GENERATION OF HOMOCLINIC OSCILLATION IN THE PHASE SYNCHRONIZATION REGIME IN COUPLED CHUA’S OSCILLATORS

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An experimental method for generating homoclinic oscillations using two nonidentical Chua’s oscillators coupled in unidirectional mode is described here. A homoclinic oscillation is obtained at the response oscillator in the weaker coupling limit of phase synchronization. Different phase locking phenomena of homoclinic oscillations with external periodic pulse have been observed when the frequency of the pulse is close to the natural frequency of the homoclinic oscillation or its subharmonics.

Keywords: Chua’s circuit; homoclinic oscillation; phase synchronization; phase locking.

1. Introduction

The generation of homoclinic chaos of Shil’nikov type [Silva, 1993] in lasers has been reported in [Allaria et al., 2001] and its phase synchronization (PS) with sinusoidal forcing has been confirmed. Its possible application in information encoding in the interspike interval [Mariño et al., 2003] of homoclinic chaos has also been explored recently for the purpose of secure communication. Encoding [Mariño et al., 2001] uses PS [Rosenblum et al., 1996] of message signal in the time sequence of spiking homoclinic chaos. In PS, the shape of spikes may be changed by channel noise but the time sequence of spike trains is not disturbed. On the other hand, complete synchronization (CS) in master-slave coupled chaotic oscillators, as proposed [Carroll et al., 1991] earlier for secure communication, is susceptible to channel noise [Sushchik et al., 1997], that causes loss of synchronization. Thus homoclinic chaos has an advantage over other deterministic form of chaos in communication applications, particularly, in the context of chaotic pulse position modulation scheme (CPPM) [Sushchik et al., 2000]. Moreover, the dynamics of interspike intervals of chaotic signal is a recent thrust area [Pavlov et al., 2001] in chaotic dynamics. The question still remains unanswered how sensory system of neuron assembly encodes external information in the form of interspike intervals. The spiking train of homoclinic chaos has similarity with the spiking neurons [Izhikevich, 2000; Meucci et al., 2002] in response to external stimuli. Studies on PS of homoclinic chaos with external stimulation may help understand how neurons communicate information with each other. In this context, we have attempted an experiment with two coupled Chua’s oscillators for the generation of homoclinic oscillations as reported in this paper. A homoclinic orbit is a bounded dynamical trajectory asymptotic to the equilibrium point of a model flow in both forward and backward times. The homoclinic oscillation is defined here.
as repeated cycles of homoclinic trajectory, which tends to one of the mirror symmetric stable focus of Chua’s oscillator and moves spirally away from it.

Numerical methods [Doedel & Kernévez, 1986] are available for locating homoclinic orbits, which rely either on continuation of a limit cycle to large period as it approaches homoclinic orbit or on numerical integration of shooting orbits in the stable and unstable manifolds of the equilibrium, and then computing a distance between them. Further improvements on these methods have been included in the numerical tool HOMCONT [Champneys et al., 1996], but it involves the rigors of numerical algorithm. Experimental studies on imperfect homoclinic bifurcation using van der Pol oscillator has also been found in literature [Glendinning et al., 2001]. It is often found [Postnov et al., 1999a; Sherman & Rinzel, 1992] in coupled neural oscillators that two limit cycle oscillations lead to PS in the weaker coupling limit when the limit cycles are close to the homoclinic bifurcation. Homoclinic bifurcation has been established in [Postnov et al., 1999b] as a mechanism of PS. Two coupled Chua’s oscillators in drive-response mode can, indeed, generate homoclinic oscillations in the weaker coupling limit of PS as shown in our experiment. The homoclinic orbit at the response repeats its trajectory with a time period of the limit cycle oscillation of the driver. Experimental evidence of phase locking on homoclinic oscillation to external periodic pulses is also presented. In the next section, the experimental setup of generating homoclinic oscillations is elaborated. The mechanism of homoclinic oscillations in coupled Chua’s Oscillator is explained in Sec. 3. In Sec. 4, phase locking of homoclinic oscillations with external periodic pulse is described with a conclusion in Sec. V.

2. Experimental Set Up

Existence of homoclinic orbits has been proved [Chua et al., 1986] numerically in Chua’s circuit as a rigorous proof of chaos in such system. The homoclinic orbit has been identified numerically as bisymptotic to a saddle focus in a 3-D space of Chua’s circuit [Silva, 1993; Rulkov & Sushchik, 1997] and in Colpitts oscillator [Feo et al., 2000]. However, it exists for a moderate amount of time before the trajectory veers away from the homoclinic orbit. The experimental circuit with two unidirectionally coupled nonidentical Chua’s oscillators, as shown in Fig. 1, can generate homoclinic oscillation, which repeats its homoclinic cycles for long periods of time. Each Chua’s oscillator consists of energy storing elements as one inductor $L_{1,2}$ with leakage resistance $R_{01,02}$, two capacitors $C_{1,3}, C_{2,4}$ and a nonlinear resistance. The governing equations of the coupled oscillators are given by

\begin{align}
\frac{dV_{C_1}}{dt} &= \frac{G_1}{C_1}[(V_{C_2} - V_{C_1}) - R_d f(V_{C_1})] \quad (1a) \\
\frac{dV_{C_2}}{dt} &= \frac{1}{C_2} [I_{L_1} - G_1(V_{C_2} - V_{C_1})] \quad (1b) \\
\frac{dI_{L_1}}{dt} &= -\frac{1}{L_1}(V_{C_2} + R_{01}I_{L_1}) \quad (1c) \\
\frac{dV_{C_3}}{dt} &= \frac{G_2}{C_3}[(V_{C_4} - V_{C_3}) - R_r g(V_{C_3})] \\
&\quad + \frac{1}{C_3 R_c} (V_{C_1} - V_{C_3}) \quad (1d) \\
\frac{dV_{C_4}}{dt} &= \frac{1}{C_4} [I_{L_2} - G_2(V_{C_4} - V_{C_3})] \quad (1e) \\
\frac{dI_{L_2}}{dt} &= -\frac{1}{L_2}(V_{C_4} + R_{02}I_{L_2}) \quad (1f)
\end{align}

where $f(*)$ and $g(*)$ represent the nonlinear resistances of the driver and the response circuits respectively as given by

\begin{align}
f(V_{C_1}) &= G_b V_{C_1} + 0.5(G_a - G_b)(|V_{C_1} + 1| - |V_{C_1} - 1|) \quad (1g) \\
g(V_{C_3}) &= G'_b V_{C_3} + 0.5(G'_a - G'_b)(|V_{C_3} + 1| - |V_{C_3} - 1|) \quad (1h)
\end{align}

and $G_1 = 1/R_d$, $G_2 = 1/R_c$ and $f(V_{C_1})$, $g(V_{C_3})$ are the piecewise-linear function representing the DP characteristics of the nonlinear resistances of the driver and the response, respectively. $Ga(Ga')$ and $Gb(Gb')$ are the inner and outer slopes of the characteristics of nonlinear resistance of the driver (response).

The parameters of the driver oscillator are selected for single scroll oscillations in the period-doubling bifurcation regime [Chua et al., 1993]. In this regime, the driver oscillation is single scroll periodic (period-3, period-4, period-5 even higher period) of large amplitude, which crosses from either (depending upon the initial point) of the mirror symmetric outer regions to the inner region near origin [Chua et al., 1986; Kennedy, 1993] as shown in Fig. 2. The large amplitude of single scroll limit cycles is
Fig. 1. Two Chua’s oscillators coupled in drive-response mode: $R_d$ and $R_r$ are the control parameters for different regimes of oscillation in the driver and response, respectively. $R_c$ decides the coupling strength, which is strong for lower value and vice versa. $V_s$ is the external pulse source.

an important criterion for the generation of homoclinic oscillations in the response oscillator. The amplitude of the limit cycle oscillation of the driver is not large enough, in the period-doubling bifurcation regime, to cross to the inner region. Before coupling, the response oscillator is kept in a nonoscillating state (stable focus) with appropriate choices of circuit parameters. For strong coupling, the response oscillator as forced by the limit cycle oscillation of the driver, produces limit cycle oscillation in complete synchrony [Roy et al., 2003] with the driver. Both the amplitudes and phases of coupled oscillators are correlated in this CS regime. As the coupling strength is made weaker and weaker, successive synchronization regimes of lag synchronization (LS), intermittent lag synchronization (ILS) and PS have been observed [Roy et al., 2003; Rosenblum et al., 1997]. The amplitudes of the coupled oscillators are uncorrelated in PS regime but the phases still remain correlated. In this PS regime, for a critical coupling strength, the response trajectory spirals away from the stable focus as driven out by the limit cycle oscillation of the driver. But it again tends to the stable focus at the instant the trajectory of the limit cycle oscillation of the driver crosses from mirror symmetric outer region to the inner region as deep into the negative region near origin. It is possible only when the driver oscillation is large enough. The mechanism of homoclinic oscillation is given in detail in the next section. The homoclinic orbits in 3-D space as shown in Fig. 3 are reconstructed using embedding technique. The homoclinic orbits are reconstructed from measured $V_{C3}(t)$ time series using delay coordinates as $V_{C3}(t-5\tau)$, $V_{C3}(t+5\tau)$ and $V_{C3}(t+15\tau)$ along $X$-axis, $Y$-axis and $Z$-axis respectively. $\tau$ is an appropriately chosen delay time. Finally, a phase velocity measure is used to confirm that the
response trajectory reaches the equilibrium. The phase velocity \( V_\phi \) is also calculated from the measured time series of \( V_{C3}(t) \) using delay coordinates as given by

\[
V_\phi = \sqrt{v_{t-\tau}^2 + v_{t+\tau}^2 + v_{t+3\tau}^2}
\]

where \( v_t = \frac{dV_{C3}(t)}{dt} \) (2)

where \( V_{C3}(t) \) is the node voltage measured at capacitor \( C_3 \) in response oscillator. The phase velocity of a trajectory is zero at equilibrium. This is taken as a basis whether a trajectory tends to equilibrium or is close to it. The response trajectory is a limit cycle close to equilibrium for coupling strength both higher and lower than a critical coupling as shown in Fig. 4(a), when the phase velocity is close to zero. The phase velocity reaches zero as shown in Fig. 4(b) when the homoclinic trajectory reaches the equilibrium at critical coupling strength. The coupling strength acts as control parameter once the driver parameters are

Fig. 2. Three eigenplanes (a) of Chua’s oscillator, two in mirror symmetric outer regions \( D_1, D_{-1} \) and one in the inner region \( D_0 \). \( E^r(P_-) \) is the eigenvector and \( E^c(P_-) \) is eigenspace corresponding to real and complex eigenvalues at equilibrium \( P_- \) in outer region \( D_{-1} \) (reproduced from [Kennedy, 1993]). (b) Phase portrait of period-4 orbit of the driver for \( R_d = 1516 \) \( \Omega \), oscilloscope display of capacitor voltages \( V_{C2} \) (CH1:200 mV/div) and \( V_{C3} \) (CH2:630 mV) with DC coupling to capacitor nodes.

Fig. 3. Homoclinic trajectory in 3-D space (in red lines): (a) period-4 homoclinic orbit, \( R_d = 1516 \) \( \Omega \), \( R_c = 1750 \) \( \Omega \), \( R_e = 10.96 \) k\( \Omega \). (b) period-5 homoclinic orbit, \( R_d = 1499 \) \( \Omega \), \( R_c = 1879 \) \( \Omega \), \( R_e = 18.36 \) k\( \Omega \). Homoclinic orbits in 3-D space have been reconstructed using delay coordinate from measured voltage \( V_{C3}(t) \) at capacitor \( C_3 \). \( V_{C3}(t-5\tau) \) along the X-axis, \( V_{C3}(t+5\tau) \) along the Y-axis and \( V_{C3}(t+15\tau) \) along the Z-axis are plotted where \( \tau \) is an appropriately chosen delay time.
Fig. 4. Phase velocity $V_\phi$ with time (ms): (period-5 homoclinic cycle, $R_d = 1499$ $\Omega$, $R_r = 1879$ $\Omega$). $V_\phi$ (in black dots) varies periodically with several minima. For coupling stronger than critical coupling (a) $R_c = 12.14$ k$\Omega$, the minima are close to zero indicating the trajectory is a limit cycle close to equilibrium. The minima (at 5.13 ms and 7 ms) are exactly at zero for critical coupling (b) $R_c = 18.36$ k$\Omega$, when homoclinic trajectory reaches equilibrium. Measured voltages $V_{C3}$ (blue line) and $V_{C4}$ (red line) reach zero simultaneously when phase velocity is zero as indicated by green arrows. Voltages are measured with AC coupling of oscilloscope when the equilibrium is set at zero.

3. Mechanism of Homoclinic Oscillation

The uncoupled response oscillator at rest has three equilibria [Chua et al., 1986; Kennedy, 1993], one unstable focus at origin as shown in Fig. 2 (in the inner region $D_0$) and two mirror symmetric stable foci (in the outer regions $D_1$ and $D_{-1}$). The unstable focus has one real positive eigenvalue and two complex conjugate eigenvalues with real negative parts. Each stable focus of the response has one real negative eigenvalue and two complex conjugate eigenvalues with negative real parts. If an external periodic pulse is forced, a spiking trajectory moves away from one of the mirror symmetric stable focus (depending upon the initial point) at the start of the pulse. At the end of the pulse, the spiking trajectory spirally tends to the stable focus due to the real negative part of the complex eigenvalue until the pulse is repeated again. Instead of forcing a periodic pulse, another self-oscillating Chua’s oscillator as driver is coupled in unidirectional mode to the response oscillator for the generation of homoclinic oscillation as discussed in Sec. 2. The driver parameters are selected for single scroll oscillation near period-4, period-5 limit cycles, when each of the mirror symmetric equilibria is a saddle focus with one real negative and two complex conjugate eigenvalues with positive parts. The eigenvalues of the coupled oscillator (period-4) are given in Table 1. The limit cycle trajectory (period-4, period-5 or higher orbits) of the driver spirals away around the equilibrium (say, in $D_1$ region) and moves to the inner region $D_0$. In this inner region, the trajectory is folded back by the real unstable eigenvector of inner region $D_0$ and ultimately comes back along the real eigenvector of $D_1$ region as shown in Fig. 2(b). Alternatively, for period-4 and higher periodic orbits, the driver trajectory may (depending upon the initial point) expand spirally from the other outer region ($D_{-1}$) to the $D_0$ region and even crosses to the positive region near origin. Subsequently, the trajectory of the driver twisted around

<table>
<thead>
<tr>
<th>Circuit</th>
<th>Inner region</th>
<th>Outer region</th>
</tr>
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<tbody>
<tr>
<td>Driver</td>
<td>$4.483, -1.291 \pm 3.098i$</td>
<td>$-4.324, -0.094 \pm 3.406i$</td>
</tr>
<tr>
<td>Response</td>
<td>$4.424, -1.234 \pm 3.548i$</td>
<td>$-4.348, 0.045 \pm 2.921i$</td>
</tr>
</tbody>
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the unstable real eigenvector of $D_0$ region, comes back to its original outer region ($D_{-1}$). Forced by the period-4 or period-5 oscillation of the driver, the response oscillator follows the driver with a time lag for weaker coupling [Roy et al., 2003; Rosenblum et al., 1997] until the driver trajectory reaches zero. At this instant, the response trajectory folded back fast to the equilibrium in outer region along its real eigenvector with negative eigenvalue as shown in red trace in Fig. 5(a). When the driver trajectory (blue trace) grows spirally again to repeat its periodic cycle, the response trajectory (red trace) is forced spirally away from the equilibrium and repeats its homoclinic cycle. Thus the coupled circuit generates a homoclinic oscillation at the response when the driver and response are phase locked. The phase difference between the driver and response remains bounded in time as shown in the phase difference plot in Fig. 5(b).

4. Choice of Coupling Strength

Complete synchronization of chaotic oscillators has been observed with strong coupling between the two nonidentical oscillators, while LS occurs with decrease in coupling strength from CS regime. PS has been found with further weakening of coupling strength [Rosenblum et al., 1997]. Intermittent lag synchronization (ILS) has also been observed [Boccaletti et al., 2000] between the regimes of LS and PS with decreasing coupling strength. The coupling strength also plays an important role in the generation of homoclinic oscillation in our coupled Chua’s oscillators since the coupled oscillators must be kept in the weaker coupling limit of PS.

For our experiment, the driver is kept either in single scroll chaotic mode while the response oscillator is kept in point attractor mode. CS to PS has been observed through LS and ILS when the coupling strength is decreased as reported elsewhere by the authors [Roy et al., 2003]. CS between the two coupled nonidentical Chua’s oscillators is obtained for strong coupling ($R_c = 345 \text{ k}\Omega$), LS ($R_c = 1.38 \text{ k}\Omega$) and ILS are obtained with further decrease in coupling strength ($R_c = 4.1 \text{ k}\Omega$). PS is obtained in the range of $R_c = 5 \text{ k}\Omega$. The PS regime has only been elaborated here, details of different synchronization regimes are given in [Roy et al., 2003]. The phase difference between the driver and response $\phi_{C1}(t) - \phi_{C3}(t)$ with time is plotted in red dots in Fig. 5(b), which is bounded to zero except for a few dots of $\pm \pi$ jumps. The phases are calculated from scalar signal using Hilbert transform. These error dots as $\pm \pi$ jumps occur when the driver signal changes the sign of its slope near zero crossing. The phases are calculated using Hilbert transform, which is sensitive to coordinate system [Roy et al., 2003].

5. Phase Locking with External Pulse

Forcing a periodic pulse at the driver capacitor
C1 (Fig. 1), the instabilities in homoclinic oscillation due to unavoidable parameter fluctuations have been stabilized in the response oscillator. The homoclinic oscillation readjusts its period to different stable homoclinic oscillations as forced by the periodic pulse. The amplitude of external forcing pulse is selected small enough while the pulse period is chosen as close to the time period of homoclinic cycle. The external pulse is phase locked to the homoclinic oscillations with different locking ratios. For the selected parameters, the natural period of the Chua’s oscillator (driver) is $T_n = 1/f_n = 390 \mu s$, $f_n$ is the natural frequency of homoclinic oscillation. The natural frequency $f_n$ of the Chua’s oscillator has been estimated from Fourier transform on measured time series data. When the pulse period is close to time period of homoclinic cycle, within a small time bound $T_s$ ($T_p \approx T_h \pm T_s$), 1:1 phase locking has been observed as shown in Fig. 6(a). For period-$n$ homoclinic cycle, time period of a homoclinic cycle is given by $T_h = nT_n$. Homoclinic cycle is a period-5 oscillation in Fig. 6(a) when the forcing pulse has an amplitude of 248 mV and time period $T_p = 1936 \mu s$. The time period of the homoclinic cycle is $T_h = 5T_n = 1950 \mu s$ ($T_s \approx 14 \mu s$). For pulse period $T_p = 1152 \mu s$ and amplitude 336 mV, the homoclinic cycle is a stable period-6 orbit in Fig. 6(b), whose time period is given by $T_h = 6T_n = 2340 \mu s$. The pulse is 1:2 phase locked to the period-6 homoclinic orbit ($T_p = T_h/2 - T_s = 1152 \mu s$ and $T_s \approx 18 \mu s$). For pulse amplitude of 420 mV and $T_p = 762 \mu s$, the homoclinic oscillation is stable at period-6 orbit in Fig. 6(c), whose time period is given by $T_h = 6T_n = 2340 \mu s$. The pulse is 1:3 phase locked to the period-6 homoclinic orbit ($T_p = T_h/3 - T_s = 762 \mu s$ and $T_s \approx 18 \mu s$). If $T_s$ is larger beyond a bounded limit, phase slips occur as shown in Fig. 7 for 1:3 phase locking with pulse of amplitude 440 mV and period $T_p = 749.7 \mu s$. 

Fig. 6. Phase locking with external pulse (a) 1:1 locking, pulse amplitude = 248 mV, $T_p = 1936 \mu s$ (b) 1:2 locking, pulse amplitude = 336 mV, $T_p = 1152 \mu s$ (c) 1:3 locking, pulse amplitude = 420 mV, $T_p = 762 \mu s$.

Fig. 7. Phase slips: pulse period $T_p = 749.7 \mu s$, 1:3 phase locking with intermediate phase slips.
\(T_p = T_h/3 - T_s = 749.7 \mu s\) and \(T_s \approx 30 \mu s\). The pulse amplitude is measured using the digital oscilloscope at source. Actual amplitude of the pulse at the capacitor \(C_1\) is much less since the pulse source \(V_s\) is connected to the capacitor with a series resistance \(R_s = 25 \text{ k}\Omega\).

6. Conclusion

An experiment using two Chua’s oscillators coupled in the PS regime is described, which can generate homoclinic oscillation for a long time. A phase velocity measure is used to confirm that the 3-D homoclinic trajectory reaches equilibrium for a critical coupling. Phase locking of homoclinic oscillation with external periodic pulse is possible and the locking ratio depends upon the frequencies of the driver and the forcing pulse. An aperiodic pulse train can be phase locked to the homoclinic oscillation if the variation in pulse intervals remain within a smaller bound. Message can be encoded in the varying time intervals of the aperiodic pulse, which problem is undertaken for our future work.

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