Effective Capacity of Correlated MISO Channels

Caijun Zhong§, Tharm Ratnarajah§ Kai-Kit Wong‡, Mohamed-Slim Alouini¶,
§The Institute of Electronics, Communications and Information Technology, Queen’s University Belfast, Belfast
¶Dept. of Electrical and Electronic Engineering, University College London, London, United Kingdom
¶Electrical Engineering, King Abdullah University of Science and Technology, Saudi Arabia

Abstract—This paper presents an analytical performance investigation of the capacity limits of correlated multiple-input single-output (MISO) channels in the presence of quality-of-service (QoS) requirements. Exact closed-form expression for the effective capacity of correlated MISO channels is derived. In addition, simple expressions are obtained at the asymptotic high and low signal-to-noise ratio (SNR) regimes, which provide insights into the impact of various system parameters on the effective capacity of the system. Also, a complete characterization of the impact of spatial correlation on the effective capacity is provided with the aid of a majorization theory result. The findings suggest that antenna correlation reduce the effective capacity of the channels. Moreover, a stringent QoS requirement causes a significant reduction in the effective capacity but this can be effectively alleviated by increasing the number of antennas.

I. INTRODUCTION

Multiple antenna technology [1, 2] has proven to be an efficient means to substantially improve the spectral efficiency over the conventional single antenna systems. The information theoretical capacity limits of multiple antenna systems operating in various practical propagation environments have been extensively studied in the literature (see e.g., [3–6] and references therein). While these works have greatly enhanced our understanding of the ultimate limit of information rate for reliable communication in the multiple antenna systems, most works adopt ergodic Shannon capacity as the performance metric, and therefore fail to capture the delay aspect, a parameter which is of paramount importance for practical system designs.

Real-time applications such as voice over IP (VOIP), multimedia streaming, mobile computing, etc, are largely delay-sensitive, meaning that the data will expire if they are not delivered successfully in a given time. In this case, a quality-of-service (QoS) metric which can reflect the end-to-end (e2e) communication delay is essential and yet conventional studies on Shannon capacity are unable to address this.

With delay constraints in the form of a delay bound with a violation probability, it has been proposed in [7] that effective capacity can serve as a suitable metric to quantify the system performance. Following this seminal work, a number of works have investigated the effective capacity of various single-antenna communication systems [8–12]. Recently, there have been some advances in understanding the effective capacity of multiple antenna systems. In particular, the effective capacity of Gaussian quasi-static block-fading multiple-input multiple-output (MIMO) systems was studied in [13], while [14] derived the optimal precoding scheme for correlated multiple-input single-output (MISO) systems. Most recently, [15] studied the effective capacity of MIMO systems with emphasis in the low and high signal-to-noise ratio (SNR) regimes. While these works have greatly improved our understanding of the subject, the impact of antenna correlation on the effective capacity is not well understood.

Motivated by this, in this paper, we provide a thorough investigation on the impact of antenna correlation on the effective capacity of multiple antenna systems. The main contributions of the paper include exact closed-form expression for the effective capacity of correlated MISO channels, which enable efficient evaluation of the effective capacity of the systems with arbitrary numbers of antennas and arbitrary correlation structures, for arbitrary SNRs. Moreover, we present closed-form expressions of the high SNR slope and power offset of the effective capacity, which characterize the high SNR scaling behavior of the effective capacity. Further, we obtain closed-form expressions for the minimum transmit energy per information bit and the wideband slope, which help understand the spectral efficiency in the low SNR regime. In the end, we provide a complete characterization on the impact of antenna correlation on the effective capacity of the systems, and show that spatial correlation always reduces the effective capacity.

II. SYSTEM MODEL

Consider a MISO system with \( N_t \) transmit antennas, the e2e input and output relationship can be expressed as

\[
y = h x + n,
\]

where \( h \) denotes the MISO channel matrix, and can be expressed as \( h = h_w R^\frac{1}{2} \), where \( R \) denotes the transmit correlation matrix satisfying \( \text{trace}(R) = N_t \) and \( h_w \in \mathbb{C}^{N_t \times N_t} \) is an uncorrelated complex Gaussian random vector with zero mean and unit variance entries. \( x \) is the transmit vector with covariance \( \mathbb{E}\{xx^\dagger\} = Q \), which is subjected to a sum-power constraint \( \text{trace}(Q) \leq P \), and \( n \) is the complex additive white Gaussian noise (AWGN) vector with zero-mean entries of variance \( N_0 \).

A. Effective Capacity

The notion of effective capacity of a wireless link was initially developed in [7] for time varying fading channels. Formally, it is defined as the maximum arrival rate that a wireless system can support subject to a statistical QoS requirement. Assuming block fading channels, the effective capacity can be described as [13]

\[
\alpha(\theta) = -\frac{1}{\theta T} \ln \mathbb{E}\left\{ e^{-\theta TC} \right\} \text{ bits/s},
\]

where \( T \) is the block-length, \( C \) is the transmission rate which is a random variable, and the expectation is over \( C \).
parameter $\theta$ describes the asymptotic decay-rate of the buffer occupancy and is given by

$$\theta = -\lim_{x \to \infty} \log P_r \left( L > x \right),$$

(3)

where $L$ is the equilibrium queue-length of the buffer at the transmitter [7]. The delay requirement of a specific application is reflected by the constraints on $\theta$ in the form of $\theta \geq \theta_0$, where $\theta_0$ is the target decay rate. A larger $\theta_0$ implies more stringent delay requirement. When $\theta = 0$, i.e., there is no delay constraint, the effective capacity reduces to the ergodic capacity of the corresponding wireless channel.

In this paper, we assume that the transmitter adopts an equal power allocation policy, i.e., $Q = \frac{P}{N_t}I$. Therefore, the mutual information achieved by a Gaussian input with covariance $Q = \frac{P}{N_t}I$ is given by

$$C = B \log_2 \left( 1 + \frac{P}{N_t} h_w^* R h_w \right) \text{ bits/s,}$$

(4)

where $B$ denotes the bandwidth of the system, and $\rho \triangleq \frac{P}{BN_0}$ is regarded as the SNR.

From the definition of effective capacity given in (2), we therefore have

$$\alpha(\theta, \rho) = \frac{1}{A} \log_2 \left( 1 + \frac{P}{N_t} h_w^* R h_w \right)^{\frac{1}{A}} \text{ bits/s/Hz,}$$

(5)

where $A = \theta TB \log_2 e$.

B. Performance Measures in the High and Low SNR Regimes

At high SNRs, the effective capacity can be captured by two key parameters, i.e., the high-SNR slope $S_\infty$, and the power offset $\mathcal{L}_\infty$, which are defined as

$$S_\infty = \lim_{\rho \to \infty} \frac{\alpha(\theta, \rho)}{\log_2 \rho},$$

(6a)

$$\mathcal{L}_\infty = \lim_{\rho \to \infty} \left( \log_2 \rho - \frac{\alpha(\theta, \rho)}{S_\infty} \right).$$

(6b)

Based on these metrics, the effective capacity at high SNR can be well approximated by

$$\alpha(\theta, \rho) = S_\infty \left( \log_2 \rho - \mathcal{L}_\infty \right) + o(1).$$

(7)

On the other hand, at low SNRs, the effective capacity can be approximated by the second-order Taylor expansion so that

$$\alpha(\theta, \rho) = \hat{\alpha}(\theta, 0) \rho + \bar{\alpha}(\theta, 0) \frac{\rho^2}{2} + o(\rho^2),$$

(8)

where $\hat{\alpha}(\theta, 0)$ and $\bar{\alpha}(\theta, 0)$ denote, respectively, the first and second derivatives of the effective capacity with respect to $\rho$ at $\rho = 0$.

By introducing $\hat{\alpha}(\theta, 0)$ and $\bar{\alpha}(\theta, 0)$, we can investigate the energy efficiency in the low SNR regime. To do so, two important parameters of interest are the minimum transmit energy per information bit, $E_{b_{\min}}$, and the wideband slope $S_0$ [15, 17]. These two key parameters dictate the capacity behavior in the low SNR regime, and can be obtained from $\alpha(\theta, \rho)$ via [15]

$$\frac{E_{b_{\min}}}{N_0} = \lim_{\rho \to 0} \frac{\rho}{\alpha(\theta, \rho)} = \frac{1}{\bar{\alpha}(\theta, 0)},$$

(9a)

$$S_0 = -\frac{2 \left[ \bar{\alpha}(\theta, 0) \right]^2}{\alpha(\theta, 0)} \ln 2.$$  

(9b)

III. EFFECTIVE CAPACITY OF CORRELATED MISO CHANNELS

In this section, we investigate the effective capacity of correlated MISO channels. We first present the exact expression for the effective capacity and then derive expressions for the asymptotic performance measures in the high and low SNR regimes. In addition, we characterize the impact of correlation on the effective capacity with the aid of a majorization theory result. We have the following key result.

**Theorem 1:** For correlated MISO channels with correlation matrix $R$, let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{N_t})$ be the diagonal matrix containing the eigenvalues of $R$. Then, the effective capacity can be expressed as

$$\alpha(\theta, \rho) = -\frac{1}{A} \log_2 \left( \sum_{i=1}^{N_t} \sum_{j=1}^{N_t} \chi_{i,j}(\Lambda) \left( \frac{N_t}{\rho \lambda_{i,j}} \right)^j u(i,j) \right),$$

(10)

where $u(i,j) \triangleq U(j, j + 1 - A, \frac{N_t}{\lambda_{i,j} \rho})$, $\eta(\Lambda)$ is the number of distinct diagonal elements of $\Lambda$, $\eta_{[i]} > \eta_{[2]} > \cdots > \eta_{[k]}$ are the distinct diagonal elements in decreasing order, $\tau_{[i]}(\Lambda)$ is the multiplicity of $\lambda_{i,j}$, $\chi_{i,j}(\Lambda)$ is the $(i,j)$th characteristic coefficient of $\Lambda$, and $U(a, b, x)$ denotes the confluent hypergeometric function of the second kind [18].

**Proof:** With the help of the following integration formula [18, Eq. (3.383.5)] and the probability density function (p.d.f.) expression of $h_w^* R h_w$ given in [19], the desired result can be obtained after some algebraic manipulations. □

Theorem 1 provides the exact characterization of the effective capacity of correlated MISO channels, and is valid for arbitrary number of transmit antennas and arbitrary correlation structure. In the following, we consider some special cases, where simple expressions can be obtained.

**Corollary 1:** When all the eigenvalues of $R$ are equal, i.e., $\lambda_i = 1$, for $i = 1, \ldots, N_t$, then the effective capacity expression reduces to

$$\alpha(\theta, \rho) = -\frac{1}{A} \log_2 \left( \left( \frac{N_t}{\rho} \right)^{N_t} U \left( \left( N_t, N_t + 1 - A, \frac{N_t}{\rho} \right) \right) \right).$$

(11)

**Proof:** When $\lambda_i = 1$ and $\eta(\Lambda) = 1$, the characteristic coefficients of $\Lambda$ become $[19]$.

$$\chi_{i,j}(\Lambda) = \begin{cases} 0 & \text{for } j = 1, \ldots, N_t - 1, \\ 1 & \text{for } j = N_t. \end{cases}$$

(12)

Substituting this into Eq. (10) yields the desired result. □

Note that Corollary 1 in essence gives the effective capacity of an uncorrelated MISO channel.

**Corollary 2:** When all the eigenvalues of $R$ are distinct, i.e., $\lambda_i \neq \lambda_j$, for $i \neq j$, then the effective capacity expression reduces to

$$\alpha(\theta, \rho) = -\frac{1}{A} \log_2 \left( \sum_{i=1}^{N_t} \frac{K_i N_t}{\lambda_{i,\rho}} U \left( \left( 1, 2 - A, \frac{N_t}{\lambda_{i,\rho}} \right) \right) \right).$$

(13)
where \( K_i = \prod_{j=1}^{N_t} \left( \frac{\lambda_i}{\lambda_j - \lambda_i} \right) \).

**Proof:** In this case, the characteristic coefficients of \( \Lambda \) become \( \Lambda_{i,1}(\Lambda) = K_i \) [19], and the desired result follows. \( \square \)

Fig. 1 plots the effective capacity of correlated MISO channels according to Theorem 1 under different QoS requirements with \( N_t = 3 \). The Monte-Carlo simulation results were obtained by averaging over 10^6 independent channel realizations, and we assumed that \( T = 1 \text{ms} = 10^{-3} \text{s} \) and \( B = 100\text{kHz} = 10^2\text{Hz} \) in all simulations. Results show that our analytical results coincide with the Monte-Carlo simulation results, as predicted by Theorem 1. Also, we observe that when the QoS requirement becomes more stringent, i.e., \( A \) becomes larger, the effective capacity of the system decreases gradually.

![Effective capacity of correlated MISO channels](image)

Fig. 1. Effective capacity of correlated MISO channels when \( N_t = 3 \), \( \Lambda = \text{diag}(1.5, 0.9, 0.6) \): Monte Carlo versus analytical.

Though Theorem 1, Corollaries 1 and 2 provide an efficient way for evaluating the effective capacity of correlated MISO channels, they are not useful to gain insights on the impact of key parameters such as the number of antennas, delay constraints and correlation structure on the effective capacity. Hence, it is of great interest to look into the high and low SNR regimes where simple expressions can be obtained.

**Theorem 2:** When all the eigenvalues of \( \mathbf{R} \) are equal, i.e., \( \lambda_i = 1 \), for \( i = 1, \ldots, N_t \), the high SNR slope is given by

\[
\mathcal{S}_\infty = \begin{cases} 
1 & \text{for } A \leq N_t, \\
\frac{N_t}{A} & \text{for } A > N_t,
\end{cases} 
\]

while the power offset \( \mathcal{L}_\infty \) is given by

\[
\mathcal{L}_\infty = \begin{cases} 
\log_2 N_t + \frac{1}{A} \log_2 \left( \frac{\Gamma(N_t - A)}{\Gamma(N_t)} \right) & \text{if } A < N_t, \\
\log_2 N_t + \frac{1}{A} \log_2 \left( \frac{\ln \Gamma(N_t)}{\Gamma(N_t)} \right) & \text{if } A = N_t, \\
\log_2 N_t + \frac{1}{N_t} \log_2 \left( \frac{\Gamma(A - N_t)}{\Gamma(A)} \right) & \text{if } A > N_t,
\end{cases} 
\]

where \( \psi(\cdot) \) is the digamma function and \( \gamma \) is the Euler-Mascheroni constant.

**Proof:** The proof is omitted due to space limit, and can be found in the submitted journal version [20]. \( \square \)

**Theorem 2** in essence provides a complete characterization of the high SNR scaling behavior of the effective capacity of an uncorrelated MISO channel. When \( N_t = 1 \), the result is reduced to the single-input single-output (SISO) result presented in [15, Theorems 5 & 6].

**Theorem 3:** When all the eigenvalues of \( \mathbf{R} \) are distinct, i.e., \( \lambda_i \neq \lambda_j \), for \( i \neq j \), the high-SNR slope \( \mathcal{S}_\infty \) is given by

\[
\mathcal{S}_\infty = \begin{cases} 
1 & \text{for } A \leq N_t, \\
\frac{N_t}{A} & \text{for } A > N_t,
\end{cases} 
\]

while the power offset \( \mathcal{L}_\infty \) is shown in Eq. (17) on the top of next page, where \( N \) denotes the nature number field.

**Proof:** The proof is omitted due to space limit. \( \square \)

From Theorem 2 and Theorem 3, we observe that the high SNR slope is not affected under loose delay constraints, i.e., when \( \theta < \frac{N_t}{T B \log_2 e} \). Otherwise, the high SNR slope is reduced. Moreover, more stringent delay requirement will cause severer reduction in the high SNR slope.

In addition, we see that the structure of the correlation matrix does not affect the high SNR slope of the effective capacity. Instead, it contributes to the decrease of the effective capacity by increasing the high SNR power offset. For instance, let us consider the case of \( A > N_t \). The difference of the high SNR power offset can be computed as

\[
\mathcal{L}_\infty^c - \mathcal{L}_\infty^u = \frac{1}{N_t} \log_2 \left( \prod_{i=1}^{N_t} \frac{1}{N_t^i} \right)^2.
\]

Applying the geometric-harmonic mean inequality, we have

\[
\prod_{i=1}^{N_t} \frac{1}{N_t^i} \geq \left( \frac{N_t}{\sum_{i=1}^{N_t} N_t^i} \right)^{N_t}
\]

Since \( \sum_{i=1}^{N_t} N_t^i = N_t \), then it is trivial to show that \( \mathcal{L}_\infty^c \geq \mathcal{L}_\infty^u \), which confirms that correlation reduces the effective capacity.

![Effective capacity of uncorrelated MISO channels](image)

Fig. 2. Effective capacity of uncorrelated MISO channels when \( N_t = 3 \): Exact versus high SNR approximation.

Fig. 2 plots the high SNR approximation of effective capacity for uncorrelated MISO channels using Theorem 2 when \( N_t = 3 \). Results reveal that the high SNR approximation is accurate even at moderate SNRs, e.g., \( \rho = 15\text{dB} \). In addition, the curves for \( A = 1 \) and \( A = 3 \) are parallel at high SNRs (i.e., having the same high SNR slope), while the slope of the curve for \( A = 5 \) is significantly smaller. When \( A \leq 3 \), the QoS requirement does not reduce the high SNR slope much, but penalizes the power offset. On the contrary, the high SNR slope is reduced when \( A > 3 \). Similar observations can be made from Fig. 3 where the high SNR approximation of the effective capacity for correlated MISO channels is examined.

\( ^2 \)We use \( \mathcal{L}_\infty^c \) and \( \mathcal{L}_\infty^u \) to denote the high SNR power offset of correlated MISO channels and uncorrelated MISO channels, respectively.
Now, we turn our attention to the analysis of effective capacity in the low SNR regime.

**Theorem 4:** In the low SNR regime, the minimum transmit energy per information bit, \( E_b/N_0 \), and the wideband slope \( S_0 \) can be, respectively, expressed as
\[
E_b/N_0 = \ln 2, \\
S_0 = 2/\left(1 + \frac{\text{trace}(\mathbf{R}^2)}{N_t} + \frac{\text{trace}(\mathbf{R}^2)A}{N_t}\right). 
\]

**Proof:** See Appendix I.

Theorem 4 provides a complete characterization of the low SNR spectral efficiency of a correlated MISO channel in the presence of delay constraints. When \( A = 0 \), i.e., no delay constraint, Theorem 4 reduces to a previous result presented in [21]. It is interesting to observe that the delay constraint does not affect \( E_b/N_0 \). However, it reduces the wideband slope \( S_0 \). Moreover, the impact of delay constraints on the effective capacity of the system is intimately associated with the correlation structure due to the term \( \text{trace}(\mathbf{R}^2)A \). Since \( \text{trace}(\mathbf{R}^2) = \sum_i \lambda_i^2 \), which is a Schur-convex function, it increases when the channel becomes more correlated. Hence, the effect of delay constraints becomes more pronounced when the channel becomes increasingly correlated.

In Fig. 4, results for the low SNR capacity approximation are plotted for uncorrelated MISO channels for \( N_t = 4 \) with different delay constraints. The curves indicate the accuracy of our analytical expressions and that the range for a good approximation improves if the QoS requirement loosens.

In particular, the approximation becomes extremely tight for the case \( A = 0.1 \). Also, when the QoS requirement becomes harsher, the wideband slope becomes smaller and therefore the effective capacity is reduced.

Theorem 3 and Theorem 4 have studied the impact of correlation on the effective capacity in the high and low SNR regimes, respectively. In the following, we present a general result, which is valid for arbitrary SNRs, thereby providing a complete characterization on the impact of antenna correlation.

**Theorem 5:** The effective capacity \( \alpha(\theta, \lambda) \) is a Schur-concave function with respect to \( \lambda = \{\lambda_1, \ldots, \lambda_{N_t}\} \). That is, for any two given vectors \( \lambda^1 \) and \( \lambda^2 \), if \( \lambda^1 \succ \lambda^2 \), then we have
\[
\alpha(\theta, \lambda^1) \leq \alpha(\theta, \lambda^2). 
\]

**Proof:** The proof is omitted due to space limit.

![Fig. 3. Effective capacity of correlated MISO channels when \( N_t = 3, \Lambda = \text{diag}(1.5, 0.9, 0.6) \): Exact versus high SNR approximation.](image3)

![Fig. 4. Low SNR effective capacity versus transmit \( E_b/N_0 \) for \( N_t = 4 \) with different delay constraints.](image4)

![Fig. 5. Impact of correlation on the effective capacity of correlated MISO channels when \( N_t = 3 \) and \( A = 3 \).](image5)
The impact of correlation structure on the effective capacity of MISO channels is illustrated in Fig. 5. Three different correlation matrices were considered with respective vectors of eigenvalues, \( \lambda^1 = [1, 1, 1], \lambda^2 = [2.4, 0.45, 0.15], \) and \( \lambda^3 = [2.7, 0.27, 0.03] \) when \( N_t = 3 \) and \( A = 3 \). These three vectors satisfy
\[
\lambda^1 > \lambda^2 > \lambda^1. \tag{20}
\]

Therefore, according to Theorem 5, we have \( \alpha(\theta, \lambda^1) \leq \alpha(\theta, \lambda^2) \leq \alpha(\theta, \lambda^3) \). As expected, the simulation results in Fig. 5 agree with our analysis in Theorem 5.

IV. CONCLUSION

In this paper, we have analyzed the effective capacity for correlated MISO channels and MIMO keyhole channels. Exact closed-form expressions for the effective capacity of the channels were derived. Moreover, we have studied the asymptotic high and low SNR behaviors of the effective capacity, and derived some simple expressions which enable us to gain insights on how various system parameters affect the effective capacity. We proved that spatial correlation always reduces the effective capacity of the channels. Our results also demonstrate that a tighter delay constraint implies a severe decrease of the effective capacity, which can be effectively alleviated by increasing the number of antennas.

APENDICES

I. PROOF OF THEOREM 4

The key step is to obtain exact expressions for the first and second order derivatives of \( \alpha(\theta, \rho) \) at \( \rho = 0 \). For notation convenience, we define \( g \overset{\triangle}{=} \mathbf{h}_w^\dagger \mathbf{R} \mathbf{h}_w \). The first order derivative can be computed as
\[
\dot{\alpha}(\theta, 0) = -\frac{1}{\theta T B} \left[ \log_2 e \mathbb{E} \{ g \} - N_t \mathbb{E} \{ \text{tr} (\mathbf{R}) \} \right] = \log_2 e \frac{g}{N_t} \mathbb{E} \{ g \}. \tag{21}
\]

On the other hand, the second derivative is given by
\[
\ddot{\alpha}(\theta, 0) = -\frac{1}{\theta T B} \left[ \log_2 e \mathbb{E} \{ g \} - N_t \mathbb{E} \{ \text{tr} (\mathbf{R}) \} \right] \mathbb{E} \{ \left[ g^2 + \frac{A^2}{\theta T B N_t^2} \mathbb{E} \{ g \} \right] \}^2. \tag{22}
\]

Finally, substituting (21) and (22) into (9) yields the desired result.

REFERENCES