A Gram-Schmidt Based Lattice-Reduction Aided MMSE Detection in MIMO Systems

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Abstract—This paper proposes an improved lattice-reduction aided (LRA) MMSE detection based on the Gram-Schmidt (GS) procedure. The proposed detection reduces the column vectors of the MIMO channel matrix using the LLL algorithm and the GS procedure to create mutually purely orthogonal column vectors of the reduced channel matrix. Then the decision boundary becomes the same as that for the ML detection. Compared to the conventional LRA MMSE detector, the proposed detector achieves much closer BER performances to those for the ML detector in both the 4×4 MIMO and the 8×8 MIMO systems.

Index Terms—Gram-Schmidt orthogonalization, lattice-reduction, LRA, GS, MMSE, signal estimation.

I. INTRODUCTION

RECENTLY, the lattice-reduction (LR) aided (LRA) detection has been receiving attractive attention since it achieves high channel capacity in the multiple-input multiple-output (MIMO) systems. The LR transforms the column vectors of the MIMO channel matrix close to mutually orthogonal, followed by the estimation of the transmitted signals [2]–[8]. The most popular LR algorithm is the well-known LLL algorithm introduced by Lenstra, Lenstra, and Lovász [1]. Using this algorithm, the LRA detector achieves highly reliable signal estimation and hence good bit error ratios (BERs). In particular, the LRA detector in a 4×4 MIMO system achieves BER relatively close to that with the maximum likelihood (ML) detector [2]–[6]. In contrast, the LRA detector in an 8×8 MIMO system does not achieve so good BER performance as the LRA detector in a 4×4 MIMO system does, compared to BERs for the ML detector [4], [6]. This is because the signal transmitted from each antenna is interfered by more signals transmitted from the other antennas in the 8×8 MIMO system than in the 4×4 MIMO system. This fact implies that the detection scheme used for the 4×4 MIMO system is not directly applicable to the 8×8 MIMO system and that some adequate detection schemes should be needed for the 8×8 MIMO system.

In this paper, we propose an improved LRA minimum mean square error (MMSE) detection by combing the LLL algorithm and the Gram-Schmidt (GS) orthogonalization procedure, aiming at achieving BER close to the BER with the ML detection for the 8×8 MIMO system. The LLL algorithm reduces the p-th column vector of the channel matrix by the q-th (q<p) column vectors to create the reduced channel matrix and the transform matrix. We call this reduction the forward LR (F-LR). In addition, we reduce the p-th column vector by the q-th (q>p) column vectors to create another reduced channel matrix and another transform matrix. We call this reduction the backward LR (B-LR) [6]. Furthermore, the proposed detector reduces the LLL-reduced column vectors using the GS procedure. Then the GS-reduced column vectors become mutually purely orthogonal and almost of equal length. Hence the decision boundary becomes the same as that for the ML detection. As a result, the proposed detector extremely improves BER performance, which is very close to that with the ML detection.

The remainder of this paper is organized as follows. Section II presents the system model and the conventional LRA detection. Section III presents the basic concept of the GS procedure based lattice-reduction. In Section IV, we propose a GS procedure based LRA MMSE detection which is applicable to the 8×8 MIMO systems. Section V gives the computer simulation results and makes discussion. Finally, we summarize and conclude the paper in Section VI.

II. SYSTEM MODEL AND CONVENTIONAL LRA DETECTION

A. System Model

A MIMO system with \( M \) transmit and \( N (\geq M) \) receive antennas investigated here is shown in Fig. 1. In the system, the signals are transmitted over a rich scattering flat fading channel. We assume that the receiver has perfect knowledge of the channel state information (CSI). Let \( \mathbf{H}=[h_{11}, \ldots, h_{M1}] \) be the basis channel matrix, of which the entry \( h_{nm} : n \in [1, N], m \in [1, M] \), at the n-th row and the m-th column is the complex channel gain between the m-th transmit and the n-th receive antennas. The channel gains are assumed to be mutually uncorrelated and of the complex Gaussian process with zero mean and unity variance. Let \( z_n : n \in [1, N], \) be the additive white Gaussian noise at the n-th receiver, where \( z_n s: n \in [1, N], \) are mutually uncorrelated. Each \( z_n \) is assumed to be of the complex Gaussian process with zero mean and variance of \( N_0 \), where \( N_0 \) is the one-sided noise power spectral density. Let \( \mathbf{s}=[s_1, \ldots, s_M]^T, \) \( \mathbf{y}=[y_1, \ldots, y_N]^T, \) and \( \mathbf{z}=[z_1, \ldots, z_N]^T \) be the transmit signal, receive signal, and the additive noise vectors, respectively. Then we have

\[
\mathbf{y} = \mathbf{Hs} + \mathbf{z} = h_{1s_1} + h_{2s_2} + \cdots + h_{Ms_M} + \mathbf{z} \tag{1}
\]

Fig. 1- MIMO system model.
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\( \text{TABLE I} \)

<table>
<thead>
<tr>
<th>\text{THE LLL ALGORITHM}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1) ): Begin Input ( \mathbf{H}, \mathbf{T} = \mathbf{I}_d = [t_1, \ldots, t_d] ), set ( \delta, \mathbf{h}_1 = \mathbf{h}_1 )</td>
</tr>
<tr>
<td>( (2) ): for ( p = 2 ) to ( M )</td>
</tr>
<tr>
<td>( (3) ): for ( q = p-1 ) down to ( 1 )</td>
</tr>
<tr>
<td>( (4) ): ( \mu_{p,q} = | \mathbf{h}_q \mathbf{h}_p | \mathbf{h}_p )</td>
</tr>
<tr>
<td>( \mathbf{h}_p = \mathbf{h}<em>p - \mu</em>{p,q} \mathbf{h}_q )</td>
</tr>
<tr>
<td>( (5) ): end</td>
</tr>
<tr>
<td>( (6) ): end</td>
</tr>
<tr>
<td>( (7) ): Let ( \mathbf{h}_p = \mathbf{h}_p )</td>
</tr>
<tr>
<td>( (8) ): for ( q = p-1 ) down to ( 1 )</td>
</tr>
<tr>
<td>( \mu_{p,q} = | \mathbf{h}_q \mathbf{h}_p | \mathbf{h}_p )</td>
</tr>
<tr>
<td>( \mathbf{h}_p = \mathbf{h}<em>p - \mu</em>{p,q} \mathbf{h}_q )</td>
</tr>
<tr>
<td>( (9) ): end</td>
</tr>
<tr>
<td>( (10) ): end</td>
</tr>
<tr>
<td>( (11) ): end</td>
</tr>
<tr>
<td>( (12) ): If ( \delta | \mathbf{h}<em>{p-1} |^2 \leq | \mathbf{h}<em>p + \mu</em>{p,p-1} \mathbf{h}</em>{p-1} |^2 ), let ( p = p + 1 )</td>
</tr>
<tr>
<td>( (13) ): Else, swap the ( (p-1) )-th and ( p )-th columns. Let ( p = \max { p-1, 2 } ), and ( \mathbf{h}_1 = \mathbf{h}_2 ) if ( p = 2 )</td>
</tr>
<tr>
<td>( (14) ): end</td>
</tr>
<tr>
<td>( (15) ): End</td>
</tr>
</tbody>
</table>

For the MMSE estimation, following Hassibi [9], define the extended receive signal vector \( \mathbf{v} \), the extended channel matrix \( \mathbf{H} \), and the extended additive noise vector \( \mathbf{z} \), respectively, as

\[
\mathbf{v} = \mathbf{Hs} + \mathbf{z} = (\mathbf{H})(T^{-1}s) + \mathbf{z} = \mathbf{H'}v + \mathbf{z}
\]

where \( \mathbf{H'} \) is the \( \mathbf{H} \) matrix with \( \mathbf{I}_d \) as the identity matrix, and \( \mathbf{H} \) is a \( d \times d \) matrix with all zero entries. Then input \( \mathbf{H} \), instead of \( \mathbf{H} \), into step (1) of Table I. The column vectors of \( \mathbf{H} \) are mutually purely orthogonal. Note that the algorithm in Table I is computationally-simple since it weakly reduces the column vectors of \( \mathbf{H'} \).

III. BASIC CONCEPT OF GRAM-SCHMIDT PROCEDURE BASED MMSE DETECTION

In this section, the basic concept of the proposed GS procedure based LRA MMSE detection is explained shortly. The LLA algorithm transforms the extended basis channel matrix \( \mathbf{H} \) to the reduced channel matrix \( \mathbf{H} \), of which the column vectors are nearly orthogonal to one another. The algorithm also makes the reduced column vectors almost of equal length. If the reduced column vectors were purely orthogonal to one another, the decision boundary for the receive signal should be the same as that for the ML detection. Unfortunately, the LLA algorithm does not make the column vectors of the reduced channel matrix \( \mathbf{H} \) mutually purely orthogonal. Hence we will make them mutually purely orthogonal using the GS procedure, as described below.

Table II shows a GS orthogonalization algorithm for the MMSE estimation. The algorithm weakly reduces the column vectors of the LLA-reduced channel matrix \( \mathbf{H} \) to create the GS-reduced channel matrix \( \mathbf{H} \) and the transform matrix \( \mathbf{T} \). The column vectors of \( \mathbf{H} \) are mutually purely orthogonal. Note that the algorithm in Table II is computationally-simple since it weakly reduces the column vectors of \( \mathbf{H} \).

IV. BASIC CONCEPT OF GRAM-SCHMIDT PROCEDURE BASED MMSE DETECTION

In this section, the basic concept of the proposed GS procedure based LRA MMSE detection is explained shortly. The LLA algorithm transforms the extended basis channel matrix \( \mathbf{H} \) to the reduced channel matrix \( \mathbf{H} \), of which the column vectors are nearly orthogonal to one another. The algorithm also makes the reduced column vectors almost of equal length. If the reduced column vectors were purely orthogonal to one another, the decision boundary for the receive signal should be the same as that for the ML detection. Unfortunately, the LLA algorithm does not make the column vectors of the reduced channel matrix \( \mathbf{H} \) mutually purely orthogonal. Hence we will make them mutually purely orthogonal using the GS procedure, as described below.

Table II shows a GS orthogonalization algorithm for the MMSE estimation. The algorithm weakly reduces the column vectors of the LLA-reduced channel matrix \( \mathbf{H} \) to create the GS-reduced channel matrix \( \mathbf{H} \) and the transform matrix \( \mathbf{T} \). The column vectors of \( \mathbf{H} \) are mutually purely orthogonal. Note that the algorithm in Table II is computationally-simple since it weakly reduces the column vectors of \( \mathbf{H} \).

\[
\mathbf{H} = \mathbf{H}' = (\mathbf{D}^\top \mathbf{D} + \rho \mathbf{I}_M)^{-1} \mathbf{H}' \mathbf{y}
\]

where \( \mathbf{H}' \) denotes the pseudo inverse of \( \mathbf{H} \). Then, \( \mathbf{s} \) is transformed to \( \mathbf{v} = \mathbf{T}^{-1} \mathbf{s} \) in the \( \mathbf{v} \)-domain. In the case that the entries of \( \mathbf{s} \) are of the commonly used quadrature amplitude modulation (QAM) mapping, proper shifting and scaling of \( \mathbf{s} \) is necessary before deriving \( \mathbf{v} \). The detailed explanation on the shifting and scaling operation is given in [7]. We express this operation as \( \mathbf{s} := \mathbf{D}^{-1} \mathbf{s} \). After that, the entries of \( \mathbf{v} \) are rounded as \( \mathbf{v} = \mathbf{v} \). Next, \( \mathbf{v} \) is transformed to \( \mathbf{s} = \mathbf{T} \mathbf{v} \) in the \( \mathbf{s} \)-domain. Then, \( \mathbf{s} \) is shifted back and scaled back in the above case. We express this operation as \( \mathbf{s} := \mathbf{C}^{-1} \mathbf{s} \).
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Now we have obtained the correct point $u_m^{(j,k)}$ in (9) and the soft estimate $\tilde{u}_m^{(j,k)}$ in (11). Since $u_m^{(j,k)}$ is not an integer, $\tilde{u}_m^{(j,k)}$ cannot be decided by quantization like the conventional LRA detector. In order to decide of $\tilde{u}_m^{(j,k)}$, we shift both $u_m^{(j,k)}$ and $\tilde{u}_m^{(j,k)}$ by $\Delta u_m^{(j,k)}(=-u_m^{(j,k)})$ such that $u_m^{(j,k)}$ should be shifted to the origin. Then the shifted $u_m^{(j,k)}$ and $\tilde{u}_m^{(j,k)}$ are expressed, respectively, as
\[
u_m^{(j,k)} = u_m^{(j,k)} + \Delta u_m^{(j,k)},
\]
\[
\tilde{\nu}_m^{(j,k)} = \tilde{u}_m^{(j,k)} + \Delta u_m^{(j,k)} = \tilde{u}_m^{(j,k)} - u_m^{(j,k)}.
\]
Fig. 2 (a) illustrates the shifting of $u_m^{(j,k)}$ and of $\tilde{u}_m^{(j,k)}$ by $\Delta u_m^{(j,k)} (= -u_m^{(j,k)})$. In the figure, the superscript $(j,k)$ omitted.

Since $u_m^{(j,k)}$ is an integer (zero), $\tilde{u}_m^{(j,k)}$ can be rounded as
\[
\tilde{\nu}_m^{(j,k)} = \lfloor \tilde{u}_m^{(j,k)} \rfloor = \lfloor \tilde{u}_m^{(j,k)} - u_m^{(j,k)} \rfloor.
\]

After that, shift back $\tilde{\nu}_m^{(j,k)}$ by $-\Delta u_m^{(j,k)} (= u_m^{(j,k)})$ to create the estimate $\nu_m^{(j,k)}$ as
\[
\tilde{\nu}_m^{(j,k)} - \Delta u_m^{(j,k)} = \lfloor \tilde{u}_m^{(j,k)} - u_m^{(j,k)} \rfloor + u_m^{(j,k)}.
\]

Using (9), (15) is expressed in the vector form as
\[
\nu_m^{(j,k)} = \tilde{\nu}_m^{(j,k)} - \Delta u_m^{(j,k)} = \lfloor \tilde{u}_m^{(j,k)} - u_m^{(j,k)} \rfloor + u_m^{(j,k)}.
\]

Fig. 2 (b) illustrates an example of rounding of $\tilde{u}_m^{(j,k)}$ to create $\tilde{u}_m^{(j,k)}$. It also illustrates the shifting back of $\tilde{u}_m^{(j,k)}$ by $-\Delta u_m^{(j,k)} (= u_m^{(j,k)})$ to create the decided estimate $\nu_m^{(j,k)}$.

We here pre-estimate the transmitted signal $s$ in (16) using the conventional LRA MMSE detection. First, derive the soft estimate $\hat{s}$ in (5). Then let $\hat{s} = \mathbb{S}[\hat{s}]$. Next, transform $\hat{s}$ to $\tilde{\nu} = T_{(j,k)}^{-1}\hat{s}$. Then, round the entries of $\tilde{\nu}$ as $\nu = \lceil \tilde{\nu} \rceil$. Finally, transform $\nu$ to $\hat{s}$ as
\[
\hat{s} = T_{(j,k)} \nu.
\]

Substitute $\hat{s}$ into (16) to revise $\hat{u}$.

Note that the cross-correlation of $T_{(j,k)}$ and $T_{(j',k')}$ of (5) and (6) are weak as shown in (6). Hence it is unlikely that all $S_{(j,k)}$ : $j \in [1,4]$, are in error at the same time. Similarly, the cross-correlation of $\tilde{T}_{(j,k)}$ and $\tilde{T}_{(j',k')}$ is weak, where $j \neq j'$ and/or $k \neq k'$. Hence it is unlikely that all $S_{(j,k)}$ are in error at the same time. As a result, better BER should be expected by selecting the most reliable estimate $\hat{s}$. $\hat{s}$.

D. List of $\hat{u}$ and Estimation of $s$

We here derive the estimate of the transmitted signal $s$ using the proposed GS-based LRA MMSE detection. Replacing $s$ in (16) by $\hat{s}$ in (17), we express the revised $\hat{u}$ as $\hat{u} = \hat{u}^{(p,j,k)}$, which is first listed. Since $\hat{u}^{(p,j,k)} = \tilde{u}^{(p,j,k)}$ in the high $E_b/N_0$ region, we further create $\hat{u}^{(p,j,k)}$ : $p \in [1,M]$, by replacing the $p$-th entry of $\hat{u}^{(p,j,k)}$ by $\hat{u}^{(p,j,k)}$ in (11). And add them to the list. By adding $\hat{u}^{(p,j,k)} : p \in [1,M]$, to the list, more reliable estimate of the transmitted signal $s$ is expected. Calculating $\hat{s}(p,j,k) = \tilde{T}(j,k) \hat{u}(p,j,k)$ : $p \in [0,M]$, then letting $\hat{s}(p,j,k) = s^{-1}(s)(p,j,k)$ and $\hat{s}(p,j,k) = \mathbb{C}[\hat{s}(p,j,k)]$, select the most reliable signal among all $\hat{s}(p,j,k) : p \in [0,M]$, $j \in [1,4]$, $k \in [1,4]$.

The above procedure is summarized in Table III, where the notations $S_{(j,k)}$ and $s^{-1}:[S]$ for $\hat{s}(p,j,k)$, $\tilde{u}^{(p,j,k)}$, $\hat{s}(i)$ and $\hat{s}$ are omitted.

### Table III: The Proposed Detection Algorithm

1. (Begin) Input $\gamma$, $\hat{u}(j,k) = [u(j,k),...,u(j,m)]$, $\tilde{T}(j,k)$, $\tilde{T}(j,k)-1$, $s(\gamma)$, H.
2. for $j = 1$ to 4
3. for $k = 1$ to 4
4. Let $\hat{s}(i) = \mathbb{S}[\hat{s}(i)]$
5. for $\delta = 0$ to 1 ($\delta$ : the number of iterations)
6. $\hat{s}(p,j,k) = \hat{u}(p,j,k) - \hat{s}(p,j,k)-1 - s(\gamma)$
7. $s(p,j,k) = \mathbb{C}[\hat{s}(p,j,k)]$
8. for $p = 1$ to $M$
9. $\hat{s}(p,j,k) = \tilde{T}(j,k) \hat{u}(p,j,k)$
10. $s(p,j,k) = \mathbb{C}[\hat{s}(p,j,k)]$
11. if $\hat{s}(p,j,k) \neq \mathbb{S}[\hat{s}(p,j,k)]$, go to (13).
12. Else, $\hat{s}(p,j,k) = \mathbb{S}[\hat{s}(p,j,k)]$
13. (end (p-loop)) $p = p+1$
14. Let $\hat{s}(i) = \mathbb{S}[\hat{s}(i)]$
15. Let $\hat{s}(i) = \mathbb{S}[\hat{s}(i)]$
16. $i = i+1$, go to (17).
17. if $\hat{s}(i) \neq \mathbb{S}[\hat{s}(i)]$ for any one of $i'$ : $i \in [0, i-1]$, then let $\tilde{s}(i,k) = \mathbb{S}[\hat{s}(i,k)]$, and go to (20).
18. Else, let $\tilde{s}(i,k) = \mathbb{S}[\hat{s}(i,k)]$
19. (end (i-loop)) $i = i+1$
20. if $k = 1$, go to (23).
21. if $\tilde{s}(i,k) \neq \mathbb{S}[\hat{s}(i,k)]$, go to (23).
22. Else, $\tilde{s}(i,k) = \mathbb{S}[\hat{s}(i,k)]$
23. (end (k-loop)) $k = k+1$
24. if $k = 1$, go to (27).
25. if $\tilde{s}(i,k) \neq \mathbb{S}[\hat{s}(i,k)]$, go to (27).
26. Else, $\tilde{s}(i,k) \neq \mathbb{S}[\hat{s}(i,k)]$
27. (end (j-loop)) $j = j+1$
28. Let $\hat{s} = \mathbb{S}[\hat{s}(j = 2) or 4, k = 4]$ and let $\hat{s} = \mathbb{S}[\hat{s}]$ (GS-estimate)
29. (End)

In Table III, we obtain the estimate $\hat{s}$ at step (28). We here call the $\hat{s}$ the GS-estimate. We also call this detection procedure the Gram-Schmidt based combined forward and backward LRA (GS-F&B-LRA) MMSE list detection. In order to achieve more reliable GS-estimate $\hat{s}$, we replace $\hat{s}(i)$ at step (6) by the updated estimate $\hat{s}(p = M, j, k)$ at step (15) for each $j$ and $k$ iteratively, where $j$ is the iteration number and $I$ is the number of iterations.

### V. SIMULATION RESULTS AND DISCUSSIONS

Computer simulations were carried out for QPSK, 16QAM and 64QAM in the 4x4 and 8x8 MIMO systems to estimate the transmitted signals using the proposed GS-F&B-LRA MMSE detection without forward error correction. Each channel is assumed to be non-frequency selective slow-varying fading channel. The receiver is assumed to have perfect
knowledge of the CSI. In the simulations, channel gains are generated using independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance of 1/2 for each dimension. Additive noise at each receive antenna is generated using i.i.d. Gaussian random variables with zero mean and variance of $N_0/2$ for each dimension.

Before calculating BERs for the proposed detection, we first determine the suitable values of $\delta$ in Table I and the suitable number of iterations $I$ in Table III. After that, we analyze the BER performances and the computational complexity.

### A. Suitable Values of Factor $\delta$ and of Number of Iterations $I$

We here determine the suitable values of $\delta$ and the suitable number of $I$ for both the conventional and the proposed detections in the 4×4 and the 8×8 MIMO systems. We look for suitable value of $\delta$ that should achieve BER of $10^{-4} - 10^{-5}$ at the lowest $E_b/N_0$.

Figs. 3 (a), (b) and (c) show the $\delta$ vs. BER characteristics for various numbers of $I$, and the $\delta$ vs. the number of swapping times characteristics for QPSK, 16QAM and 64QAM at $E_b/N_0$ of 16dB, 21dB and 26dB over the 4×4 MIMO channel, respectively. Figs. 3 (d), (e) and (f) show those two characteristics for QPSK, 16QAM and 64QAM at $E_b/N_0$ of 12dB, 17dB and 22dB over the 8×8 MIMO channel, respectively.

### TABLE IV

**Suitable Values of $\delta$ and $I$**

<table>
<thead>
<tr>
<th></th>
<th>QPSK</th>
<th>16QAM</th>
<th>64QAM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conventional detection</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4×4 MIMO</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>8×8 MIMO</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td><strong>Proposed detection</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j \in {1,2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$I=1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I=2$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$I=6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j \in {1,4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>$I=1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I=5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I=6$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figs. 4 (a), (b) and (c) show the number of iterations $I$ vs. BER characteristics for the proposed detection $\delta$ vs. BER characteristics for the proposed detection for QPSK, 16QAM and 64QAM at $E_b/N_0$ of 16dB, 21dB and 26dB over the 4×4 MIMO channel, respectively. Figs. 4 (d), (e) and (f) show those characteristics for the simulated column-swapping at $h_p$ entries each, and those entries each are $M=1$, $M=2$ and $M=5$, respectively. Table IV summarizes the suitable values of $\delta$ and those of $I$ for the three modulation types in both the 4×4 and the 8×8 MIMO systems.

Remark that the parameter $j \in \{1,2\}$ achieves good BERs for the 4×4 MIMO system, while the parameter $j \in \{1,4\}$ is necessary to achieve good BERs for the 8×8 MIMO system.

### B. BER Performances

Figs. 5 (a)–(f) show the $E_b/N_0$ vs. BER characteristics for QPSK, 16QAM and 64QAM in the 4×4 and the 8×8 MIMO systems, setting $\delta$ shown in Table IV. In each figure, BER curves with legends (1) and (2) are derived using the conventional and the proposed detections, respectively.

Figs. 5 (a), (b) and (c) show the BER characteristics for QPSK, 16QAM and 64QAM in the 4×4 MIMO system, respectively. BER curve (2) for QPSK agreed and those for 16QAM and 64QAM almost agreed with the BER curves for the ML detection. As seen in Figs. 5 (b) and (c), the BER slopes are as steep as those for the ML detection. Hence we can obtain the same BER as that for the ML detection in the high $E_b/N_0$ region by increasing the transmit signal energy by 0.05dB and 0.15dB for 16QAM and 64QAM, respectively.

Figs. 5 (d), (e) and (f) show the BER characteristics for QPSK, 16QAM and 64QAM in the 8×8 MIMO system, respectively. BER curve (2) for QPSK agreed and those for 16QAM and 64QAM almost agreed with the BER curves for the ML detection. As seen in Figs. 5 (e) and (f), the BER slopes are as steep as those for the ML detection. Hence we can obtain the same BER as that for the ML detection in the high $E_b/N_0$ region by increasing the transmit signal energy by 0.1dB and 0.4dB for 16QAM and 64QAM, respectively.

As a consequence, the proposed detection dramatically improves the BER performances, which achieves near-ML BER performances, in particular, for the 8×8 MIMO system.

### C. Computational Complexity Analysis

We count up the number of calculations of the multiplication of two complex values in Tables I–III and the number of those calculations of eqs. (5), (11) and (17) and their related calculations, since those calculations dominantly contribute to the computational complexity.

In Table I, the number of calculations of $h_p^h_p$ in step (4) and of $h_p^h_p$ in step (9) at $p$ with $q \in \{1,p-1\}$ is a total of $2(p-1)$. Hence, with weak reduction, the total number of those calculations over $p \in \{2,M\}$ is $\sum_{p=2}^{M} 2(p-1) = M(M-1)$. Since both $h_p$ and $h_p$ have $2M$ entries each, we have $2M(M-1)$ multiplications for the above calculations. The number of divisions in both steps (4) and (9) is $M(M-1)$. For the squared norms $\|h_p\|^2$s in steps (4) and (9) at $p$ with $q \in \{1,p-1\}$, we need to calculate only $\|h_p\|^2$ at $p$, since the other $\|h_p\|^2$s for $q \in \{1,p-2\}$ have already been calculated before $p$ in the for-loop of $p$. Hence, with weak reduction, we have a total of $2M(M-1)$ multiplications for $\|h_p\|^2$ over $p \in \{2,M\}$. The number of calculations in both steps (5) and (10) at $p$ is a total of $3(p-1)$. With weak reduction, we have a total of $2M(M-1)3(p-1)=3M^2(M-1)$ multiplications over $p \in \{2,M\}$.

If a column-swapping occurs at $p$, then $p$ goes back to $(p-1)$. The number of calculations of both $h_p^h_p$ and $h_p^h_p$ in step (4) and of $h_p^h_p$ in step (9) at $p$ and the number of divisions of them by $\|h_p\|^2$ at $p$ are $2(p-1)$ each, and those at $(p-1)$ are $2(p-2)$ each. Hence we have $2M\{2(2p-1)+2(p-2)\}=(4M(2p-3))$ multiplications and $2(p-1)2(p-2)\{2(2p-3)\}=(2(2p-3))$ divisions due to the column-swapping at $p$. The number of multiplications in both steps (5) and (10) at both $p$ and $(p-1)$ is a total of $2M\{3(p-1)+3(p-2)\}=6M(2p-3)$. Similarly, the number of multiplications for $\|h_p\|^2$ at $p$ and for $\|h_{p-2}\|^2$ at $(p-1)$ in both steps (4) and (9) is a total of $2M^2-2M^2$. Let $a$ be the average column-swapping times. Assuming that the column-swapping occurs uniformly with respect to $p$, the total number of calculations in Table I is

$$A_p(M,a)=4M(M-1+2a)+(M-1)(2M^2+2M+2M^3)$$

$$+\frac{a}{M}\sum_{p=2}^{M}(4M+2+6M)(2p-3)+4M$$

$$=M(M-1)(5M+7)+2a(5M^2+2M-1)$$

on average. Here the term $4M(M-1)$ is the number of multiplications for the squared norm of the right hand side of the inequality at step (12). Note that the left hand side $\|h_p\|^2$
Fig. 4- The number of iterations $I$ vs. BER characteristics for the proposed detection; (a) QPSK: 4×4 MIMO $E_b/N_0=16$dB, (b) 16QAM: 4×4 MIMO $E_b/N_0=21$dB, (c) 64QAM: 4×4 MIMO $E_b/N_0=26$dB, (d) QPSK: 8×8 MIMO $E_b/N_0=12$dB, (e) 16QAM: 8×8 MIMO $E_b/N_0=17$dB, (f) 64QAM: 8×8 MIMO $E_b/N_0=22$dB; $\delta=0$.

Fig. 3- The $\delta$ vs. BER characteristics and the $\delta$ vs. the number of swapping times characteristics; (a) QPSK: 4×4 MIMO $E_b/N_0=16$dB, (b) 16QAM: 4×4 MIMO $E_b/N_0=21$dB, (c) 64QAM: 4×4 MIMO $E_b/N_0=26$dB, (d) QPSK: 8×8 MIMO $E_b/N_0=12$dB, (e) 16QAM: 8×8 MIMO $E_b/N_0=17$dB, (f) 64QAM: 8×8 MIMO $E_b/N_0=22$dB; Curves (1) and (2) are BERs for the conventional and the proposed detections, respectively. Curves (3) are the number of swapping times.

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in step (12) has already been calculated in step (4). Also note that the column-swapping practically more often occurs at the smaller $p$ than at the larger $p$. Therefore, the actual number of calculations in Table I should be smaller than $A(M,a)$. 

In Table II, the column vectors of $H'$ are weakly reduced. Hence the total number of multiplications for $H'H_y$ and that for $\|h_x\|^2$ in step (4) over $p \in [2,M]$ are $M'(M-1)$ and $2M(M-1)$, respectively. The number of divisions in step (4) is $M(M-1)/2$. The total number of multiplications in step (5) is $2M'(M-1)$. As a result, the total number of multiplications in Table II is $A_2(M) = (M-1)(M^2 + 2M + \frac{1}{3}M + 2M^2)$.

To derive the inverse of an $M \times M$ matrix, we used the LU decomposition method, which requires $4M^3/3$ multiplications [10]. Hence, the calculation of $(H'H + \rho I_d)^{-1}H'y$ in (5) requires the number of multiplications of $S(M) = \frac{1}{3}M^2(M+1) + \frac{4}{3}M^2 + 2M^2$. Both $s^{(j)} = T^{(j)}\xi^{(j)}$ in (17) and $\xi^{(j)} = T^{(j)-1}s$ require the total number of multiplications of $T(M) = 2M^2 + \frac{4}{3}M^3$ for each $j$. Both $\hat{T}^{(j,k)} = T^{(j)}\hat{T}^{(j,k)}$ and $\hat{u}^{(j,k)} = T^{(j,k)-1}s$ in (11) require the total number of multiplications of $U(M) = \frac{1}{2}M^2(M+1) + (\frac{4}{3}M^3 + M^2)$ for each $j$ and $k$.

In the proposed detection algorithm of Table III, the equations in steps (11), (21) and (25) hold in high probabilities
either of more than 80, 95 and 99 percent at BER=10−5, respectively. The equation in step (17) holds in high probabilities of more than 97 and 98 percent for i=l and i>2, respectively. Hence the steps (12), (18), (22) and (26) are seldom needed to calculate. This fact implies that they negligibly contribute to the computational complexity.

Since \( \mathbf{T}^{(j,k)-1} \) has already been derived, the calculation for \( \mathbf{T}^{(j,k)-1} \mathbf{s}^{(i)} \) in step (6) requires \( M^2 \) multiplications for each \( j, k \) and \( i \). For the 4×4 MIMO system with \( j \in \{1,2\} \), \( k \in \{1,4\} \), the \( \mathbf{T}^{(j,k)-1} \mathbf{s}^{(i)} \) should be calculated \( 8(=2M) \) times for each \( i \). For the 8×8 MIMO system with \( \{j,k\} \in \{1,4\} \), it should be calculated \( 16(=2M) \) times for each \( i \). Hence, the total number of multiplications for \( \mathbf{T}^{(j,k)-1} \mathbf{s}^{(i)} \) is \( M^2 2M(\bar{j}+1)=2M^3(\bar{j}+1) \). Similarly, \( \mathbf{T}^{(j,k)} \mathbf{u}^{(0,j,k)} \) in step (7) and \( \mathbf{T}^{(j,k)} \mathbf{u}^{(p,j,k)} \) in step (10) require \( 2M^3(\bar{j}+1) \) and \( 2M^3(\bar{j}+1) \) multiplications, respectively. As a result, the total number of multiplications in Table III is \( A_{\text{III}}(M, I) = (4M^3+2M^4)I(\bar{j}+1) \). Note that all the calculations in steps (6), (7) and (10) are not done since the process skips out of the for-loop of \( i \) at \( i=\bar{j} \) in step (17). This process decreases the iteration times. Hence the actual number of multiplications in Table III is smaller than \( A_{\text{III}}(M, I) \).

We first count up the number of calculations for 64QAM in the 4×4 MIMO system, where \( \bar{M}=4 \). As we set \( \delta=0.75 \) for the conventional detection in Table IV, the average swapping times are \( a=3.6 \) at \( E_b/N_0=26dB \) from Fig. 3 (c). Hence the total number of multiplications is \( N_c=4(3,6)+8(4)+7(4)=1225 \) for the conventional detection.

For the proposed detection, we set \( \delta=0.5 \) and \( I=2 \) for 64QAM in Table IV. From Fig. 3 (c), the average swapping times are \( a=1.8 \) at \( E_b/N_0=26dB \). As we chose \( j \in \{1,2\} \) for the 4×4 MIMO system, the LLL algorithm is used twice. Hence the total number of multiplications in Table I is \( 2A(4,1,8) \). We further reduced the columns of \( \mathbf{H}^{(j,k)}: j \in \{1,2\} \), \( k \in \{1,4\} \), eight times using Table II. Hence the number of multiplications in Table II is \( 8A(4) \). The number of multiplications in Table III is \( A_{\text{III}}(4,2) \), where the skipping-out process at step (17) in the for-loop of \( i \) is not taken into account. The number of multiplications for (5), (11), (17), \( \mathbf{T}^{(j,k)} \mathbf{u}^{(0,j,k)} \) and \( \mathbf{T}^{(j,k)-1} \mathbf{s} \) is a total of \( 8(4)+27(4)+8(4) \). As a result, the total number of multiplications for the proposed detection is \( N_c=2A(4,1,8)+8A(4)+A_{\text{III}}(4,2)+8(4)+27(4)+8(4) \). Since \( N_c/N_0=5 \), the proposed detection has at most five times larger number of calculations than the conventional detection.

In the similar manner, we counted up the number of multiplications for the 8×8 MIMO system, where \( \bar{a}=11.4 \) for \( \delta=0.75 \) at \( E_b/N_0=22dB \) from Fig. 1 (d) with setting \( j=6 \) in Table IV. And it was found that the proposed detection has at most 12 times larger number of calculations than the conventional detection.

As a result, the computational complexity for the proposed detection is at most five times and at most 12 times larger than the conventional LRA MMSE detection in the 4×4 and the 8×8 MIMO systems, respectively.

VI. CONCLUSIONS

In this paper, we proposed a Gram-Schmidt based LRA MMSE list detector. First we forward and backward reduced the column vectors of the extended channel matrix using the LLL algorithm to create two reduced channel matrices. To achieve more reliable estimate, we further created two more LLL-reduced channel matrices by rearranging the order of the columns of the channel matrix. Those LLL-reduced column vectors are forward and backward reduced using the GS procedure to create the column vectors purely orthogonal to one another. After that, we created eight or 16 estimates of the transmitted signal. Among them, we selected the most reliable estimate.

The proposed detector dramatically improved the BER performances for QPSK, 16QAM and 64QAM in both the 4×4 and 8×8 MIMO systems. It achieved near-ML BER performances. The BER curves almost agreed with those for the ML detector. BER slopes are as steep as or steeper than those for the ML detector in the high \( E_b/N_0 \) region. This is because the GS procedure creates the column vectors of the reduced channel matrix to be mutually purely orthogonal. Hence the decision boundary became the same as that of the ML detection.

As a consequence, the proposed detector is worthy for applying to both the 4×4 and the 8×8 MIMO systems.

REFERENCES