A variant of Steffensen–King's type family with accelerated sixth-order convergence and high efficiency index: Dynamic study and approach

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Abstract

First, it is attempted to derive an optimal derivative-free Steffensen–King's type family without memory for computing a simple zero of a nonlinear function with efficiency index $4^{1/3} \approx 1.587$. Next, since our without memory family includes a parameter in which it is still possible to increase the convergence order without any new function evaluations. Therefore, we extract a new method with memory so that the convergence order rises to six without any new function evaluation and therefore reaches efficiency index $6^{1/3} \approx 1.817$. Consequently, derivative-free and high efficiency index would be the substantial contributions of this work as opposed to the classical Steffensen's and King's methods. Finally, we compare some of the convergence planes with different weight functions in order to show which are the best ones.

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1. Introduction

Construction and development of Multipoint methods without memory, for computing a simple or multiple roots of a given nonlinear function, is based on Kung and Traub's conjecture [12], proved in some special cases [22,25,26]. It is supposed that any multipoint method without memory by using $n$ function evaluations per iteration can reach optimal convergence order $2^n$. The most well known optimal method is Newton's method [2] but we are interested in two-point methods. There are some legendary optimal two-point methods without memory like King's [10], Ostrowski's [16] and Jarratt's [7,8,14] methods in which they consume three function evaluations per iteration with optimal convergence four. In spite being optimal, they require evaluation of the first derivative at one or two points and hence cannot be used for nonsmooth functions. Needless to say, King's and Ostrowski's type methods have been considerably used to create higher optimal order methods without memory [1,3–6,9,11,17,18,20,23]; on the other hand, based our best knowledge, there is no higher optimal order of Jarratt's type method.

In this work, King's family is modified to avoid using derivative evaluation and consequently can be applied to nonsmooth functions. To this end, its first step is replaced by an Steffensen's type method [19] and its second step is altered by considering the idea of weight function. It is attempted to preserve optimality, too. Until now, this modified family can be called a

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without memory method. In addition, we go on to rise the convergence order as high as possible from four to six according to introducing an accelerator parameter approximated by Newton’s interpolation without using any new function evaluations. Indeed, we extract a with memory family. As a result, efficiency index, defined by EI(p, n) = \( p^{1/n} \) where \( p \) and \( n \) are convergence order and function evaluations, increases from 1.587 to 1.817; which is much better than optimal three-, four-, five-point methods without memory with efficiency indexes \( 8^{1/4} \approx 1.681, 16^{1/5} \approx 1.741 \) and \( 32^{1/6} \approx 1.781 \), respectively. Finally, derivative-free Ostrowski’s type method with or without memory will be acquired too.

The rest of this paper is organized as follows: Section 2 concerns with optimal derivative-free extraction of King’s method and its convergence order analysis. Further development to acquire a method with memory is dealt with in Section 3. Some numerical aspects is illustrated in Section 4. Section 5 contains the convergence planes of the method with the optimal Steffenson–King’s type family applied to the quadratic polynomial \( p(z) = z^2 - 1 \). Section 6 includes conclusion remarks and suggestion for future research.

2. Construction of the optimal Steffensen–King’s type families

Let the scalar function \( f : D \subset \mathbb{R} \to \mathbb{R} \) and \( f(x) = 0 \neq f'(x) = c_1 \). In other words, \( x \) is a simple zero of \( f(x) = 0 \). In this section our primal goal is to modify the classic King’s method [10] (iteration indexes has been removed)

\[
\begin{align*}
  y &= x - \frac{f(x)}{f'(x)}, \\
  z &= y - \frac{f(x) + \beta (y)}{f'(x) + \beta f'(y)} \frac{f(y)}{f'(y)}, \quad \beta \in \mathbb{R}.
\end{align*}
\]

Also, the error equation of (2.1) is given by

\[
e_{n+1} = \left( \frac{(2\beta + 1)c_2^2 - c_1 c_2 c_3}{c_1} \right) e_n^4 + O(e_n^5),
\]

where \( c_k = f^{(k)}(x)/k! \), \( k = 1, 2, \ldots \). Note that \( \beta = 0 \) gives the celebrity Ostrowski’s method.

We suggest the following Steffensen–King’s type family for (2.1):

\[
\begin{align*}
  w &= x + \gamma f(x), \quad 0 \neq \gamma \in \mathbb{R}, \\
  y &= x - \frac{f(x)}{f'(w)}, \\
  z &= y - g(t) \frac{f(x) + \beta (y)}{f'(x) + \beta f'(y)} \frac{f(y)}{f'(y)}, \quad \beta \in \mathbb{R},
\end{align*}
\]

where \( f(x, y) = f(x) - f(y)/(x - y) \), and \( t = f(y)/f(x) \). Moreover, \( g(t) \) is a real-valued weight function to be determined afterwards.

To find the suitable weight function \( g \) in (2.2), providing optimal order four, we will use the method of undetermined coefficients and Taylor’s series about zero, since \( t \), tends to zero when \( x \) tends to \( x \). We have

\[
g(t) = g(0) + g'(0) t + \frac{g''(0)}{2} t^2 + \cdots.
\]

By using Taylor’s expansion of \( f(x) \) about \( x \) and taking into account that \( f(x) = 0 \), we obtain

\[
f(x) = c_1 e + c_2 e^2 + c_3 e^3 + c_4 e^4 + O(e^5),
\]

and

\[
f'(x) = c_1 + 2c_2 e + 3c_3 e^2 + 4c_4 e^3 + O(e^4),
\]

where \( e = x - x \).

We refrain ourselves of retyping the widely practiced approach in the literature, and put forward the self-explained Mathematica code used to provide some suitable conditions on \( g \) in such a way that the proposed family (2.2) gets optimal fourth-order.

\[
f[e] := \sum_{k=1}^{4} c_k \star e^k; \\
e w \star e = e + \gamma f[e] \quad (\star ew = w - x*); \\
f[x, y] := \frac{f[y] - f[x]}{y - x}; \\
e y = e - Series \{ f[e] / f'[e, ew], \{e, 0, 4\} \}; \quad (\star ey = y - x*) \\
t = f[ey] / f[e]; \\
g[t] = g[0] + g[1] \cdot t; \quad (\star g[0] = g(0), \quad g[1] = g'(0)*) \\
ez = ey - g[t] \star \frac{f[x] + \beta (y)}{f'[x] + \beta f'(y)} \frac{f(y)}{f'(y)}; \quad (\star ez = z - x*)
\]

In fact, the expression of the asymptotic error of \( ez = z - x \) can be presented as
\[ ez = a_1 e + a_2 e^2 + a_3 e^3 + a_4 e^4 + O(e^5). \]  
(2.5)

The family (2.2) will have the order of convergence equal to four if the coefficients \( a_1, a_2, \) and \( a_3 \) in (2.5) all vanish. First, for \( a_1 \) we have

\[ a_1 = \text{Coefficient}[ez,e]/\text{Simplify} \]
\[ \text{Out}[a_1] = 0; \]
\[ a_2 = \text{Coefficient}[ez,e^2]/\text{Simplify} \]
\[ \text{Out}[a_2] = - \frac{c_1}{c_2} \]

Comment: This condition vanishes the coefficient of \( e^2 \).

\[ a_3 = \text{Coefficient}[ez,e^3]/\text{Simplify} \]
\[ \text{Out}[a_3] = - \frac{c_2}{c_3} \]

Comment: These conditions vanishes the coefficient of \( e^3 \).

\[ a_4 = \text{Coefficient}[ez,e^4]/\text{Simplify} \]
\[ \text{Out}[a_4] = \frac{c_1 (\gamma c_1 + 1) (2 \beta (2 \beta - 1) \gamma c_1 + 1) - c_1 c_3}{c_1} \]

Comment: These conditions gives the coefficient of \( e^4 \).

Hence, according to the above analysis, the general error equation of (2.2) is given by

\[ e_{n+1} = \frac{c_1 (\gamma c_1 + 1) (2 \beta (2 \beta - 1) \gamma c_1 + 1) - c_1 c_3}{c_1} e_n^4 + O(e_n^5). \]  
(2.6)

Summarizing, we have

**Theorem 2.1.** Assume that \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) is a real function, with a simple root \( z \) in an open set \( D \), and \( x_0 \) is an initial value close to \( z \). Then, the Steffensen–King’s type family (2.2) has optimal fourth-order convergence provided that

\[ g(0) = 1, \quad g'(0) = -1. \]  
(2.7)

Some simple weight functions satisfying conditions (2.7) are

\[ g_1(t) = 1 - t, \quad g_2(t) = \frac{1}{1+t}, \quad g_3(t) = \cos(t) - \sin(t), \quad g_4(t) = e^{-t}. \]  
(2.8)

We consider these weight functions in without and with memory methods (2.2) and (3.3).

3. Development and construction with memory family

This section concerns with extraction an efficient with memory method from (2.2) since its error equation contains the parameter \( \gamma \) which can be approximated in such a way that increases the local order of convergence. So, we set \( \gamma = \gamma_k \) as the iteration proceeds by the formula \( \gamma_k = \frac{-1}{f'(z)} \) for \( k = 1, 2, \ldots \), where \( f'(z) \) is an approximation of \( f'(z) \). We have a method through the following forms of \( \gamma_k \)

\[ \gamma_k = - \frac{1}{f'(z)} = - \frac{1}{N_3(x_k)} \]  
(3.1)

where \( N_3(t) \) is the Newton’s interpolation polynomial of third degree, set through four available approximations \( x_k, y_{k-1}, x_{k-1}, w_{k-1} \), i.e.,

\[ N_3(t) = f(x_k) + f[x_k, y_{k-1}](t - x_k) + f[x_k, y_{k-1}, w_{k-1}](t - x_k)(t - y_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - y_{k-1})(t - w_{k-1}), \]

and

\[ N'_3(x_k) = f[x_k, y_{k-1}] + f[x_k, y_{k-1}, w_{k-1}](x_k - y_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](x_k - y_{k-1})(x_k - w_{k-1}). \]  
(3.2)

Hereby, the with memory extension of (2.2) can be presented as follows

\[
\begin{align*}
    g(t), \quad &\beta, \quad x_0, \quad \gamma_0 \quad \text{are chosen suitably,} \\
    w_k &= x_k + \gamma_0 f(x_k), \quad k = 0, 1, 2, \ldots, \\
    y_k &= x_k - \frac{f(x_k)}{f'_k(x_k)}, \\
    t_k &= \frac{f(x_k)}{f'_k(x_k)}, \\
    y_{k+1} &= y_k - g(t_k) \frac{f(x_k) + \beta f(y_k)}{f'_k(x_k) + \beta f'_k(y_k)} f'_k(x_k), \\
    \gamma_{k+1} &= - \frac{1}{N'_3(x_{k+1})}.
\end{align*}
\]  
(3.3)
If \( \lim h(x_k) = c \) as \( k \to \infty \), we write \( h \to c \). If \( \frac{f}{g} \to c \), where \( c \) is a nonzero constant, we shall write \( f \sim g \) \[21\].

We need:

**Lemma 3.1** [13]. Assuming (3.1) and (3.2), we have

\[
1 + \gamma_k f'(x) \sim e_{k-1} e_{k-1,y} e_{k-1,w}.
\]

where \( \gamma_k = \frac{1}{N_{\gamma_k}(x)} \), \( e_k = x_k - x \), \( e_{k,y} = y_k - x \), and \( e_{k,w} = w_k - x \).

In order to obtain the order of convergence of the Steffensen–King’s type family with memory (3.3), we establish the following theorem:

**Theorem 3.1.** If an initial approximation \( x_0 \) is sufficiently close to the zero \( x \) of \( f(x) \) and the parameter \( \gamma_k \) in the iterative scheme (3.3) is recursively calculated by the forms given in (3.1), then the order of convergence is 6.

**Proof.** Let \( \{x_k\} \) be a sequence of approximations generated by the iterative method with memory (3.3). If this sequence converges to the zero \( x \) of \( f \) with the order \( r \) then we write

\[
e_{k-1} \sim e^r_k, \quad e_k = x_k - x.
\]

Thus

\[
e_{k-1} \sim (e^r_{k-1})' = e^{r^2}_{k-1}.
\]

Moreover, assume that the iterative sequence \( w_k \) and \( y_k \) have the order \( s \) and \( p \), respectively. Then, (3.5) gives

\[
e_{k,w} \sim e^p_k \sim (e^r_{k-1})^p = e^{r^p}_{k-1}.
\]

and

\[
e_{k,y} \sim e^p_k \sim (e^r_{k-1})^p = e^{r^p}_{k-1}.
\]

Since

\[
e_{k,w} \sim (1 + \gamma_k f'(x)) e_k,
\]

\[
e_{k,y} \sim (1 + \gamma_k f'(x)) e^2_k,
\]

and

\[
e_{k-1} \sim (1 + \gamma_k f'(x))^2 e^4_k.
\]

using Lemma 3.1 and (3.10), induce

\[
e_{k,w} \sim (1 + \gamma_k f'(x)) e_k \sim (e_{k-1} e_{k-1,y} e_{k-1,w}) e_k = e^{r+s+p+1}_{k-1},
\]

\[
e_{k,y} \sim (1 + \gamma_k f'(x)) e^2_k \sim (e_{k-1} e_{k-1,y} e_{k-1,w}) e^2_k = e^{2r+s+p+1}_{k-1},
\]

and

\[
e_{k-1} \sim (1 + \gamma_k f'(x))^2 e^4_k \sim (e_{k-1} e_{k-1,y} e_{k-1,w})^2 e^4_k = e^{2r+2s+2p+2}_{k-1}.
\]

Matching the powers of \( e_{k-1} \) on the right hand sides of (3.7)–(3.12), (3.8)–(3.13), (and), (3.7)–(3.14), one can obtain

\[
\begin{align*}
rs - r - p - s &= 1, \\
rp - 2r - p - s &= 1, \\
2r - 4r - 2p - 2s &= 2.
\end{align*}
\]

The non-trivial solution of this system is \( s = 2, p = 3 \) and \( r = 6 \). This completes the proof. \( \square \)

4. Numerical examples and conclusion

In this section we examine applicability of the proposed methods (2.2) and (3.3) with different weight functions (2.8) in this work. For this purpose, two examples are tested in which one of them is nonsmooth. We use these notations. The errors \( |x_n - x| \) denote approximations to the sought zeros, and \( a(-b) \) stands for \( \alpha \times 10^{-b} \). Moreover, COC indicates computational order of convergence which has been used by researchers in order to find the theoretical order by means of experiments, and is computed by [13,24]
The software Mathematica 9, with 100 arbitrary precision arithmetic has been used in our computations. The results alongside the test functions are given in Tables 1 and 2. As it can be observed, Steffensen–King’s type methods without and with memory (2.2) and (3.3) produce desirable results for different weight functions $g_i(t), i = 1, 2, 3, 4$ (see (2.8)).

5. Convergence planes of the method (2.2) applied to the quadratic polynomial $p(z) = z^2 - 1$

In this section, we are going to compare, by means of the tool developed by the second author in [15], the convergence planes of the method (2.2) when it is applied to the polynomial $p(z) = z^2 - 1$ for 3 different weight functions $h_1(t) = 1 - t$, $h_2(t) = \frac{1}{1+t}$ and $h_3(t) = \cos(t) - \sin(t)$. In concrete, we are going to choose three different values of the starting point $x_0$ and then we are going compare how the convergence planes change with different weight functions.

The convergence plane is obtained by associating each point of the plane with a value of the parameter $\gamma$ and a value of the parameter $\beta$. That is, the tool is based on taking the vertical axis as the value of $\gamma$ and the horizontal axis as the value of the parameter $\beta$, so every point in the plane represents a value for both parameters. Once the convergence plane has been computed it is easy to distinguish the pairs $(\beta, \gamma)$ for which the iteration of the starting point is convergent to any of the roots. So this tool provides a global vision about where points converges and shows what are the best choices of the parameters $\beta$ and $\gamma$ to ensure the convergence of the greatest set of starting points.

A point is painted, after a maximum of 200 iterations and with a tolerance of $10^{-6}$, in cyan if the iteration of the method starting in $x_0$ chosen converges to the root $-1$, in magenta if it converges to the root 1 and in yellow if the iteration diverges to $\infty$. Moreover, it appears in red the convergence, to any strange fixed points, in light green the convergence to 2-cycles, in orange to 3-cycles, in dark blue to 4-cycles, in dark magenta to 5-cycles, dark yellow to 6-cycles, and in dark red the convergence to 7-cycles. The regions in black correspond to zones of convergence to other cycles or represent chaotic behaviors. As a consequence, every point of the plane which is neither cyan nor magenta is not a good choice of $\beta$ and $\gamma$ for that starting point $x_0$ in terms of numerical behavior.

In Fig. 1 we see the convergence planes associated to the method (2.2) with weight function $h_1(t) = 1 - t$ applied to the quadratic polynomial $p(z) = z^2 - 1$, in Fig. 2 with weight function $h_2(t) = \frac{1}{1+t}$ and in Fig. 3 with weight function $h_3(t) = \cos(t) - \sin(t)$.

In our focus our attention in the left and right hand of the figures (with starting point $x_0 = -2$) we observe that in Fig. 1 it appears a large zone of values of both parameters for which the iterations of $x_0 = -2$ do not converge (zones of colors different to cyan and magenta). Moreover, if we compare the cyan and magenta zones in Figs. 1–3, it seems that the greater ones appear in Fig. 3, then in Fig. 1 and the lower ones in Fig. 2, so in terms of convergence weight function $h_3$ is better than the other ones. In both cases, for greater values of $\gamma$ (in norm) the convergence to the root is not guaranteed. The differences between Figs. 3 and 1 are lower than if we compare them with Fig. 2.

Table 1

<table>
<thead>
<tr>
<th>$f_1(x) = \sin(x) e^{\frac{1}{x} \log(x \sin(x)) + 1} \log(x \sin(x) + 1)$</th>
<th>$x = 0$.</th>
<th>$x_0 = 0.35$.</th>
<th>$\gamma_0 = 0.01$.</th>
<th>$\beta = 0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steffensen–King’s methods</td>
<td>$</td>
<td>x_1 - x</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
<td>Without memory method (2.2) using $g_1(t)$</td>
<td>0.7124(-2)</td>
<td>0.7806(-9)</td>
<td>0.1056(-36)</td>
<td>4.0021</td>
</tr>
<tr>
<td>With memory method (3.3) using $g_1(t)$</td>
<td>0.9575(-2)</td>
<td>0.1702(-12)</td>
<td>0.1580(-77)</td>
<td>6.0471</td>
</tr>
<tr>
<td>Without memory method (2.2) using $g_2(t)$</td>
<td>0.6048(-2)</td>
<td>0.8614(-9)</td>
<td>0.3751(-36)</td>
<td>3.9049</td>
</tr>
<tr>
<td>With memory method (3.3) using $g_2(t)$</td>
<td>0.8546(-2)</td>
<td>0.1974(-12)</td>
<td>0.3958(-77)</td>
<td>6.0806</td>
</tr>
<tr>
<td>Without memory method (2.2) using $g_3(t)$</td>
<td>0.7078(-2)</td>
<td>0.2738(-8)</td>
<td>0.4317(-34)</td>
<td>3.9987</td>
</tr>
<tr>
<td>With memory method (3.3) using $g_3(t)$</td>
<td>0.1013(-1)</td>
<td>0.1403(-12)</td>
<td>0.4916(-78)</td>
<td>6.0255</td>
</tr>
<tr>
<td>Without memory method (2.2) using $g_4(t)$</td>
<td>0.6539(-2)</td>
<td>0.3139(-9)</td>
<td>0.1925(-38)</td>
<td>3.9899</td>
</tr>
<tr>
<td>With memory method (3.3) using $g_4(t)$</td>
<td>0.9016(-2)</td>
<td>0.1897(-12)</td>
<td>0.3067(-77)</td>
<td>6.0661</td>
</tr>
</tbody>
</table>

Table 2

| $f_2(x) = |x^2 - 9|$, | $x = 3$. | $x_0 = 2.6$. | $\gamma_0 = 0.01$. | $\beta = 0$. |
|---|---|---|---|---|
| Steffensen–King’s methods | $|x_1 - x|$ | $|x_2 - x|$ | $|x_3 - x|$ | COC |
| Without memory method (2.2) using $g_1(t)$ | 0.7854(-1) | 0.1567(-6) | 0.2618(-29) | 3.9921 |
| With memory method (3.3) using $g_1(t)$ | 0.7776(-1) | 0.2216(-6) | 0.3048(-49) | 6.3903 |
| Without memory method (2.2) using $g_2(t)$ | 0.7890(-1) | 0.2527(-7) | 0.2001(-33) | 4.0155 |
| With memory method (3.3) using $g_2(t)$ | 0.7811(-1) | 0.2036(-7) | 0.1835(-49) | 6.3807 |
| Without memory method (2.2) using $g_3(t)$ | 0.7835(-1) | 0.2211(-6) | 0.1407(-28) | 3.9909 |
| With memory method (3.3) using $g_3(t)$ | 0.7758(-1) | 0.2299(-7) | 0.3798(-49) | 6.3948 |
| Without memory method (2.2) using $g_4(t)$ | 0.7873(-1) | 0.9112(-7) | 0.1664(-30) | 3.9949 |
| With memory method (3.3) using $g_4(t)$ | 0.7795(-1) | 0.2129(-7) | 0.2400(-49) | 6.3857 |
Now, considering the center part of the figures (with starting point \(x_0 = 0\)) we observe that in Fig. 1 it appears a large zone of values of both parameters for which the iterations of \(x_0 = 0\) diverges to \(\infty\) or to different cycles, this zone is almost null in Figs. 2 and 3, so it seems again that in terms of convergence weight functions \(h_2\) and \(h_3\) are better than the other ones. Now comparing Figs. 2 and 3 we observe that the non convergent zone is lower in 3, so again in terms of convergence weight function \(h_3\) is better than the other ones.
6. Concluding remarks and suggestions

We have constructed an Steffensen–King’s type family without and with memory. Our proposed methods do not need any derivative and therefore are applicable to nonsmooth functions too. Anther merit of the proposed methods is that their without memory versions are optimal in the sense of Kung and Traub’s conjecture. In addition, it contains an accelerator parameter which rises convergence order form four to six without any new functional evaluations. In other words, the efficiency index of the with memory family is $6^{1/3} \approx 1.817$ which is much better even than six-point methods without memory with efficiency index $6^{1/2} \approx 1.811$. Also, methods with memory with less points are more stable than methods without memory with high points. Consequently, applying methods with memory with two points is suggested that methods without memory with three, four, five or six points. We finalize this work by suggesting some points for future research: first developing the proposed methods for computing multiple roots, second exploring its dynamic or basins of attractions, third applying it to systems of nonlinear equations, and finally extending it to two or more accelerators. We have seen also the convergence planes associated to different weight functions when we applied method (2.2) to the quadratic polynomial $p(z) = z^2 - 1$.

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