Positive and dead core solutions of singular Dirichlet boundary value problems with φ-Laplacian

Ravi P. Agarwal\textsuperscript{a,}*, Donal O’Regan\textsuperscript{b}, Svatoslav Staněk\textsuperscript{c}

\textsuperscript{a} Department of Mathematical Sciences, Florida Institute of Technology, 150 West University Boulevard, Melbourne, FL 32901-6975, USA
\textsuperscript{b} Department of Mathematics, National University of Ireland, Galway, Ireland
\textsuperscript{c} Department of Mathematical Analysis, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic

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Abstract

The paper discusses the existence of positive solutions, dead core solutions and pseudodead core solutions of the singular Dirichlet boundary value problem (BVP)

\begin{equation}
\phi(u'(t)))' = \lambda[f(t, u(t), u'(t)) + h(t, u(t), u'(t))],
\end{equation}

\begin{equation}
u(0) = A, \quad u(T) = A
\end{equation}

depending on the positive parameter \(\lambda\). Here \(\phi \in C^0(\mathbb{R})\) is increasing, \(f \in C^0([0, T] \times (0, A] \times \mathbb{R}),\ h \in C^0([0, T] \times [0, A] \times (\mathbb{R} \setminus \{0\}))\), \(f\) is singular at the value 0 of its first phase variable and \(h\) may be singular at the value 0 of its second phase variable.

We say that \(u \in C^1([0, T])\) is a positive solution of the BVP (1.1)\textsuperscript{\lambda} and (1.2) if \(\phi(u') \in C^1([0, T] \setminus A)\) where \(A = \{t \in [0, T] : u'(t) = 0\}\), \(u\) satisfies (1.2), \(0 < u \leq A\) on \([0, T]\) and (1.1)\textsuperscript{\lambda} holds for each \(t \in [0, T] \setminus A\).

A function \(u \in C^1([0, T])\) is said to be a dead core solution of the BVP (1.1)\textsuperscript{\lambda} and (1.2) if there exist \(0 < \alpha < \beta < T\) such that \(u(t) = 0\) for \(t \in [\alpha, \beta]\), \(0 < u \leq A\) for \(t \in [0, \alpha) \cup (\beta, T],\ \phi(u') \in C^1([0, T] \setminus [\alpha, \beta]),\ u\) satisfies (1.2) and (1.1)\textsuperscript{\lambda} holds for each \(t \in [0, T] \setminus [\alpha, \beta]\). The interval \([\alpha, \beta]\) is called the dead core of \(u\) (see e.g. [1,2]).

If \(\alpha = \beta\) in the definition of the dead core solution \(u\) of the BVP (1.1)\textsuperscript{\lambda} and (1.2), then \(u\) is said to be a pseudodead core solution of the BVP (1.1)\textsuperscript{\lambda} and (1.2) (see [3]).

* Corresponding author. Tel.: +1 321 674 7202; fax: +1 321 674 7412.
E-mail addresses: agarwal@fit.edu (R.P. Agarwal), donal.oregan@nuigalway.ie (D. O’Regan), stanek@inf.upol.cz (S. Staněk).

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Let \( \|x\| = \max\{|x(t)| : 0 \leq t \leq T\} \) denote the norm in \( C^0[0, T] \), \( L_1[0, T] \) be the set of measurable functions on \([0, T]\) and \( AC[0, A] \) be the set of absolutely continuous functions on \([0, A]\).

The aim of this paper is to discuss the existence of positive, dead core and pseudodead core solutions of the BVP (1.1) and (1.2).

Throughout the paper, the functions \( \phi, f, h \) satisfy the following assumptions

(H1) \( \phi \in C^0(\mathbb{R}) \) is odd and increasing, \( \lim_{x \to \infty} \phi(x) = \infty \),

\[
\begin{align*}
  & f \in C^0([0, T] \times [0, A] \times \mathbb{R}) \text{ is positive, } \lim_{x \to 0^+} f(t, x, y) = \infty \\
  & \text{for each } (t, y) \in [0, T] \times \mathbb{R} \text{ and } \phi(\cdot) \in C^0(\mathbb{R}).
\end{align*}
\]

(H2) \( f(t, x, y) \leq p(x)\omega(|y|) \) for \( (t, x, y) \in [0, T] \times (0, A] \times \mathbb{R} \), where

\[
\begin{align*}
  & p : (0, A] \to (0, \infty) \text{ is nonincreasing, } p \in L_1(0, A] \text{ and } \omega : [0, \infty) \to (0, \infty) \\
  & \text{is nondecreasing},
\end{align*}
\]

(H3) \( h \in C^0([0, T] \times [0, A] \times (\mathbb{R} \setminus \{0\})) \) is nonnegative and

\[
\begin{align*}
  & h(t, x, y) \leq K\gamma(|y|) \text{ for } (t, x, y) \in [0, T] \times (0, A] \times (\mathbb{R} \setminus \{0\}) \\
  & \text{where } K \in [0, \infty) \text{ and } \gamma : (0, \infty) \to (0, \infty) \text{ is nonincreasing},
\end{align*}
\]

(H4) \( \int_0^\infty \frac{e^{-(\phi^{-1}(s)) + \gamma(|\phi^{-1}(s)|)}}{\omega(\phi^{-1}(s)) \phi^{-1}(s)} \, ds = \infty \).

For each \( n \in \mathbb{N} \), define the functions \( f_n \in C^0([0, T] \times [0, A] \times \mathbb{R}) \), \( f_n^* \in C^0([0, T] \times [0, A] \times \mathbb{R}) \), \( h_n \in C^0([0, T] \times [0, A] \times \mathbb{R}) \) and \( h_n^* \in C^0([0, T] \times [0, A] \times \mathbb{R}) \) by the formulas

\[
\begin{align*}
  f_n(t, x, y) &= \begin{cases} 
    \min\{f(t, x, y), nx\} & \text{for } (t, x, y) \in [0, T] \times [0, A] \times \mathbb{R} \\
    0 & \text{for } (t, y) \in [0, T] \times \mathbb{R} \text{ and } x = 0,
  \end{cases} \\
  f_n^*(t, x, y) &= \begin{cases} 
    f_n(t, A, y) & \text{for } (t, x, y) \in [0, T] \times (A, \infty) \times \mathbb{R} \\
    f_n(t, x, y) & \text{for } (t, x, y) \in [0, T] \times [0, A] \times \mathbb{R} \\
    x & \text{for } (t, x, y) \in [0, T] \times (-\infty, 0) \times \mathbb{R},
  \end{cases} \\
  h_n(t, x, y) &= \begin{cases} 
    \min\{h(t, x, y), n|y|\} & \text{for } (t, x, y) \in [0, T] \times [0, A] \times (\mathbb{R} \setminus \{0\}) \\
    0 & \text{for } (t, x) \in [0, T] \times [0, A] \text{ and } y = 0.
  \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
  h_n^*(t, x, y) &= \begin{cases} 
    h_n(t, A, y) & \text{for } (t, x, y) \in [0, T] \times (A, \infty) \times \mathbb{R} \\
    h_n(t, x, y) & \text{for } (t, x, y) \in [0, T] \times [0, A] \times \mathbb{R} \\
    h_n(t, 0, y) & \text{for } (t, x, y) \in [0, T] \times (-\infty, 0) \times \mathbb{R}.
  \end{cases}
\end{align*}
\]

Then

\[
\begin{align*}
  & f_n(t, x, y) \leq f(t, x, y) \leq p(x)\omega(|y|), \quad (t, x, y) \in [0, T] \times (0, A] \times \mathbb{R}, \quad (1.3) \\
  & h_n(t, x, y) \leq h(t, x, y) \leq K\gamma(|y|), \quad (t, x, y) \in [0, T] \times [0, A] \times (\mathbb{R} \setminus \{0\}). \quad (1.4)
\end{align*}
\]

Since \( f_n^*(t, 0, y) = 0 \) for \( (t, y) \in [0, T] \times \mathbb{R} \), \( h_n^*(t, x, 0) = 0 \) for \( (t, x) \in [0, T] \times \mathbb{R} \), \( \lim_{n \to \infty} f_n^*(t, x, y) = f(t, x, y) \) uniformly on each compact set of \([0, T] \times (0, A] \times \mathbb{R}\) and \( \lim_{n \to \infty} h_n^*(t, x, y) = h(t, x, y) \) uniformly on each compact set of \([0, T] \times [0, A] \times (\mathbb{R} \setminus \{0\}) \), we discuss the existence of positive, dead core and pseudodead core solutions of the BVP (1.1) and (1.2) by considering solutions of the family of auxiliary regular differential equations

\[
(\phi(u'(t)))' = \lambda [f_n^*(t, u(t), u'(t)) + h_n^*(t, u(t), u'(t))], \quad (1.5)
\]

with \( n \to \infty \). We note that this technique to obtain positive, dead core and pseudodead core solutions of the BVP (1.1) and (1.2) is related to the technique presented in [3] where the authors discuss positive and dead core solutions to the BVP

\[
\begin{align*}
  u''(t) + r(t, u(t)) &= \lambda g(t, u(t)), \\
  u'(a) &= 0, \quad \beta u'(b) + \alpha u(b) = A, \quad \beta \geq 0, \quad \alpha, A > 0,
\end{align*}
\]

where \( r \in C^0((a, b] \times [0, \infty)) \) is nonnegative, \( r(t, 0) = 0 \) for \( t \in (a, b] \) and \( g \in C^0([a, b] \times (0, \frac{A}{\alpha})) \) is positive (see [3, Theorem 17]).
By a solution of the BVP (1.5)$_n^\lambda$ and (1.2) we mean a function $u_n \in C^1[0, T]$ such that $\phi(u'_n) \in C^1[0, T]$, $u_n$ satisfies (1.2) and (1.5)$_n^\lambda$ (with $u = u_n$) holds for each $t \in [0, T]$.

It is useful to introduce also the notion of a solution of the BVP (1.1)$_n^\lambda$ and (1.2). We say that $u$ is a solution of the BVP (1.1)$_n^\lambda$ and (1.2) if there exists a sequence $\{k_n\} \subset \mathbb{N}$, $\lim_{n \to \infty} k_n = \infty$ such that $\lim_{n \to \infty} u_{k_n} = u$ in $C^1[0, T]$ where $u_{k_n}$ is a solution of the BVP (1.5)$_{k_n}^\lambda$, (1.2). In Section 3 (see Theorem 3.1) we will show that any solution of the BVP (1.1)$_n^\lambda$ and (1.2) is either a positive solution or a dead core solution or a pseudodead core solution of this problem.

For the solvability of the BVP (1.5)$_n^\lambda$ and (1.2), we use the following existence principle which is the special case of the existence principle presented in [4, Theorem 2.1].

**Theorem 1.1.** Let $\lambda \in (0, \infty)$ and $n \in \mathbb{N}$. Suppose that there exist positive constants $S_0$ and $S_1$ independent of $\mu$ such that $\|u\| < S_0$ and $\|u'\| < S_1$ for any solution $u$ to the one parameter differential equations

$$
(\phi(u'(t)))' = \mu \lambda [f_n^\lambda(t, u(t), u'(t)) + h_n^\lambda(t, u(t), u'(t))], \quad \mu \in [0, 1]
$$

satisfying (1.2). Then the BVP (1.5)$_n^\lambda$ and (1.2) has a solution.

The rest of the paper is organized as follows. Section 2 is devoted to the regular BVP (1.5)$_n^\lambda$ and (1.2). Using Lemmas 2.1 and 2.4 and Theorem 1.1, it is proved that this problem has a solution for each positive $\lambda$ and $n \in \mathbb{N}$ (Lemma 2.5). Lemmas 2.6 and 2.7 give properties of solutions to the BVP (1.5)$_n^\lambda$ and (1.2) which are used in the next section. The main results are presented in Section 3. Under assumptions (H$_1$)-(H$_4$), for each positive $\lambda$, the BVP (1.1)$_n^\lambda$ and (1.2) has a solution. This solution is either a positive solution or a pseudodead core solution or a dead core solution (Theorem 3.1). For $\lambda$ sufficiently small, the BVP (1.1)$_n^\lambda$ and (1.2) has only positive solutions (Corollary 3.2). If $\lambda$ is sufficiently large, then the BVP (1.1)$_n^\lambda$ and (1.2) has only dead core solutions (Corollary 3.3). Two examples demonstrate the application of our existence results.

### 2. Auxiliary regular BVP (1.5)$_n^\lambda$ and (1.2)

Properties of solutions to the BVP (1.5)$_n^\lambda$ and (1.2) are given in the following lemma.

**Lemma 2.1.** Let $\lambda > 0$ and let $u_n$ be a solution of the BVP (1.5)$_n^\lambda$ and (1.2). Then

$$
0 \leq u_n(t) \leq A \quad \text{for} \quad t \in [0, T],
$$

there exist $\alpha_n, \beta_n$, $0 < \alpha_n \leq \beta_n < T$, such that

$$
\begin{align*}
&u_n'(t) < 0 \quad \text{for} \quad t \in [0, \alpha_n), \\
&u_n'(t) = 0 \quad \text{for} \quad t \in [\alpha_n, \beta_n], \\
&u_n'(t) > 0 \quad \text{for} \quad t \in (\beta_n, T]
\end{align*}
$$

and $u_n'$ is increasing on the intervals $[0, \alpha_n]$ and $[\beta_n, T]$. If $\alpha_n < \beta_n$ then

$$
u_n(t) = 0 \quad \text{for} \quad t \in [\alpha_n, \beta_n],$$

**Proof.** Suppose that $\min\{u_n(t) : 0 \leq t \leq T\} = u_n(\tau) < 0$. Then $\tau \in (0, T)$, $u_n'(\tau) = 0$ and $(\phi(u_n'(t)))'_{t=\tau} \geq 0$, which contradicts $(\phi(u_n'(t)))'_{t=\tau} = \lambda f_n^\lambda(\tau, x_n(\tau), 0) = \lambda u_n(\tau) < 0$. Hence $u_n' \geq 0$ on $[0, T]$. Therefore $(\phi(u_n'(t)))'_{t=0} = \lambda [f_n^\lambda(0, u_n(0), u_n'(0)) + h_n^\lambda(0, u_n(0), u_n'(0))] \geq 0$ for $t \in [0, T]$ and since $\phi$ is increasing on $\mathbb{R}$ by (H$_1$), $u_n'$ is nondecreasing on $[0, T]$. From $(\phi(u_n'(t)))'_{t=0} > 0$ and $u_n'(\xi_n) = 0$ for some $\xi_n \in (0, T)$ which follows from $u_n(0) = u_n(T) = A$, we have $u_n'(0) < 0$ and $u_n'(T) > 0$. Hence there exist $0 < u_n < \beta_n < T$ such that (2.2) is true and also $u_n < A$ on $[0, T]$. Since $0 < u_n < A$ on $[0, \alpha_n) \cup (\beta_n, T]$, we have

$$
(\phi(u_n'(t)))'_{t=\tau} = \lambda [f_n^\lambda(\tau, u_n(\tau), u_n'(\tau)) + h_n^\lambda(\tau, u_n(\tau), u_n'(\tau))] > 0
$$

for $t \in [0, \alpha_n) \cup (\beta_n, T]$, and from $u_n'(\alpha_n) = 0 = u_n'(\beta_n)$ and (2.2) we see that $\phi(u_n')$ is increasing on the intervals $[0, \alpha_n]$ and $[\beta_n, T]$ and the same is true for $u_n'$.

Assume that $\alpha_n < \beta_n$. Then $u_n(t) = B$ for $t \in [\alpha_n, \beta_n]$ where $B \in [0, A)$. If $B > 0$ then

$$
f_n(t, u_n(t), u_n'(t)) + h_n(t, u_n(t), u_n'(t)) = f_n(t, B, 0) + h_n(t, B, 0)
$$

$$
= \min\{f(t, B, 0), nB\}.
for $t \in [\alpha_n, \beta_n]$ and therefore for these $t$ we have

$$ (\phi(u'_n(t)))' = \lambda \min\{f(t, B, 0), nB\} > 0, $$

which is impossible. Hence (2.3) holds. □

**Remark 2.2.** Lemma 2.1 shows (see (2.1)) that any solution $u_n$ of the BVP (1.5)$_n^\lambda$ and (1.2) satisfies

$$ (\phi(u'_n(t)))' = \lambda [f_n(t, u_n(t), u'_n(t)) + h_n(t, u_n(t), u'_n(t))] $$

for $t \in [0, T]$.

**Remark 2.3.** From the proof of Lemma 2.1 it follows that if a solution $u_n$ of the BVP (1.5)$_n^\lambda$ and (1.2) satisfies $u_n > 0$ on $[0, T]$ then $u'_n$ has a unique zero on $[0, T]$. We now give *a priori* bounds for solutions of the BVP (1.5)$_n^\lambda$ and (1.2).

**Lemma 2.4.** There exists a positive constant $S$ depending on $\lambda > 0$ and independent of $n$ such that

$$ \|u'_n\| < S $$

(2.4)

for any solution $u_n$ of the BVP (1.5)$_n^\lambda$ and (1.2).

**Proof.** Let $u_n$ be a solution of the BVP (1.5)$_n^\lambda$ and (1.2). By Lemma 2.1, $u_n$ satisfies (2.1) and (2.2) with some $0 < \alpha_n \leq \beta_n < T$ and $u'_n$ is nondecreasing on $[0, T]$. Hence

$$ \|u'_n\| = \max\{|u'_n(0)|, u'_n(T)\}. $$

(2.5)

From (1.3), (1.4) and (2.2) and Remark 2.2, it follows that

$$ (\phi(u'_n(t)))' \leq \lambda [p(u_n(t))\omega(-u'_n(t)) + K\gamma(-u'_n(t))], \quad t \in [0, \alpha_n) $$

and

$$ (\phi(u'_n(t)))' \leq \lambda [p(u_n(t))\omega(u'_n(t)) + K\gamma(u'_n(t))], \quad t \in (\beta_n, T]. $$

Now integrating

$$ \frac{(\phi(u'_n(t)))' u'_n(t)}{\omega(-u'_n(t)) + \gamma(-u'_n(t))} > \lambda [p(u_n(t)) + K |u'_n(t)\| \tag{2.6} $$

from 0 to $\alpha_n$ and

$$ \frac{(\phi(u'_n(t)))' u'_n(t)}{\omega(u'_n(t)) + \gamma(u'_n(t))} < \lambda [p(u_n(t)) + K |u'_n(t)\| \tag{2.7} $$

from $\beta_n$ to $T$ yields

$$ \int_0^{\phi(u'_n(0))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} ds \leq \lambda \int_{u_n(\alpha_n)}^A (p(s) + K) ds \leq \lambda \int_0^A (p(s) + K) ds \tag{2.8} $$

and

$$ \int_0^{\phi(u'_n(T))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} ds \leq \lambda \int_{u_n(\beta_n)}^A (p(s) + K) ds \leq \lambda \int_0^A (p(s) + K) ds. \tag{2.9} $$
By (H₄), there exists a positive constant L (depending on λ) such that
\[\int_0^L \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} \, ds > \lambda \int_0^A (p(s) + K) \, ds.\]

From this, (2.8) and (2.9) imply \(\max\{\phi(|u'_n(0)|), \phi(u''_n(T))\} < L\). Consequently max\(|u'_n(0)|, u''_n(T)\} < \phi^{-1}(L)\) and using (2.5), (2.4) is true with \(S = \phi^{-1}(L)\). \(\Box\)

**Lemma 2.1** and **Theorem 1.1** yield the following existence result for the BVP \((1.2)\) and \((1.2)\).

**Lemma 2.5.** Let \(\lambda > 0\). Then for each \(n \in \mathbb{N}\), there exists a solution \(u_n\) of the BVP \((1.2)\) and \((1.2)\) satisfying (2.1) and (2.2) with some \(0 < \alpha_n \leq \beta_n < T\).

**Proof.** Fix \(n \in \mathbb{N}\). The existence of a solution of the BVP \((1.2)\) and \((1.2)\) will be proved if, by **Theorem 1.1**, there exist positive constants \(S_0\) and \(S_1\) (depending on \(\lambda\)) such that \(|u| < S_0\) and \(|u''| < S_1\) for any solution \(u\) of the family of the BVPs \((1.2)\) and \((1.2)\) where \(\mu \in [0, 1]\). If \(\mu = 0\) then \(u = A\) is the unique solution of the BVP \((1.6)\) and \((1.2)\). Let \(\mu \in (0, 1]\) and \(S\) be a positive constant in **Lemma 2.4**. From the proofs of **Lemma 2.1** and **Lemma 2.4**, it follows that \(0 \leq u \leq A\) on \([0, T]\) and \(|u''| < S\) for any solution \(u\) of the BVP \((1.6)\) and \((1.2)\). Hence the assumptions of **Theorem 1.1** are satisfied with \(S_0 = A + 1\) and \(S_1 = S\). Consequently there exists a solution \(u_n\) of the BVP \((1.2)\) and \((1.2)\) and, by **Lemma 2.1**, \(u_n\) satisfies (2.1) and (2.2). \(\Box\)

**Lemma 2.5** shows that for any \(\lambda > 0\), the BVP \((1.2)\) and \((1.2)\) has a solution \(u_n\) for all \(n \in \mathbb{N}\). The next three lemmas give some properties of solutions \(u_n\), which will be used in Section 3.

**Lemma 2.6.** Let \(\lambda > 0\) and let \(u_n\) be a solution of the BVP \((1.2)\) and \((1.2)\). Then \(\{u'_n\}\) is equicontinuous on \([0, T]\).

**Proof.** By **Lemma 2.1** and **Lemma 2.4**, there exists a positive constant \(S\) and there exist \(0 < \alpha_n \leq \beta_n < T\) such that for each \(n \in \mathbb{N}\),
\begin{align*}
0 &\leq u_n(t) \leq A, \quad t \in [0, T], \quad (2.10) \\
\|u'_n\| &< S, \quad (2.11)
\end{align*}
\(u'_n\) is nondecreasing on \([0, T]\), \(u'_n < 0\) on \([0, \alpha_n)\), \(u'_n = 0\) on \([\alpha_n, \beta_n]\) and \(u'_n > 0\) on \((\beta_n, T]\). In addition, if \(\alpha_n < \beta_n\) for some \(n \in \mathbb{N}\), then \(u_n(t) = 0\) for \(t \in [\alpha_n, \beta_n]\).

In order to show that \(\{u'_n\}\) is equicontinuous on \([0, T]\), define the functions \(G \in C^0[0, \infty)\) and \(P \in AC[0, A]\) by the formulas
\begin{align*}
G(v) &= \int_0^v \frac{\phi(u'(s))}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} \, ds, \quad v \in [0, \infty), \quad (2.12) \\
P(v) &= \int_0^v (p(s) + K) \, ds, \quad v \in [0, A], \quad (2.13)
\end{align*}
where \(K > 0\) appears in (H₃). Let \(0 \leq t_1 < t_2 \leq T\). If \(t_2 \leq \alpha_n\), integrating (see (2.6))
\[\frac{(\phi(u'_n(t)))(u'_n(t))}{\omega(-u'_n(t)) + \gamma(-u'_n(t))} > \lambda[p(u_n(t)) + K]u'_n(t), \quad t \in [0, \alpha_n]\]
over \([t_1, t_2]\) yields
\[0 < G(-u'_n(t_1)) - G(-u'_n(t_2)) < \lambda[p(u_n(t_1)) - p(u_n(t_2))]. \quad (2.15)\]
If \(t_1 \geq \beta_n\), integrating (see (2.7))
\[\frac{(\phi(u'_n(t)))(u'_n(t))}{\omega(u'_n(t)) + \gamma(u'_n(t))} < \lambda[p(u_n(t)) + K]u'_n(t), \quad t \in (\beta_n, T]\]
over \([t_1, t_2]\) gives
\[0 < G(u'_n(t_2)) - G(u'_n(t_1)) < \lambda[p(u_n(t_2)) - p(u_n(t_1))]. \quad (2.17)\]
If $t_1 < \alpha_n < t_2$, integrating (2.14) from $t_1$ to $\alpha_n$ yields

$$0 < G(-u'_n(t_1)) - \lambda \left[ P(u_n(t_1)) - P(u_n(\alpha_n)) \right]$$

and if $t_1 < \beta_n < t_2$, integrating (2.16) from $\beta_n$ to $t_2$ gives

$$0 < G(u'_n(t_2)) - \lambda \left[ P(u_n(t_2)) - P(u_n(\beta_n)) \right].$$

Finally, if $\alpha_n < \beta_n$ and $\alpha_n \leq t_1 < t_2 \leq \beta_n$, we have $u'_n(t_1) = u'_n(t_2) = 0$. Since $\{u_n\}$ is equicontinuous on $[0, T]$, which follows from the boundedness of $\{u_n\}$ in $C^1[0, T]$ and $P$ is increasing and absolutely continuous on $[0, A]$, we see that $\{P(u_n)\}$ is equicontinuous on $[0, A]$. From this, given $\varepsilon > 0$ we can find $\delta > 0$ such that for each $0 < t_1 < t_2 \leq T, t_2 - t_1 < \delta$, we have

$$\left| P(u_n(t_2)) - P(u_n(t_1)) \right| < \varepsilon, \quad n \in \mathbb{N}.$$  

We now show that $\{\hat{G}(u'_n)\}$ is equicontinuous on $[0, T]$ where $\hat{G} \in C^0(\mathbb{R})$ is defined by the formula

$$\hat{G}(v) = \begin{cases} G(v) & \text{for } v \in [0, \infty) \\ -G(-v) & \text{for } v \in (-\infty, 0). \end{cases}$$

Let $0 \leq t_1 < t_2 \leq T, t_2 - t_1 < \delta$. Then from (2.15), (2.17) and (2.20) it follows that

$$0 < \hat{G}(u'_n(t_2)) - \hat{G}(u'_n(t_1)) < \lambda \varepsilon \quad \text{if } 0 \leq t_1 < t_2 \leq \alpha_n$$

$$0 < \hat{G}(u'_n(t_2)) - \hat{G}(u'_n(t_1)) < \lambda \varepsilon \quad \text{if } \beta_n \leq t_1 < t_2 \leq T.$$

If $\alpha_n = \beta_n$ then (see (2.18)–(2.20))

$$0 < \hat{G}(u'_n(t_2)) - \hat{G}(u'_n(t_1)) < 2\lambda \varepsilon \quad \text{if } t_1 < \alpha_n = \beta_n < t_2$$

and if $\alpha_n < \beta_n$ and either $t_1 < \alpha_n < t_2$ or $\alpha_n \leq t_1 < t_2 \leq \beta_n$ or $t_1 < \beta_n < t_2$, we have

$$0 \leq \hat{G}(u'_n(t_2)) - \hat{G}(u'_n(t_1)) < 2\lambda \varepsilon.$$

Hence $0 \leq \hat{G}(u'_n(t_2)) - \hat{G}(u'_n(t_1)) < 2\lambda \varepsilon$ for $t_1, t_2 \in [0, T], 0 < t_2 - t_1 < \delta$, and consequently $\{\hat{G}(u'_n)\}$ is equicontinuous on $[0, T]$. Since $\hat{G}$ is continuous and increasing on $\mathbb{R}$ and $\| u'_n \| < S$ by (2.11), $\{u'_n\}$ is equicontinuous on $[0, T]$. \hfill \Box

**Lemma 2.7.** There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$

$$\sup \{\| u'_n \| : n \in \mathbb{N} \} < \frac{A}{T}$$

where $u_n$ is a solution of the BVP (1.5)$_\lambda$ and (1.2).

**Proof.** Let $\lambda_0 > 0$ satisfy

$$\lambda_0 < \left( \int_0^A (p(s) + K) \, ds \right)^{-1} \int_0^{\phi(A/T)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} \, ds.$$

Then

$$\lambda_0 = \left( \int_0^A (p(s) + K) \, ds \right)^{-1} \int_0^{\phi(A/T - \varepsilon)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} \, ds.$$
where \( \varepsilon \in (0, \frac{A}{T}) \). Let \( \lambda \in (0, \lambda_0] \) and \( u_n \) be a solution of the BVP (1.5)_n^\lambda and (1.2). Since (see (2.5), (2.8) and (2.9)) \( \|u_n'\| = \max\{|u_n'(0)|, u_n'(T)| \) and
\[
\int_0^1 \phi(u_n') \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} ds < \lambda \left( \int_0^A (p(s) + K) ds \right)
\leq \lambda_0 \left( \int_0^A (p(s) + K) ds \right),
\]
we have \( \|u_n'\| < \frac{A}{T} - \varepsilon \). Hence (2.22) is true.
\[ \square \]

**Remark 2.8.** Let \( \lambda_0 > 0 \) be given in Lemma 2.7. If \( \lambda \in (0, \lambda_0] \) then \( u_n > 0 \) on \([0, T]\) for any solution \( u_n \) of the BVP (1.5)_n^\lambda and (1.2). Suppose not. Then there exist \( \lambda_1 \in (0, \lambda_0], n_0 \in \mathbb{N} \) and a solution \( u_{n_0} \) of the BVP (1.5)_n^\lambda and (1.2) such that \( u_{n_0}(\xi) = 0 \) for some \( \xi \in (0, T) \). Hence \( A = u_{n_0}(T) - u_{n_0}(\xi) = u_{n_0}'(\eta)(T - \xi) \) where \( \eta \in (\xi, T) \). Therefore \( u_{n_0}'(\eta) = \frac{A}{T - \xi} > \frac{A}{T} \), contrary to \( \|u_n'\| < \frac{A}{T} \).

**Lemma 2.9.** For each \( c \in (0, T) \), there exists \( \lambda_c > 0 \) such that for all \( \lambda > \lambda_c \),
\[
\lim_{n \to \infty} u_n(c) = 0
\tag{2.23}
\]
where \( u_n \) is a solution of the BVP (1.5)_n^\lambda and (1.2).

**Proof.** Fix \( c \in (0, T) \). Let \( \varepsilon \in (0, A) \) and set \( \varrho = \min\{c, T - c\} \),
\[
A = \inf \left\{ f(t, x, y) : (t, x, y) \in [0, T] \times (0, A] \times \left[ -\frac{2A}{\varrho}, \frac{2A}{\varrho} \right] \right\} > 0,
\]
\[
\lambda_c = \frac{2}{A \varrho} \phi \left( \frac{2A}{\varrho} \right).
\]
Fix \( \lambda \in (\lambda_c, \infty) \) and let \( u_n \) be a solution of the BVP (1.5)_n^\lambda and (1.2). We claim that
\[
u_n(c) < \varepsilon \quad \text{for} \quad n > \frac{A}{\varepsilon}.
\tag{2.24}
\]
If (2.24) is not true, there exists \( n_0 > \frac{A}{\varepsilon} \) such that \( u_{n_0}(c) \geq \varepsilon \). The next part of the proof is divided into two cases.

**Case 1.** Suppose \( u_{n_0}'(c) \leq 0 \). Since \( u_{n_0}' \) is nondecreasing on \([0, T]\) by Lemma 2.1, \( u_{n_0}' \leq 0 \) on \([0, c] \). If \( u_{n_0}'(\frac{c}{2}) < -\frac{2A}{c} \) then \( u_{n_0}' < -\frac{2A}{c} \) on \([0, \frac{c}{2}] \) and consequently
\[
u_{n_0}(0) = u_{n_0} \left( \frac{c}{2} \right) - \int_0^{c/2} u_{n_0}'(t) dt > u_{n_0} \left( \frac{c}{2} \right) - A > A,
\]
which is impossible. Hence
\[
u_{n_0} \left( \frac{c}{2} \right) \geq -\frac{2A}{c}, \quad 0 \geq u_{n_0}'(t) \geq -\frac{2A}{c} \quad \text{for} \quad t \in \left[ \frac{c}{2}, c \right].
\tag{2.25}
\]
Since \( n_0 u_{n_0}' \geq n_0 \varrho > A \) for \( t \in [0, c] \), we have
\[
u_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \geq A \quad \text{for} \quad t \in \left[ \frac{c}{2}, c \right].
\]
Then from
\[
(\phi(u_{n_0}'(t)))' = \lambda \left[ f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) + h_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \right] \geq \lambda f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t))) \geq \lambda A \geq \lambda_c A
\]
for \( t \in \left[ \frac{c}{2}, c \right] \), it follows that \( \phi(u_{n_0}'(c)) - \phi(u_{n_0}'(\frac{c}{2})) \geq \frac{\lambda_c A c}{2} \). From this
\[
\phi \left( -u_{n_0}' \left( \frac{c}{2} \right) \right) > -\phi(u_{n_0}'(c)) + \frac{\lambda_c A c}{2} \geq \frac{\lambda_c A c}{2} = c \phi \left( \frac{2A}{\varrho} \right) \geq \phi \left( \frac{2A}{c} \right).
\tag{2.26}
\]
Therefore \(-u''_n(t) > \frac{2A}{T-c}\), contrary to (2.25).

Case 2. Suppose \(u'_n(c) > 0\). Then \(u''_n\) is positive and increasing on \([c, T]\) by Lemma 2.1. If \(u''_n \left(\frac{T+c}{2}\right) > \frac{2A}{T-c}\), then \(u''_n > \frac{2A}{T-c}\) on \([\frac{T+c}{2}, T]\) and \(u_n(T) = u_n(\frac{T+c}{2}) + \int_{\frac{T+c}{2}}^{T} u''(t)\,dt > A\), which is impossible. Hence

\[
0 < u''_n(t) \leq \frac{2A}{T-c} \quad \text{for} \quad t \in \left[\frac{T+c}{2}, T\right].
\]

(2.27)

Since \(n_0u_n \geq n_0\varepsilon > A\) on \([c, T]\), it follows that \(f_n(t, u_n(t), u''_n(t)) \geq A\) for \(t \in \left[\frac{T+c}{2}, T\right]\) and therefore (2.26) holds on \([c, \frac{T+c}{2}]\). Integrating \(\phi(u'_n(t))' > \lambda_c A\) from \(c\) to \(\frac{T+c}{2}\) yields

\[
\phi \left( u'_n \left( \frac{T+c}{2} \right) \right) - \phi(u'_n(c)) = \lambda_c A \left( \frac{T-c}{2} \right).
\]

and therefore

\[
\phi \left( u'_n \left( \frac{T+c}{2} \right) \right) > \phi(u'_n(c)) + \frac{\lambda_c A}{2} \left( \frac{T-c}{2} \right) \geq \phi \left( \frac{2A}{T-c} \right).
\]

Hence \(u''_n \left( \frac{T+c}{2} \right) > \frac{2A}{T-c}\), contrary to (2.27).

We have proved that for each \(\varepsilon > 0\) and \(n > \frac{A}{\varepsilon}\), \(u_n(c) < \varepsilon\). Since \(u_n(c) \geq 0\) for \(n \in \mathbb{N}\), (2.23) is true. \(\square\)

3. Main results and examples

**Theorem 3.1.** Let assumptions \((H_1)-(H_4)\) be satisfied. Then for every \(\lambda > 0\), the BVP (1.1) \(^\lambda\) and (1.2) has a solution. Moreover, any solution of the BVP (1.1) \(^\lambda\) and (1.2) is either a positive solution or a pseudodead core solution or a dead core solution of this problem.

**Proof.** Fix \(\lambda > 0\). Let \(u_n\) be a solution of the BVP (1.5) \(^\lambda\) and (1.2). The existence of \(u_n\) for each \(n \in \mathbb{N}\) follows from Lemma 2.5. By Lemmas 2.1, 2.4 and 2.5, there exists a positive constant \(S\) and there exist \(0 < \alpha_n \leq \beta_n < T\) such that for each \(n \in \mathbb{N}\),

\[
0 \leq u_n(t) < A, \quad t \in (0, T),
\]

\[
\|u'_n\| < S, \quad \|u''_n\| < S.
\]

(3.1)

\(u'_n\) is increasing on the intervals \([0, \alpha_n]\) and \([\alpha_n, T]\), \(u''_n < 0\) on \([0, \alpha_n]\), \(u''_n = 0\) on \([\alpha_n, \beta_n]\) and \(u''_n > 0\) on \((\beta_n, T]\). In addition, if \(\alpha_n < \beta_n\) then \(u_n = 0\) on \([\alpha_n, \beta_n]\). Also \(u'_n\) is equicontinuous on \([0, T]\) by Lemma 2.6. Going if necessary to a subsequence, we can assume, by the Arzelà–Ascoli theorem, that \(u_n\) converges in \(C^1[0, T]\) to some \(u \in C^1[0, T]\). Hence \(u\) is a solution of the BVP (1.1) \(^\lambda\) and (1.2).

Let now \(u\) be a solution of the BVP (1.1) \(^\lambda\) and (1.2). Without loss of generality we can assume that \(u = \lim_{n \to \infty} u_n\) in \(C^1[0, T]\) where \(u_n\) is a solution of the BVP (1.5) \(^\lambda\) and (1.2) and \(u_n\) has the properties presented in the opening part of the proof of this theorem. Then \(u(0) = u(T) = A\) and \(u'\) is nondecreasing on \([0, T]\). The next part of the proof is divided into three cases.

Case 1. Suppose \(\min\{u(t) : 0 \leq t \leq T\} > 0\). Then there exist \(\varepsilon > 0\) and \(n_0 \in \mathbb{N}\) such that

\[
\min\{u_n(t) : 0 \leq t \leq T\} \geq \varepsilon \quad \text{for} \quad n \geq n_0.
\]

(3.2)

Also (see Remark 2.3) \(\alpha_n = \beta_n\) and \(\alpha_n\) is the unique zero of \(u'_n\) for \(n \geq n_0\). Put

\[
\mu = \min\{f(t, x, y) : (t, x, y) \in [0, T] \times [\varepsilon, A] \times [-S, S]\}.
\]

Then \(\mu > 0\) by (H2) and if \(n_1 \in \mathbb{N}\) satisfies \(n_1 \geq \max\{\frac{\mu}{\varepsilon}, n_0\}\) we have

\[
(\phi(u'_n(t)))' \geq \lambda f_n(t, u_n(t), u''_n(t)) \geq \lambda \mu
\]

(3.3)
for \( t \in [0, T] \) and \( n \geq n_1 \). Hence \(-\phi(u_n'(t)) = \phi(u_n'(\alpha_n)) - \phi(u_n'(t)) \geq \lambda \mu(\alpha_n - t) \) and therefore

\[
\phi(u_n'(t)) \leq -\phi^{-1}(\lambda \mu(\alpha_n - t)) \quad \text{for } t \in [0, \alpha_n] \text{ and } n \geq n_1. \tag{3.4}
\]

An analogous reasoning shows that

\[
u_n'(t) \geq -\phi^{-1}(\lambda \mu(t - \alpha_n)) \quad \text{for } t \in [\alpha_n, T] \text{ and } n \geq n_1. \tag{3.5}
\]

Passing if necessary to a subsequence we can assume that \( \{\alpha_n\} \) is convergent and let \( \lim_{n \to \infty} \alpha_n = \alpha \). Letting \( n \to \infty \) in (3.4) and (3.5) yields

\[
u'(t) \leq -\phi^{-1}(\lambda \mu(\alpha - t)), \quad t \in [0, \alpha], \tag{3.6}
\]

\[
u'(t) \geq -\phi^{-1}(\lambda \mu(t - \alpha)), \quad t \in [\alpha, T]. \tag{3.7}
\]

Hence \( \alpha \) is the unique zero of \( \nu' \) and \( u(0) = u(T) = A \) shows that \( \alpha \in (0, T) \). Furthermore,

\[
\lim_{n \to \infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t)), \quad t \in [0, T],
\]

\[
\lim_{n \to \infty} h_n(t, u_n(t), u_n'(t)) = h(t, u(t), u'(t)), \quad t \in [0, T] \setminus \{\alpha\}
\]

and (see (1.3), (3.1) and (3.2))

\[
0 < f_n(t, u_n(t), u_n'(t)) \leq p(\varepsilon)\omega(S), \quad t \in [0, T] \text{ and } n \geq n_1.
\]

From (1.4), (3.1), (3.6) and (3.7), it follows that for \( a \in (0, \alpha) \), \( b \in (\alpha, T) \) and sufficiently large \( n \),

\[
0 \leq h_n(t, u_n(t), u_n'(t)) \leq K\gamma \left( \frac{1}{2} \phi^{-1}(\lambda \mu(\alpha - a)) \right), \quad t \in [0, a]
\]

and

\[
0 \leq h_n(t, u_n(t), u_n'(t)) \leq K\gamma \left( \frac{1}{2} \phi^{-1}(\lambda \mu(b - \alpha)) \right), \quad t \in [b, T].
\]

Thus taking the limit as \( n \to \infty \) in

\[
\phi(u_n'(t)) = \phi(u_n'(0)) + \lambda \int_0^t f_n(s, u_n(s), u_n'(s)) + h_n(s, u_n(s), u_n'(s)) \, ds
\]

and

\[
\phi(u_n'(t)) = \phi(u_n'(T)) - \lambda \int_T^t f_n(s, u_n(s), u_n'(s)) + h_n(s, u_n(s), u_n'(s)) \, ds,
\]

we get, by the Lebesgue dominated convergence theorem and using the fact that \( a \in (0, \alpha) \) and \( b \in (\alpha, T) \) are arbitrary,

\[
\phi(u'(t)) = \phi(u'(0)) + \lambda \int_0^t [f(s, u(s), u'(s)) + h(s, u(s), u'(s))] \, ds, \quad t \in [0, \alpha),
\]

and

\[
\phi(u'(t)) = \phi(u'(T)) + \lambda \int_T^t [f(s, u(s), u'(s)) + h(s, u(s), u'(s))] \, ds, \quad t \in (\alpha, T].
\]

Hence \( \phi(u') \in C^1([0, T] \setminus \{\alpha\}) \) and \( u \) satisfies (1.1)\(^3\) for \( t \in [0, T] \setminus \{\alpha\} \). We have proved that \( u \) is a positive solution of the BVP (1.1)\(^3\) and (1.2).

**Case 2.** Suppose \( \min_{[0, T]} u(t) = 0 \), \( u(\alpha) = u(\beta) = 0 \) for some \( 0 < \alpha < \beta < T \) and \( u > 0 \) on \([0, \alpha) \cup (\beta, T]\). Since \( u' \) is nondecreasing on \([0, T]\), \( u' < 0 \) on \([0, \alpha) \) and \( u' > 0 \) on \((\beta, T]\). Let \( t_0 \in (0, \alpha) \) and set \( t_1 = \frac{\alpha + t_0}{2} \). Then there exists \( n_2 \in \mathbb{N} \) such that \( u_n(t) \geq u_n(t_1) \geq \frac{u(t_1)}{2} \) and \( u_n'(t) < 0 \) for \( t \in [0, t_1] \) and \( n \geq n_2 \). Put

\[
\varrho = \min \left\{ f(t, x, y) : (t, x, y) \in [0, T] \times \left[ \frac{u(t_1)}{2}, A \right] \times \{-S, S\} \right\}.
\]
Then \( \varrho > 0 \) by (H2). Let \( n_3 \in \mathbb{N}, n_3 \geq \max\{\frac{2p}{a_1(t_1)}, n_2\} \). Then \( nu_n(t) \geq \frac{nu(t_1)}{2} \geq \varrho \) for \( t \in [0, t_1] \) and \( n \geq n_3 \). Therefore for these \( t \) and \( n \), \( f_n(t, u_n(t), u_n'(t)) \geq \varrho \) and \( \phi(u_n') \geq \lambda \varrho \). It follows that

\[
\phi(u_n'(t)) = \phi(u_n'(t_1)) + \int_{t_1}^{t} (\phi(u_n'(s)))' ds < -\lambda \varrho (t_1 - t) \quad \text{for } t \in [0, t_1] \text{ and } n \geq n_3.
\]

Hence \( u_n'(t) < -\varrho^{-1}(\lambda \varrho (t_1 - t)) \) for \( t \in [0, t_0] \) and \( n \geq n_3 \). Letting \( n \to \infty \) gives \( u'(t) \leq -\varrho^{-1}(\lambda \varrho (t_1 - t_0)) \) for \( t \in [0, t_0] \). Consequently

\[
\lim_{n \to \infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t))
\]

and

\[
\lim_{n \to \infty} h_n(t, u_n(t), u_n'(t)) = h(t, u(t), u'(t))
\]

for \( t \in [0, t_0] \). Next it follows from (1.3) and (1.4) that

\[
\sup\{f_n(t, u_n(t), u_n'(t)) : t \in [0, t_0], n \geq n_3\} \leq p \left( \frac{\|u(t_1)\|}{2} \right)\omega(S)
\]

and

\[
\sup\{h_n(t, u_n(t), u_n'(t)) : t \in [0, t_0], n \geq n_3\} \leq K \gamma [\varphi^{-1}(\lambda \varrho (t_1 - t_0))].
\]

Now taking the limit as \( n \to \infty \) in (3.8), we get

\[
\phi(u'(t)) = \phi(u'(0)) + \lambda \int_{0}^{t} [f(s, u(s), u'(s)) + h(s, u(s), u'(s))] ds
\]

for \( t \in [0, t_0] \) by the Lebesgue dominated convergence theorem. Since \( t_0 \in (0, \alpha) \) is arbitrary, the equality (3.9) holds for \( t \in [0, \alpha) \). Therefore \( \phi(u') \in C^1[0, \alpha) \) and \( u(t) \) satisfies (1.1) for \( t \in [0, \alpha) \).

Essentially the same reasoning we can apply on the interval \((\beta, T]\) to obtain that \( \phi(u') \in C^1(\beta, T] \) and \( u(t) \) is a solution of (1.1) on \((\beta, T] \). Summarizing, we have shown that \( u(t) \) is a pseudodead core solution of the BVP (1.1) and (1.2) and \( u(t) = 0 \) for \( \alpha, \beta \).

Case 3. Suppose \( \min\{u(t) : 0 \leq t \leq T\} = 0 \) and \( u(\xi) = 0 \) for a unique \( \xi \in (0, T) \). On the intervals \([0, \xi) \) and \((\xi, T]\), we can proceed analogously to Case 2 (where the intervals \([0, \alpha) \) and \((\beta, T]\) are considered) in order to prove that \( \phi(u') \in C^1([0, T] \setminus \{\xi\}) \) and \( u(t) \) satisfies (1.1) on \([0, T] \setminus \{\xi\} \). Therefore \( u(t) \) is a pseudodead core solution of the BVP (1.1) and (1.2).

**Corollary 3.2.** Let assumptions (H1)–(H4) be satisfied. Then there exists \( \lambda_0 > 0 \) such that the BVP (1.1) and (1.2) has for \( \lambda \in (0, \lambda_0] \) only positive solutions.

**Proof.** Let \( \lambda_0 > 0 \) be given in Lemma 2.7. Let \( \lambda \in (0, \lambda_0] \) and set \( V = \sup\{\|\sigma_n\| : n \in \mathbb{N}\} \) where \( \sigma_n \) is a solution of the BVP (1.5) and (1.2). Then \( V < \frac{1}{A} \) by Lemma 2.7 and therefore \( 0 < A - VT \leq u_n(t) \leq A \) for \( t \in [0, T] \) and \( n \in \mathbb{N} \). Let \( \sigma_n \) be a solution of the BVP (1.1) and (1.2). Since \( u = \lim_{n \to \infty} u_n \) is \( C^1[0, T] \) where \( \{k_n\} \) is a subsequence of \( \{n\} \), we see that \( 0 < A - VT \leq u \leq A \) on \([0, T] \). Hence all solutions of the BVP (1.1) and (1.2) are positive solutions for each \( \lambda \in (0, \lambda_0] \).

**Corollary 3.3.** Let assumptions (H1)–(H4) be satisfied. Then for sufficiently large value of \( \lambda \), the BVP (1.1) has only dead core solutions. Moreover, to given \( c_1, c_2, 0 < c_1 < c_2 < T \), the BVP (1.1) and (1.2) has for sufficiently large value of \( \lambda \) dead core solutions \( u \) such that \( u(t) = 0 \) for \( t \in [c_1, c_2] \).

**Proof.** Let \( 0 < c_1 < c_2 < T \). By Lemma 2.9, there exists \( \lambda_\ast > 0 \) such that for all \( \lambda > \lambda_\ast \),

\[
\lim_{n \to \infty} u_n(c_j) = 0, \quad j = 1, 2,
\]

where \( u_n \) is a solution of the BVP (1.5) and (1.2). Let \( \lambda > \lambda_\ast \) and \( u \) be a solution of the BVP (1.1) and (1.2). Then there exists a subsequence \( \{k_n\} \) of \( \{n\} \) such that \( u = \lim_{n \to \infty} u_{k_n} \) in \( C^1[0, T] \). Since \( u' \) is nondecreasing on \([0, T] \) and

\[
\text{(3.10)}
\]

gives \( u(c_1) = u(c_2) = 0 \), we have \( u(t) = 0 \) for \( t \in [c_1, c_2] \). Hence \( u \) is a dead core solution. We have proved that for all \( \lambda > \lambda_\ast \), any solution of the BVP (1.1) and (1.2) is a dead core solution.
**Example 3.4.** Consider the differential equation

\[
(|u'|^{p-2}u')' = \lambda \left( \frac{1}{u^\alpha} + tu^\beta |u'|^\delta + \frac{e^u}{|u'|^\nu} \right), \quad \lambda > 0,
\]

where \( p > 1, \alpha \in (0, 1), \beta, \nu \in (0, \infty) \) and \( \delta \in (0, p) \). The Eq. (3.11) is the special case of (1.1)\(^\lambda\) with \( f(u) = |u|^{p-2}u \) satisfying (H\(_1\)) and \( f(t, x, y) = \frac{1}{x^\alpha} + tx^\beta |y|^\delta, h(t, x, y) = \frac{e^u}{|y|^\nu} \). From \( \frac{1}{x^\alpha} + tx^\beta |y|^\delta \leq \left( \frac{e^u}{x^\alpha} + Tx^\beta \right)(1 + |y|^\delta) \) we see that (H\(_2\)) is satisfied with \( p(x) = \frac{1}{x^\alpha} + TA^\delta \) and \( \omega(z) = 1 + z^\delta \). Also (H\(_3\)) is satisfied with \( K = e^\delta \) and \( \gamma(z) = \frac{1}{z^\delta} \). Since \( \delta \in (0, p) \), we have

\[
\int_0^\infty \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} ds > \int_1^\infty \frac{\frac{1}{s^{\frac{1}{\nu}}}}{1 + s\frac{1}{\nu} + s^{-\frac{1}{\nu}}} ds = \int_1^\infty \frac{s^{\frac{1}{\nu}}}{1 + s\frac{1}{\nu} + s^{-\frac{1}{\nu}}} ds = \frac{1}{3} \int_1^\infty s^{-\frac{1}{\nu}} ds = \infty.
\]

By **Theorem 3.1**, for each \( A > 0 \), the BVP (3.11) and (1.2) has a solution which is either a positive solution or a pseudodead core solution or a dead core solution. If \( \lambda \) is sufficiently small then all solutions of the BVP (3.11) and (1.2) are positive solutions by **Corollary 3.2**, and if \( \lambda \) is sufficiently large then all solutions are dead core solutions by **Corollary 3.3**.

**Example 3.5.** Consider the singular BVP

\[
\begin{align*}
  u'' &= \frac{\lambda}{\sqrt{u}}, \quad \lambda > 0, \\
  x(0) &= 1, \\
  x(T) &= 1.
\end{align*}
\]

Eq. (3.12) is the special case of (1.1)\(^\lambda\) with \( f(u) = u, f(t, u) = \frac{1}{\sqrt{u}} \) and \( h = 0 \) satisfying assumptions (H\(_1\))–(H\(_4\)) (where \( p(u) = \frac{1}{\sqrt{u}}, |u| = 1, K = 0 \) and \( \gamma = 1 \)). Hence **Theorem 3.1** guarantees that any solution of the BVP (3.12) and (3.13) is either a positive solution or a pseudodead core solution or a dead core solution of this problem. Assume that \( u \) is a dead core solution or a pseudodead core solution of the BVP (3.12) and (3.13) with some \( \lambda_0 > 0 \) in (3.12). Then there exist \( 0 < t_1 < t_2 < T \) such that \( u(t_1) = u'(t_1) = u(t_2) = u'(t_2) = 0, u = 0 \) on \( [t_1, t_2], u' < 0 \) on \( [0, t_1) \) and \( u' > 0 \) on \( (t_2, T] \). Integrating \( u''(t)u'(t) \) gives

\[
(u'(t))^2 = 4\lambda_0\sqrt{u(t)} + a \quad \text{for} \quad t \in [0, t_1]
\]

where \( a \) is a constant. From \( u(t_1) = u'(t_1) = 0 \) we obtain \( a = 0 \). Hence \( u'(t) = -2\sqrt{\lambda_0}\sqrt{u(t)} \) for \( t \in [0, t_1] \).

Integrating the last equality over \( [0, t] \subset [0, t_1] \) and using \( u(0) = 1 \) yields

\[
u(t) = \left( 1 - \frac{3}{2}\sqrt{\lambda_0}t \right)^{\frac{3}{2}} \quad \text{for} \quad t \in [0, t_1].
\]

An analogous reasoning shows that

\[
u(t) = \left( 1 - \frac{3}{2}\sqrt{\lambda_0}(T - t) \right)^{\frac{3}{2}} \quad \text{for} \quad t \in [t_2, T].
\]

From (3.14) and (3.15) it follows that \( t_1 = t_2 \) if and only if \( \lambda_0 = \frac{16}{9T^2} \) and for \( \lambda_0 > \frac{16}{9T^2} \) we have \( t_2 - t_1 = T - \frac{4}{3\sqrt{\lambda_0}} > 0 \).

Summarizing, we have proved that the BVP (3.12) and (3.13) has a positive solution for \( \lambda \in (0, \frac{16}{9T^2}) \), a unique pseudodead core solution for \( \lambda = \frac{16}{9T^2} \) and dead core solutions for \( \lambda > \frac{16}{9T^2} \). The function \( u(t) = \left( 1 - \frac{2}{T} \right)^{\frac{3}{2}} \) is the
unique pseudodead core solution of the BVP (3.12) and (3.13) (with $\lambda = \frac{16}{9T^2}$ in (3.12)) and

$$u(t) = \begin{cases} 
\left(1 + \frac{3}{2}\sqrt{\lambda} t\right)^{\frac{4}{3}} & \text{for } t \in \left[0, \frac{2}{3\sqrt{\lambda}}\right], \\
0 & \text{for } t \in \left[\frac{2}{3\sqrt{\lambda}}, T - \frac{2}{3\sqrt{\lambda}}\right], \\
\left(1 + \frac{3}{2}\sqrt{\lambda}(T - t)\right)^{\frac{4}{3}} & \text{for } t \in \left[T - \frac{2}{3\sqrt{\lambda}}, T\right].
\end{cases}$$

is the unique dead core solution of the BVP (3.12) and (3.13) for $\lambda > \frac{16}{9T^2}$ and the interval $\left[\frac{2}{3\sqrt{\lambda}}, T - \frac{2}{3\sqrt{\lambda}}\right]$ is its dead core interval.

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