(\delta_{m(k)})}^2 + \| \Pi_m(\bar{v}_l - \bar{v}_k) \|_{H^2(\delta_{m(k)})}^2
\]
From [15], we can bound the second term as
\[ \| \Pi_m (\tilde{v}_l - \tilde{v}_k) \|^2_{H^2(\Omega)} \leq c \left( h_l^2 |u|_{H^2(\Omega)}^2 + h_k^2 |u|_{H^2(\Omega)}^2 \right). \]

For the second term we use a trace inequality to have
\[ \| \Pi_m (J_m \tilde{v}_l - \tilde{v}_l) \|^2_{H^2(\Omega)} \leq c h_l^{-1} \| \Pi_m (J_m \tilde{v}_l - \tilde{v}_l) \|^2_{L^2(\Omega)} \]
\[ = c h_l^{-1} \sum_{E \subseteq \gamma_m(\tau)} \| J_m \tilde{v}_l - \tilde{v}_l \|^2_{L^2(E)} \]
\[ \leq c h_l^{-1} \sum_{E \subseteq \gamma_m(\tau)} h_l \sum_{\tau \in B_E} |\tilde{v}_l|_{H^1(\tau)}^2. \]

For each \( E \subseteq \gamma_m(\tau) \), define \( Q^2 u \) inside \( B_E \), where \( Q^2 u \) is the averaged Taylor polynomial of order 2 of \( u \) as defined in Chapter 4 of [12]. We note that \( (J_m I_h u - I_h u) \)
\[ = J_m I_h (u - Q^2 u) - I_h (u - Q^2 u) \] on \( E \), and hence using the Bramble-hilbert lemma, we have, cf. [12],
\[ \| J_m \tilde{v}_l - \tilde{v}_l \|^2_{L^2(E)} \leq c h_l \sum_{\tau \in B_E} |I_h (u - Q^2 u)|_{H^1(\tau)}^2 \]
\[ \leq c h_l \sum_{\tau \in B_E} |u - Q^2 u|_{H^1(\tau)}^2 \]
\[ \leq c h_l^3 \sum_{\tau \in B_E} |u|_{H^2(\tau)}^2. \]

Hence,
\[ \| \Pi_m (J_m \tilde{v}_l - \tilde{v}_l) \|^2_{H^2(\Omega)} \leq c h_l^3 |u|_{H^2(\Omega)}^2. \]

Theorem 3.6 gives an estimate for the error in the finite element solution in the broken \( H^1 \) seminorm. In the following we remark on the \( L^2 \) norm estimate of the error, which can be obtained using the same arguments as is in [11] for the conforming \( P_1 \) finite element. Using (3.14) and (3.15) and a duality technique in Lemma III.1.4 of [8], we can get the estimate:
\[ \| u - u_h \|_{L^2(\Omega)} \leq c h^2 |u|_{H^2(\Omega)}, \]
where \( h = \max_k h_k \). A careful manipulation of the estimates will yield the following estimate:
\[ \| u - u_h \|_{L^2(\Omega)} \leq C h \left( \sum_{k=1}^N h_k^2 |u|_{H^2(\Omega_k)}^2 \right)^{\frac{1}{2}}, \]
which includes explicitly the mesh sizes of the subdomains. However, the factor \( h \) appearing in the estimate is due to the use of the global regularity assumption:
\[ |u|_{H^2(\Omega)} \leq c |f|_{L^2(\Omega)}. \]
4. An additive Schwarz method. Recently, an additive Schwarz method for the CR mortar finite element, which uses the standard mortar condition, has been proposed in [19]. A P1 mortar finite element version of the method can be found in [6]. In this section, we propose a similar method for our discrete problem (2.4), which is formed as a natural extension of the original method from the standard mortar case to the new approximate mortar case. This is done by a simple adjustment in the definition of the subspaces involved in the decomposition of the discrete space, so that the subspaces adapt naturally to the new situation. In the standard mortar case, we recall from the definition of the subspaces that part of the subdomain interior nodes, that is, the set of edge midpoints those lying closest to a mortar side were treated as if they were on the mortar side, in other words, the mortar side becomes thicker. Note that due to the new mortar condition we do not require this arrangement.

So, following the general framework for additive Schwarz methods (cf. [20]), we can decompose $V_h$ as $V_h = V^S + V^0 + \sum_{i=1}^N V^i$. For $i = 1, \ldots, N$, $V^i$ is the restriction of $V_h$ to $\Omega_i$, with functions vanishing at subdomain boundary edge midpoints $\partial \Omega_{ih}$ as well as on the remaining subdomains.

$V^S$ is the space of functions given by their values on the skeleton edge midpoints $S^{CR}_h = \bigcup_{m=1}^{N_{mh}} S^{CR}_h$, $V^0 = \{ v \in V^h : v(x) = 0, x \in \Omega_{ih} \setminus S^{CR}_h \}$. We assume that there are no corner triangles, that is, triangles having more than one edge on a subdomain boundary. It is then easy to see that the corresponding stiffness matrix is a block diagonal matrix with each block being associated to one mortar side only.

The coarse space $V^0$ is a special space having a dimension equal to the number of subdomains. It is defined using the function $\chi_i \in X_h(\Omega_i)$ associated with subdomain $\Omega_i$. $\chi_i$ is defined by its nodal values as: $\chi_i(x) = 1/\sum_j \rho_j(x)$ at $x \in \Omega_{ih}$, where the sum is taken over the subdomains $\Omega_j$ to which $x$ belongs, and $\rho_j = 1$, $\forall j$. Note that the $\rho_j$’s may represent physical parameters with jumps across interfaces, see [19]. $V^0$ is given as the span of its basis functions, $\Phi_i, i = 1, \ldots, N$, i.e., $V^0 = \text{span}\{ \Phi_i : i = 1, \ldots, N \}$, where $\Phi_i$ associated with $\Omega_i$, is defined as follows.

\[
\Phi_i(x) = \begin{cases} 
1, & x \in \Omega_{ih}^{CR} \\
\rho_i \chi_i(x), & x \in \Omega_{ih}^{CR} \\
\rho_i Q_m(j_m \chi_j)(x), & x \in \Omega_{ih}^{CR} \setminus \delta_{m(i)}^{\gamma(i)} \\
\rho_i Q_m(j_m \chi_j)(x), & x \in \Omega_{ih}^{CR} \setminus \delta_{m(i)}^{\gamma(i)} \\
\rho_i \chi_j(x), & x \in \Omega_{ih}^{CR} \setminus \delta_{m(i)}^{\gamma(i)} \\
0, & x \in \partial \Omega_{ih}^{CR} \setminus \partial \Omega
\end{cases}
\]

and $\Phi_i(x) = 0$ at all other $x$ in $\Omega_{ih}^{CR}$. We use exact bilinear forms for all our subproblems. The projection-like operators $T^i : V_h \rightarrow V^i$ are defined in the standard way, i.e., for $i \in \{S, 0, \ldots, N\}$ and $u \in V_h$, $T^i u \in V^i$ is the solution of $a_h(T^i u, v) = a_h(u, v), \forall v \in V^i$. Let $T = T^S + T^0 + T^1 + \ldots + T^N$. The problem (2.4) is now replaced by an equivalent system, cf. [20],

\[
Tu_h^* = g,
\]

where $g = T^S u_h^* + \sum_{i=0}^N T^i u_h^*$. Let $c$ and $C$ represent constants independent of the mesh sizes $h = \inf_i h_i$ and $H = \max_i H_i$, then the following theorem holds.

**Theorem 4.1.** For all $u \in V_h$,

\[
c \frac{h}{H} a_h(u, u) \leq a_h(T u, u) \leq C a_h(u, u).
\]
The theorem can be shown in the same way as the proof in [19], which uses the general theory for Schwarz methods, cf. [20]. It follows from the theorem that the condition number of the operator $T$ grows as $\frac{n}{k^2}$.

5. Implementation issues. The best way to understand how to implement the method, is to look into the matrix representation of the method. When it comes to implementation, the mortar method of this paper differs from the one in [18] mainly in the mortar condition. We consider therefore only the mortar condition, and present its matrix representation here. For the rest we refer to the matrix formulation section of [18].

For each mortar $\gamma_m$, let $\{E_k : E_k \in T_h(\gamma_m)\}_{k=1,...,n}$ and $\{E_o : E_o \in T_h(\delta_m)\}_{o=1,...,p}$ as the sets of $n$ and $p$ triangle edges along the mortar and the corresponding nonmortar side, respectively. Let $J_m$ be the matrix representation of the interpolation operator $J_m : X_h(\gamma_m) \to Z_h(\gamma_m)$, in the sense that if $u_{\gamma_m}$ is the vector containing the values of the function $u \in X_h$ at the nodes of $\gamma_m$, then $J_m u_{\gamma_m}$ will return the values of the function $J_m u$ at the edge left- and right-endpoint $\gamma_m = \{ x : x = x_{E_k}^l, x_{E_k}^l, E_k \in T_h(\gamma_m) \}$. We note that $\gamma_m$ contains edge endpoints which geometrically occupy the same space as those of $\gamma_m$. In the set $\gamma_m$, each edge endpoint in the interior of $\gamma_m$ occurs twice, once as the left endpoint of an edge and once as the right endpoint of the neighboring edge, or vice versa (the superscript ‘LR’ stands for ‘Left and Right’). Let $I_m$ be the matrix representation of the interpolation operator $I_m : X_h(\gamma_m) \to Y_h(\gamma_m)$, in the following sense. Let $u_{\gamma_m}$ be the same vector as above, then $I_m u_{\gamma_m}$ will be the vector containing the values of the function $I_m u$ at the nodes of $\gamma_m$, that is the set of edge endpoints of $\gamma_m$. Let $h_k$ be the length of the edge $E_k$, then

$$ I_m = \begin{bmatrix}
\frac{2h_1+h_2}{h_1+h_2} & \frac{-h_1}{h_1+h_2} & 1 \\
1 & \frac{h_1}{h_1+h_2} & 1 \\
\frac{h_2}{h_1+h_2} & \frac{h_1}{h_2} & 1 \\
\frac{h_3}{h_2} & \frac{h_1}{h_2+h_3} & * \\
* & \frac{h_{n-2}}{h_{n-2}+h_{n-1}} & * \\
* & \frac{h_{n-1}}{h_{n-1}+h_n} & * \\
\frac{h_n}{h_{n-1}+h_n} & 1 & \frac{h_{n-1}}{h_{n-1}+h_n} \\
\frac{h_n}{h_{n-1}+h_n} & 1 & \frac{h_{n-1}}{h_{n-1}+h_n}
\end{bmatrix}. $$

We follow Definition 2.2, and introduce an intermediate matrix $K_m$ so that we have

$$ J_m = K_m I_m. $$

$K_m$ is a block matrix consisting of $n$ rectangular blocks, each corresponding to an edge $E_k \in T_h(\gamma_m)$ and being equal to the following $2 \times 3$ rectangular matrix

$$ \begin{bmatrix}
\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2} & 1 & \frac{1}{2}
\end{bmatrix}. $$

The columns of this rectangular matrix correspond to the left endpoint $x_{E_k}^l$, the midpoint $x_{E_k}^m$, and the right endpoint $x_{E_k}^r$ of the edge $E_k$, respectively, and the rows...
correspond to the endpoints \( x^E_k \) and \( x^F_k \) of the edge \( E_k \), respectively. Consequently, the columns of \( K_m \) correspond to the set \( \tau_m \), while the rows correspond to the set \( \tau^L_{mh} \). We remark that the extra work involved in our algorithm, compared to the one presented in [18], is associated with the application of \( K_m \).

Let \( \varphi_k \) and \( \varphi_l \) be the standard CR basis functions associated with the edge midpoint \( x^E_m \in \gamma^{CR} (E_k \in T_h (\gamma_m)) \) and \( x^F_m \in \delta^{CR} (E_l \in T_h (\delta_m)) \), respectively. Let \( \psi_o \) be the basis function of the \( M^h (\delta_m) \), associated with the edge midpoint \( x^o_m \) (\( E_o \in T_h (\delta_m) \)), defined as \( \psi_o = 1 \) in \( E_o \) and zero otherwise. Finally, let \( \xi_q \) be the basis function of \( P^1 (E_q) \), associated with one of the edge endpoints \( x^E_q \) (left) and \( x^F_q \) (right).

Define the master matrix as \( M_{\gamma_m} = \{ (J_m \varphi_k, \psi_o) \}_{L^2 (\delta_m)} \), and the corresponding slave matrix as \( S_{\delta_m} = \{ (\varphi_l, \psi_o) \}_{L^2 (\delta_m)} \), where \( x^E_m \in \gamma^{CR} \) and \( x^F_m \in \delta^{CR} \). Now, let \( u_{\gamma_m} \) be the vector defined as above, and let \( u_{\delta_m} \) be the corresponding vector containing the values of the function \( u \in X_h \) at the edge midpoints \( \delta^{CR} \), then

\[
S_{\delta_m} u_{\delta_m} = M_{\gamma_m} u_{\gamma_m} + N_{\delta_m} K_m I_m u_{\gamma_m}
\]

is the matrix (discrete) representation of the mortar condition, where the supporting master matrix \( N_{\gamma_m} = \{ (\xi_q, \psi_o) \}_{L^2 (\delta_m)} \) with \( x^E_m \in \gamma^{CR} \) and \( x^F_m \in \delta^{CR} \). We note that \( S_{\delta_m} \) is a diagonal matrix containing the lengths of the triangle edges \( E_o \in T_h (\delta_m) \) along the nonmortar side, as entries.

6. Numerical results. The numerical results are presented here. We consider our model problem on a unit square domain with the function \( f = 2 \pi^2 \sin (\pi x) \sin (\pi y) \) and homogeneous Dirichlet boundary condition. The exact solution \( u \) equals to the function \( \sin (\pi x) \sin (\pi y) \). For all our experiments, as a general rule, we decompose the domain into a \( d \times d = d^2 \) number of square subdomains (subregions), and then uniformly triangulate each subdomain. In order to get nonmatching grids across all interfaces, each pair of subdomains sharing an interface are triangulated using a fixed \( 2m^2 \) and \( 2n^2 \) number of right angle triangles, where \( m \) is different from \( n \). Note that the number \( d \) is inversely proportional to the subdomain size \( H \), whereas the numbers \( m \) and \( n \) are proportional to \( \frac{d}{K} \), the ratio between the subdomain size and the mesh size \( h \). All sides of a subdomain are chosen to be either mortar or nonmortar. For comparisons, we consider both the standard mortar condition of [15] and the new approximate mortar condition of the present paper, for the lowest order CR finite element for solving the boundary value problem. The two resulting algebraic systems, which differ from each other due to different mortar conditions, are then solved using the Conjugate Gradients (CG) method.

In our first experiment, we look at the accuracy of the computed solution \( u_h \) for the approximate mortar condition, and compare it with that for the standard mortar condition. The \( L^2 \)-norm and the \( H^1_h \)-seminorm of the error \( u - u_h \) for different mesh sizes are calculated and shown in Table 6.1. As we can see from the table that the errors in both mortar cases are very close, and they vary as \( h^2 \) in the \( L^2 \)-norm and as \( h \) in the \( H^1_h \)-seminorm, which is in accordance with the theory.

In our next experiment, we study the convergence behavior of their corresponding additive Schwarz methods, those are the one proposed for the approximate mortar condition in Section 4, and the one proposed for the standard mortar condition in [19]. The additive Schwarz methods are used as preconditioners for the CG method for their respective algebraic systems. The condition number estimates and the number of iterations required to reduce the discrete \( L^2 \)-norm of the residual by a factor of \( 10^{-6} \)
are shown in the Table 6.2. As seen from the table the condition number estimates as well as the iteration counts remain bounded for fixed $\frac{H}{h}$ ratio, which is as predicted by the theory. Again, the condition number estimates in both mortar cases are very close, however, the approximate mortar case seems to require slightly fewer iteration for the same tolerance.

We conclude by saying that both approaches, the standard mortar and the proposed approximate mortar, exhibit a very similar numerical behavior. The approximate mortar approach has, however, the advantage that the mortar condition uses only the nodal values on the interface, thereby making all its algorithms comparatively simpler and less intricate.

**Acknowledgements.** The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the paper. The work of the third author has been supported by the special funds for major state basic research projects (973) under 2005CB321701 and the National Sciences Foundation (FSF) of China (10731060).

**REFERENCES**


