Identifying codes in some subgraphs of the square lattice

Marc Daniel, Sylvain Gravier*, Julien Moncel
CNRS-UJF, ERTé “Maths à Modeler”, Groupe de Recherche GeoD, Laboratoire Leibniz,
46 Avenue Félix Viallet, Grenoble Cedex F-38031, France

Abstract

An identifying code of a graph is a subset of vertices $C$ such that the sets $B(v) \cap C$ are all nonempty and different. In this paper, we investigate the problem of finding identifying codes of minimum cardinality in strips and finite grids. We first give exact values for the strips of height 1 and 2, then we give general bounds for strips and finite grids. Finally, we give a sublinear algorithm which finds the minimum cardinality of an identifying code in a restricted class of graphs which includes the grid.

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1. Introduction

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we denote by $B(v)$ the set $N(v) \cup \{v\}$, where $N(v)$ denotes the neighborhood of $v$.

An identifying code of $G$ is a subset $C \subseteq V$ such that

- $\forall v \in V, B(v) \cap C \neq \emptyset$,
- $\forall v \neq w \in V, B(v) \cap C \neq B(w) \cap C$.

For a given vertex $v \in V$, the set $B(v) \cap C$ is called the identifying set of $v$. If this set is nonempty then we will say that $v$ is covered by the code $C$.

If this set is distinct from the identifying set of every other vertex of $G$, then we will say that $v$ is identified by the code $C$.

* Corresponding author.
E-mail address: sylvain.gravier@imag.fr (S. Gravier).

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The concept of identifying code first appeared in [9]. An application of identifying codes is the fault diagnosis of multiprocessor systems. Let us model a multiprocessor system as a graph $G = (V, E)$ whose vertices are processors and edges are links between these processors. Assume that each processor may report to a central controller the state of its neighborhood by sending a 1-bit information (e.g. ‘0’ if a processor of its neighborhood is faulty, ‘1’ otherwise). The processors corresponding to an identifying code of $G$ constitutes a set $C$ of processors such that:

- If all the processors of $C$ returns the bit ‘1’, then we know that none of the processor of the network is faulty,
- assuming that at most one of the processors of the network is faulty, if some of the processors of $C$ return the bit ‘0’, then we are able to determine which processor of the network is faulty.

A direct consequence of the definition is that $G$ has an identifying code if and only if the sets $B(v)$ are all different, and in this case $V$ itself is an identifying code of $G$. The associated optimization problem is therefore to find an identifying code of minimum cardinality (when there exists one). This problem has been shown to be NP-Hard in [4].

When the graph $G$ is infinite then we define the density of an identifying code of $G$ as follows. Let $v_0$ be an arbitrary vertex of $G$, and for all $n \in \mathbb{N}$ let $B_n$ be the set $\{x \mid \delta(x, v_0) \leq n\}$, where $\delta(x, v_0)$ is the number of edges in any shortest path between $x$ and $v_0$. Then the density $d(C, G)$ of an identifying code $C$ of $G$ is defined as the following limit:

$$d(C, G) = \limsup_{n \to \infty} \frac{|C \cap B_n|}{|B_n|}.$$ 

We denote by $d^*(G)$ be the infimum density of an identifying code of $G$.

The grid denotes the graph with vertex set $V = \mathbb{Z} \times \mathbb{Z}$ and edge set $\{(u, v) \in V^2 \mid u - v \in \{(0, \pm 1), (\pm 1, 0)\}\}$. We call strip of height $k$ the subgraph $\mathcal{S}_k$ of the grid induced by the vertex set $\{1, \ldots, k\} \times \mathbb{Z}$, and semi-strip of height $k$ the subgraph $\mathcal{S}_k^+$ of the grid induced by the vertex set $\{1, \ldots, k\} \times \mathbb{N}$. We call $k \times n$ grid the subgraph $\mathcal{G}_{k \times n}$ of the grid induced by the vertex set $\{1, \ldots, k\} \times \{1, \ldots, n\}$.

Grid graphs are widely used structures for VLSI systems.

It is easy to see that $d^*(\mathcal{S}_k^+) = d^*(\mathcal{S}_k)$ for all $k \geq 1$.

Indeed, if $C$ is an identifying code of a strip $\mathcal{S}_k$, then it suffices to make a finite number of modifications on the trace of $C$ on $\{1, \ldots, k\} \times \mathbb{N}$ so that it becomes an identifying code of the semi-strip $\mathcal{S}_k^+$. Hence $d^*(\mathcal{S}_k^+) \leq d^*(\mathcal{S}_k)$. On the other hand, let $C$ be an identifying code of the semi-strip $\mathcal{S}_k^+$. By pasting two copies of $\mathcal{S}_k^+$ on their first columns we obtain a strip $\mathcal{S}_k$. It suffices to make a finite number of modifications on the two copies of $C$ so that it becomes an identifying code of the strip $\mathcal{S}_k$. Hence $d^*(\mathcal{S}_k^+) \geq d^*(\mathcal{S}_k)$.

Thus all the results stated in Section 2 concerning strips can also be stated for semi-strips.

In Section 2, we give the exact values of $d^*(\mathcal{S}_1)$ and $d^*(\mathcal{S}_2)$ and some upper and lower bounds for $d^*(\mathcal{S}_k)$, for all $k \geq 3$. We also give some upper and lower bounds for $d^*(\mathcal{G}_{k \times n})$. 

In Section 3, we give an algorithm which computes $d^*(\mathcal{G}_{k \times n})$ in time $O(\log n)$ whenever $k$ is fixed. The algorithm works even on a more general class of graphs, namely the fasciagraphs.

2. Strips and finite grids

2.1. Strips of height 1 and 2

**Proposition 1.** We have $d^*(\mathcal{S}_1) = \frac{1}{2}$ and $d^*(\mathcal{S}_2) = \frac{3}{7}$.

**Proof.** Let $C$ be an identifying code of $\mathcal{S}_1$. For any set $X = \{u, v, w, t\}$ of four consecutive vertices of $\mathcal{S}_1$, we have $|C \cap X| \geq 2$. Indeed if $|C \cap X| = 1$, then either the vertices $v, w$ have same identifying set (cases $C \cap X = \{v\}$ or $C \cap X = \{w\}$), or one of the vertices $v, w$ is not covered (cases $C \cap X = \{u\}$ or $C \cap X = \{t\}$). If $C \cap X = \emptyset$ then both $v$ and $w$ are not covered. Thus $d^*(\mathcal{S}_1) \geq \frac{1}{2}$.

To conclude we exhibit in Fig. 1 an identifying code of $\mathcal{S}_1$ of density $\frac{1}{2}$.

Let $C$ be an optimal identifying code of $\mathcal{S}_2$. We first may assume that its density is $< \frac{1}{2}$. Indeed one can easily exhibit an identifying code of $\mathcal{S}_2$ with density $< \frac{1}{2}$ (see for example Fig. 5).

Let us call column of $\mathcal{S}_2$ any pair of vertices $(u, v)$ such that $u - v = (0, \pm 1)$. A column $(u, v)$ is said to be of type $k$ if $|C \cap \{u, v\}| = k$, for $k = 0, 1, 2$. Let us call isolated column of $\mathcal{S}_2$ a column of type 0 which is nonadjacent to any column of type 2.

We may assume that

**Claim 2.** There are no three consecutive columns of type 2 in $\mathcal{S}_2$.

Indeed, assuming the contrary, let $(u, v)$ be a column of type 2 adjacent to two columns of type 2. In this case we can remove the vertex $u$ from $C$ to obtain another identifying code of $\mathcal{S}_2$. Repeating this for any such column of type 2, we obtain another identifying code of $\mathcal{S}_2$ satisfying Claim 2 with density $\leq d(C, \mathcal{S}_2)$.

In addition, we may claim that

**Claim 3.** There is no column of type 2 adjacent to a column of type 2 and a column of type 1.

Indeed, assuming the contrary, let $(u, v)$ be a column of type 2 adjacent to a column $(x, y)$ of type 2 and a column $(z, a)$ of type 1. Without loss of generality assume that $a \notin C$ and $a - v = (1, 0)$. Then, depending on the other column adjacent to $(z, a)$, we can either move or remove $v$ from $C$ to obtain another identifying code of $\mathcal{S}_2$. Indeed,

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Fig. 1. A periodic identifying code of $\mathcal{S}_1$ of density $\frac{1}{2}$.
let \((b, c)\) be the other column adjacent to \((z, a)\) and let us assume that \(b - z = (1, 0)\).
If \(b \in C\) then \(C \setminus \{v\}\) is still an identifying code of \(S_2\). If not, then \(C \setminus \{v\} \cup \{a\}\) is an identifying code of \(S_2\). Note that doing this does not add any new column of type 2 adjacent to a column of type 2 and a column of type 1. Repeating this for any such column of type 2, we obtain another identifying code of \(S_2\) satisfying Claim 3 with density less than or equal to \(d(C, S_2)\).

Finally, we may assume that

**Claim 4.** Each column of type 2 is adjacent to a column of type 0.

Thanks to Claims 2 and 3, we know that the only case to deal with is the case of a column of type 2 adjacent to two columns of type 1. Assume that \((u, v)\) is such a column, and let \((x, y)\) be the column adjacent to \((u, v)\) such that \(y \in C\) and \(y - v = (1, 0)\). We will use the so-called *rightshift principle*, which consists in removing \(y\) from \(C\) and eventually adding to \(C\) a new vertex belonging to the right part of the strip \(S_2\) (see Fig. 2). This process transforms \((x, y)\) into a column of type 0, adjacent to \((u, v)\).

Let us call \(a, z, b, c, r, d, s, t\) the vertices of the four consecutive columns next to \((x, y)\) as in Fig. 2. If \(C \setminus \{y\}\) is still an identifying code of \(S_2\) then we can remove \(y\) from \(C\) and we are done.

Else, we claim that either \(C \setminus \{y\} \cup \{z\}\) or \(C \setminus \{y\} \cup \{b\}\) is an identifying code of \(S_2\). Indeed, if neither \(C \setminus \{y\}\) nor \(C \setminus \{y\} \cup \{z\}\) is an identifying code of \(S_2\), it means that \(y\) is necessary to identify \(z\) from one of the vertices \(a\) or \(c\). In this case, replacing \(y\) by \(b\) identifies \(z\) and leads to an identifying code of \(S_2\).

We wish to repeat this process for any column of type 2 surrounded by two columns of type 1, to obtain an identifying code of \(S_2\) satisfying Claim 4. However, this process may create new columns of type 2 surrounded by two columns of type 1 (see for instance the case of Fig. 2). In this case we re-apply the process, and so on if re-applying the process creates new columns of type 2 surrounded by two columns of type 1. We claim that we re-apply this process only a finite number of times.

Indeed, it suffices to notice that this process creates a new column of type 2 surrounded by two columns of type 1 if and only if we are in the case described in Fig. 2, where we replace \(y\) by \(b\) to identify \(z\) from \(a\) (indeed, replacing \(y\) by \(z\) cannot create such a column, and if we desire identify \(z\) from \(c\) then we know that
In this case $(r, d)$ becomes a new column of type 2 surrounded by two columns of type 1.

If re-applying this process does not end, it means that there is a succession of an infinite number of such “blocks” $u, v, x, y, a, z, b, c$. As the density of such a block is $\geq \frac{1}{2}$, then we obtain a semi-strip $S_2^+ \subseteq S_2$ such that $C \cap S_2^+$ has density $\geq \frac{1}{2}$. This yields a contradiction since $C$ has density $< \frac{1}{2}$ (indeed, it means that the complementary of $S_2^+$ has a density $d < d(C, S_2) = d^*(S_2)$, which contradicts since $d^*(S_2) = d^*(S_2^+)$).

Therefore, the neighborhood $N$ of an isolated column of type 0 is necessarily as described in Fig. 3 (up to symmetry). This can be obtained by an easy exploration of cases (last and first columns are obtained thanks to Claim 4).

The last claim we need is the following:

**Claim 5.** There is no column of type 2 adjacent to two columns of type 0.

Indeed, in the opposite case, the two vertices $u, v$ of such a column would have same identifying set $\{u, v\}$.

By Claim 5, we know that we can partition $S_2$ into blocks $N$ and other blocks as described in Fig. 4. As $d(C, N) = \frac{3}{7}$ and $d(C, B) \geq \frac{1}{7}$ for any block $B$ as described in Fig. 4, we have $d(C, S_2) \geq \frac{3}{7}$.

To conclude we give in Fig. 5 an identifying code of $S_2$ of density $\frac{3}{7}$.

**Remark 6.** It is straightforward that, up to translations, the code given in Fig. 1 is the unique periodic code of minimum density of $S_1$. We may state an analogous result for $S_2$, up to reflexions of the copies of $N$. 
Remark 7. The same technique can be used to find optimal codes of the paths $P_n$ for $n \geq 1$. We obtain $d^* = \lceil \frac{n+1}{2} \rceil / n$.

2.2. General bounds

In this section, we will give some general lower and upper bounds for $d^*(\mathcal{G}_k \times n)$ and $d^*(\mathcal{S}_k)$ for all large enough $k$ and $n$. We will need the following result:

Theorem 8. We have $\frac{15}{43} \leq d^*(\mathbb{Z}^2) \leq \frac{7}{20}$.

The upper bound is due to Cohen et al. (see [5]), and the lower bound is due to Cohen et al. (see [6,7]). The authors have conjectured in [5] that $d^*(\mathbb{Z}^2) = \frac{7}{20}$.

We will use a “cut-and-paste” technique, based on the knowledge of the bounds on $d^*(\mathbb{Z}^2)$, to find upper and lower bounds for $d^*(\mathcal{G}_k \times n)$ and $d^*(\mathcal{S}_k)$. This technique is more or less standard (see for example [8]).

Theorem 9. For all $k \geq 6, n \geq 2$ we have

$$\frac{1}{43k} (15k - 41) \leq d^*(\mathcal{S}_k) \leq \min \left( \frac{2}{5}, \frac{1}{20k} (7k + 38) \right)$$

and

$$\frac{1}{43kn} (15kn - 41(k + n) - 80) \leq d^*(\mathcal{G}_k \times n) \leq \frac{1}{20kn} (7kn + 38(k + n) - 308).$$

Proof. As the technique used is the same for $\mathcal{S}_k$ and $\mathcal{G}_k \times n$, we will hereunder give only the details of the proof for the case $\mathcal{S}_k$. The proof for $\mathcal{G}_k \times n$ can be easily deduced from the proof of the case $\mathcal{S}_k$.

The upper bound $\frac{2}{5}$ comes from the fact that the vertex set $\{(u,v) \in \{1, \ldots, k\} \times \mathbb{Z} | v \mod 5 \in \{1, 3\}\}$ is always an identifying code of $\mathcal{S}_k$ for all $k \geq 3$ (see Fig. 6).

For the other upper bound, let $C$ be an optimal identifying code of $\mathbb{Z}^2$. We know, by Theorem 8, that its density is $\leq \frac{7}{20}$. Thus, for any $k \geq 6$ there exists a strip $\mathcal{S}_{k-6}$ of height $k - 6$ included in $\mathbb{Z}^2$ such that $C \cap \mathcal{S}_{k-6}$ has density $\leq \frac{7}{20}$. However, the set $C \cap \mathcal{S}_{k-6}$ is possibly not an identifying code of $\mathcal{S}_{k-6}$. Even so, from $C \cap \mathcal{S}_{k-6}$ we

\[ \begin{array}{c}
\text{Fig. 6. A periodic identifying code of } \mathcal{S}_3 \text{ of density } \frac{2}{5}. \end{array} \]
construct an identifying code $C_k$ of $\mathcal{S}_k$ in the following way:  
\[ C_k \cap (\{4, \ldots, k-3\} \times \mathbb{Z}) = C \cap \mathcal{S}_{k-3}, \]
\[ (\{1,3\} \times \mathbb{Z}) \subset C_k \]

and
\[ (\{k-2,k\} \times \mathbb{Z}) \subset C_k. \]

It is easy to see that $C_k$ is an identifying code of $\mathcal{S}_k$. Its density is therefore greater or equal to $d^*(\mathcal{S}_k)$. The upper bound follows from the fact that $d(C_k, \mathcal{S}_k) \leq (\frac{1}{20}(k-6) + \frac{2}{3})/k$. 

For the lower bound, let us consider $C_k$ an optimal identifying code of $\mathcal{S}_k$. From $C_k$ we construct $C$ a $(0,k+3)$-periodic identifying code of $\mathbb{Z}^2$ as follows (see Fig. 7):  
\[ C \cap (\{1,\ldots,k\} \times \mathbb{Z}) = C_k \]

and
\[ (\{k+1,k+3\} \times \mathbb{Z}) \subset C. \]

It is easy to see that $C$ is an identifying code of $\mathbb{Z}^2$. Its density is therefore greater than or equal to $d^*(\mathbb{Z}^2) \geq \frac{15}{33}$. The lower bound follows from the fact that $d(C) = (d^*(\mathcal{S}_k)k + \frac{2}{3}3)/k + 3$.

**Remark 10.** The upper bound $d^*(\mathcal{S}_k) \leq \frac{2}{3}$ is better than $(7k + 38)/20k$ for all $k < 38$.

3. An algorithm for fasciagraphs

In this section we will give an algorithm which enables us to find the minimum cardinality of an identifying code of $\mathcal{G}_{k \times n}$ in time $O(\log n)$, provided $k$ is fixed. This algorithm works on fasciagraphs, which are an important class of polygraphs including grid graphs. Polygraphs were introduced in [1] as a model for polymers.
3.1. Fasciagraphs: definition

Let $G_1, \ldots, G_n$ be $n$ disjoints graphs and $X_1, \ldots, X_n$ be a sequence of sets of edges such that for all $i = 1, \ldots, n$, an edge of $X_i$ joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$, where $G_{n+1}$ denotes the graph $G_1$. The polygraph $\Omega_n(G_1, \ldots, G_n; X_1, \ldots, X_n)$ over the monographs $G_1, \ldots, G_n$ is defined in the following way:

- $V(\Omega_n) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$,
- $E(\Omega_n) = E(G_1) \cup X_1 \cup E(G_2) \cup \cdots \cup X_{n-1} \cup E(G_n) \cup X_n$.

The monographs $G_j$ are called the fibers of the polygraph (Fig. 8).

Suppose that for all $i = 1, \ldots, n$, $G_i$ is isomorphic to a fixed graph $G$. In addition, let the sets $X_i$, $i = 1, \ldots, n - 1$, be equal to a fixed edge set $X$ and $X_n = \emptyset$. This special polygraph is called a fasciagraph, and we denote it $\Psi_n(G, X)$.

A grid graph is a typical example of a fasciagraph, and is a widely used interconnection structure in multiprocessor VLSI systems. In the next subsection, we will give an algorithm which finds the minimum cardinality of an identifying code of a fasciagraph $\Psi_n(G, X)$ in time $O(\log n)$, provided $G$ has a fixed number of vertices.

3.2. Algorithm for fasciagraphs

Our algorithm uses the “regular” inductive structure of fasciagraphs and the fact that being an identifying code is a “local” property, in the sense that given $C$ an identifying code of a fasciagraph $\Psi_n(G, X)$, the identifying set of a vertex $v$ depends only on the trace of $C$ on the three fibers surrounding $v$. For more details about local properties, see [10].

We need an additional definition. Let $G = (V, E)$ be a graph and let $W \subseteq V$. We will say that a subset $C \subseteq V$ is a $W$-identifying code of $G$ if all the vertices of $W$ have distinct nonempty identifying sets. Thus, an identifying code of $G$ is a $V$-identifying code of $G$.

Let $G$ be a graph on $k$ vertices $v_1, \ldots, v_k$ and let $\Psi_n(G, X)$ be a fasciagraph over fiber $G$. For all $i$ and $j$ we will denote by $v_i^j$ the vertex $v_i$ in the $j$th fiber $G_j$ of $\Psi_n(G, X)$. We simply denote by $\Psi_m$ the subgraph of $\Psi_n(G, X)$ induced by the $m$th first fibers $G_1, \ldots, G_m$ of $\Psi_n(G, X)$. Moreover for all $i, j$ we denote by $G_{i,j}$ the set of vertices of $\bigcup_{t=i}^{m} G_t$.

A matrix $M \in \mathcal{M}_{k \times 5}(\{0, 1\})$ is said to be valid if there exists a $G_{2,4}$-identifying code $C$ of $\Psi_5$ such that

$$M_{ij} = 1 \iff v_i^j \in C.$$

A valid matrix simply encodes the trace of a $G_{2,4}$-identifying code of $\Psi_5$. 
We will hereunder construct an auxiliary digraph $\mathcal{G}(G,X)$.

- The vertices of $\mathcal{G}(G,X)$ are the valid matrices of $M_{k \times 5}(\{0,1\})$, plus two additional vertices $B$ and $E$, that we call respectively begin and end vertices of $\mathcal{G}(G,X)$.
- There is an arc between $B$ and a valid matrix $M \neq E$ if and only if $M$ encodes a $G_{1,4}$-identifying code of $\Psi_5$.
- There is an arc between a matrix $M \neq B$ and $E$ if and only if $M$ encodes a $G_{2,5}$-identifying code of $\Psi_5$.
- There is an arc between two valid matrices $M$ and $N \neq B,E$ if and only if there is a $G_{2,5}$-identifying code $C$ of $\Psi_6$ such that $M$ is the trace of $C$ on $\bigcup_{i=1,5} G_i$ and $N$ is the trace of $C$ on $\bigcup_{i=2,6} G_i$.

More formally, there is an arc between $M$ and $N$ if and only if there exists a $G_{2,5}$-identifying code $C$ of $\Psi_6$ such that $M_{ij} = 1 \iff v^j_i \in C$ and $N_{ij} = 1 \iff v^j_{i+1} \in C$.

For instance, if the case where $\Psi_n(G,X)$ is the fasciagraph $G_{3 \times n}$, one can easily check that there is an arc between

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(see Fig. 9).

**Remark 11.** There is a one to one mapping between the identifying codes of $\Psi_n(G,X)$ and the paths on $n-2$ vertices from $B$ to $E$ in $\mathcal{G}(G,X)$.

In addition, the graph is labeled in the following way:

- For any $M,N \neq E$, the arc $(M,N)$ is labeled by the number of ‘1’ of the fifth column of $N$.

![Fig. 9. An example of arc in $\mathcal{G}(G,X)$](image-url)
For any \( M \neq E \), the arc \((B,M)\) is labeled by the number of ‘1’ of the first, second, third and fourth columns of \( M \).

For any \( M \), the arc \((M,E)\) is labeled by 0.

**Remark 12.** For a given path on \( n - 2 \) vertices \( P = (B,M_5,M_2,\ldots,M_n,E) \) in \( \mathcal{G}(G,X) \), the length of \( P \) is the cardinality of the corresponding identifying code of \( \Psi_n(G,X) \).

Thus, determining the minimum cardinality of an identifying code of \( \Psi_n(G,X) \) is equivalent to compute the minimum length of a \((B,E)\)-path on \( n - 2 \) vertices in \( \mathcal{G}(G,X) \). This can be done in \( O(K \log n) \) time, where \( K \) depends only on the size of \( \mathcal{G}(G,X) \) (for this more or less standard result see for example [2] or [12]).

**Theorem 13.** For a fixed \( k \), one can compute the minimum cardinality of an identifying code of a fasciagraph \( \Psi_n(G,X) \) in time \( O(\log n) \) whenever \( |V(G)| \leq k \).

**Proof.** First observe that the size of \( \mathcal{G}(G,X) \) depends only on \( k \), since \( \mathcal{G}(G,X) \) has at most \( 2^{5k} \) vertices. \( \square \)

It remains, now, to prove that one can construct \( \mathcal{G}(G,X) \) in time \( O(\log n) \). This comes from the fact that determining if a matrix in \( M \in \mathcal{M}_{k \times 5}(\{0,1\}) \) is valid can be done by computing all the identifying codes of \( \Psi_5(G,X) \); and checking if there is an arc between two valid matrix can be done by computing all the identifying codes of \( \Psi_6(G,X) \).

This algorithm was inspired by another one, devised to compute the domination number of fasciagraphs and rotagraphs. That algorithm and some general theory about this approach can be found in [10,11].

4. Conclusion and perspectives

After giving the exact values of \( d^*(\mathcal{S}_1) \) and \( d^*(\mathcal{S}_2) \), we have proved some general bounds for \( d^*(\mathcal{S}_k) \) and \( d^*(\mathcal{G}_{k \times n}) \). Our “cut-and-paste” technique is based on the knowledge of the bounds for \( d^*(\mathbb{Z}^2) \). Using the same technique, analogous results can be obtained for strips and finite grids in other lattices, provided we have bounds (or, what is better, exact values) on the optimal density of an identifying code in these lattices, which is the case. For instance, in [3] one can find bounds for the optimal density of an identifying code in the king lattice, the triangular lattice and the hexagonal grid (for some of these lattices we can even find the exact values).

Our algorithm can also be used for subgraphs of such lattices whenever these subgraphs are fasciagraphs (e.g. strips, grids, etc.).

Moreover, there exists a more general concept of identifying code: for a given integer \( t \geq 1 \), a subset \( C \subseteq V \) of the vertices of a graph \( G = (V,E) \) is a \( t \)-identifying code of \( G \) if all the sets \( B_t(v) \cap C \) are nonempty and different, where \( B_t(v) \) denotes the ball of radius \( t \) centered on the vertex \( v \). In this paper we have only considered 1-identifying
codes. As in [3] they also considered $t$-identifying codes for $t \geq 1$. Hence, we can adapt our “cut-and-paste” technique to find bounds for $t$-identifying codes.

Finally, the algorithm can be easily modified so as to find the minimum cardinality of a $t$-identifying code of a fasciagraph $Ψ_n(G, X)$ in time $O(\log n)$ provided the size of $G$ is bounded by a fixed constant $k$. Indeed, it suffices to consider as valid a matrix of $M_{k \times (2t+1)}(\{0, 1\})$ if and only if there exists a $G_{2, 2t}$-identifying code of $Ψ_{2t+1}(G, X)$ such that $M_{ij} = 1 \iff v_i \in C$, and the algorithm remains the same.

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