EXPERIMENTAL REALIZATION OF STRANGE NONCHAOTIC ATTRACTIONS IN A NONLINEAR SERIES LCR CIRCUIT WITH NONSINUSOIDAL FORCE

K. Srinivasan*, D. V. Senthilkumar†, R. Suresh, K. Thamilmaran and M. Lakshmanan‡
Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirappalli 620 024, India
lakshman@cnld.bdu.ac.in
†Centre for Dynamics of Complex Systems and Potsdam Institute for Climate Impact Research, Potsdam, Germany
*Department of Physics, National Institute of Technology, Tiruchirappalli 620 015, India

Received April 7, 2009; Revised May 6, 2009

We have identified several prominent routes, namely, fractalization, fractalization followed by intermittency, intermittency and Heagy–Hammel routes, for the birth of strange nonchaotic attractors (SNAs) in a quasiperiodically forced electronic system with nonsinusoidal (square wave) force as one of the quasiperiodic forces [Senthilkumar et al., 2008]. In addition, a new bubbling route has also been identified in this circuit. Although some of these prominent routes have been reported experimentally [Thamilmaran et al., 2006] in a quasiperiodically forced electronic circuit with both the forcings as sinusoidal forces, experimental identification of all these routes is reported here in a quasiperiodically forced electronic circuit with one of the forcings as a nonsinusoidal (square wave) force. The birth of SNAs by these routes are characterized from both the experimental and numerical data by the maximal Lyapunov exponents and their variance, Poincaré maps, Fourier amplitude spectra, spectral distribution functions and the distribution of finite-time Lyapunov exponents.

Keywords: Strange nonchaotic attractors; various routes; experimental realization in nonlinear series LCR circuit.

1. Introduction

Very often, nonlinear dynamical systems under the action of quasiperiodic forces exhibit attractors which are neither regular nor chaotic. They are termed as strange nonchaotic attractors (SNA) [Grebogi et al., 1984; Bondeson et al., 1985; Romeiras & Ott, 1987; Romeiras et al., 1987; Ding et al., 1989; Kapitaniak & Wojewoda, 1993; Lai, 1996; Staglino et al., 1996; Yalcinkaya & Lai, 1996; Venkatesan & Lakshmanan, 1997; Prasad et al., 1998; Venkatesan & Lakshmanan, 1998; Venkatesan et al., 2000]. Typically they show a fractal nature, which is the characteristic property of chaotic attractors, and so they are geometrically strange. On the other hand, they do not show

*Author for correspondence
such nonsinusoidal quasiperiodic forces can be a wave) quasiperiodic force and demonstrated that ear series LCR circuit with a nonsinusoidal (square modulations can generate SNAs [Prasad et al., 2001; Prasad et al., 2007, and references therein]. In this connection, we have recently reported a new route for the birth of SNA, namely, the bubbling route [Senthilkumar et al., 2008], in a nonlinear series LCR circuit with a nonsinusoidal (square wave) quasiperiodic force and demonstrated that such nonsinusoidal quasiperiodic forces can be a source for generating new routes for the birth of SNAs. In this manuscript, we show both numerically and experimentally that the most common scenarios (prominent routes) for the birth of SNAs, namely fractalization, intermittency and Heagy–Hammel routes, besides the bubbling route, can occur in quasiperiodically forced nonlinear series LCR circuit with one of the quasiperiodic forcings being a nonsinusoidal force (square wave).

We consider the quasiperiodically driven negative conductance series LCR circuit with a diode investigated in [Thamilmaran et al., 2006; Senthilkumar et al., 2008] and unravel the dynamics of the circuit with one of the forcings as a square wave force (nonsinusoidal) for suitable range of parameter values. The main reason for choosing the square wave as one of the driving forces is its bistable nature. Bistability is responsible for hysteresis in many physical and technical systems. Bistability and hysteresis are increasingly recognized as major properties of regulatory networks governing numerous biological phenomena [Graziani et al., 2004; Dubnau & Losick, 2006]. It is known that bistable systems are also good approximations to the dynamics of certain neurons [Bulsa et al., 1991; Longtin et al., 1994]. In particular, they have recently been shown to be powerful building blocks for neural networks performing associative memory tasks [Chinarm & Menzinger, 2000]. The most important application of the square wave is its suitability for digital applications and secure communication in view of its bistable nature. Very recently, we have also shown that square wave quasiperiodic forces can lead to the birth of SNAs via new scenarios and mechanisms [Senthilkumar et al., 2008]. In view of the above applications of square wave pulse, it is of considerable importance to study its effects.

In particular, in this paper, we have considered a nonlinear series LCR circuit with square wave force as one of the quasiperiodic forces and demonstrate the existence of the different scenarios (fractalization, fractalization followed by intermittency, intermittency and Heagy–Hammel) for the formation of SNA in this circuit along with a new route, namely the bubbling route, which has been recently reported [Senthilkumar et al., 2008]. The organization of this manuscript is as follows. In Sec. 2, we present a brief account of the experimental realization of the quasiperiodically forced negative conductance series LCR circuit with diode using a sinusoidal force and a nonsinusoidal (square wave).
wave) force as quasiperiodic forces. In Sec. 3, we present a two-parameter phase diagram of the circuit, where the regions corresponding to the different dynamical transitions to SNA are delineated as a function of the parameters based on our numerical analysis. In Sec. 4, the creation of SNA through fractalization route is discussed. Fractalization followed by intermittency, and intermittency routes to SNA are illustrated in Secs. 5 and 6, while in Sec. 7 the Heagy–Hammel route to SNA is demonstrated. The bubbling route to SNA is briefly presented in Sec. 8. The various scenarios are first demonstrated using numerical analysis, which is then corroborated by the corresponding experimental results. Finally, we summarize our results in Sec. 9.

2. Circuit Realization

We consider a simple second-order nonlinear, dissipative, nonautonomous negative conductance series LCR circuit with a sinusoidal voltage generator, \( f_1(t) \), reported by us recently [Thamilmaran et al., 2006; Thamilmaran et al., 2005] along with a second nonsinusoidal force, \( f_2(t) \), as shown in Fig. 1(a) [Senthilkumar et al., 2008].

The circuit consists of a series LCR network, forced by a sinusoidal voltage generator \( f_1(t) \) and a nonsinusoidal (square wave) voltage generator \( f_2(t) \) (HP 33120A series), respectively. Two extra components, namely a p-n junction diode (D) and a linear negative conductor \( g_D \), are connected in parallel to the forced series LCR circuit. The negative conductor used here is a standard op-amp based negative impedance converter (NIC). The diode operates as a nonlinear conductance, limiting the amplitude of the oscillator. In Fig. 1(a), \( v, i_L \) and \( i_D \) denote the voltage across the capacitor \( C \), the current through the inductor \( L \) and the current through the diode \( D \), respectively. The actual \( v - i \) characteristic of the diode [Fig. 1(b)] is approximated by the usual two-segment piecewise-linear function [Fig. 1(c)] which facilitates the numerical analysis considerably. The state equations governing the circuit (Fig. 1) are a set of two first-order nonautonomous differential equations,

\[
\begin{align}
C \frac{dv}{dt} &= i_L - i_D + g_D v, \\
\frac{di_L}{dt} &= -R_L v + E_{f1} \sin(\omega f_1 t) \\
&\quad + E_{f2} \text{sgn}[\sin(\omega f_2 t)],
\end{align}
\]

where

\[
i_D(v) = \begin{cases} 
  g_D(v-V), & v \geq V, \\
  0, & v < V.
\end{cases}
\]

Here \( g_D \) is the slope of the characteristic curve of the diode, \( E_{f1} \) and \( E_{f2} \) are the amplitudes, \( \omega f_1 \) and \( \omega f_2 \) are the angular frequencies of the forcing functions \( f_1(t) = E_{f1} \sin(\omega f_1 t) \) and \( f_2(t) = E_{f2} \text{sgn}[\sin(\omega f_2 t)] \), respectively. In the absence of \( f_2(t) \), the circuit [Fig. 1(a)] has been shown to exhibit chaos and also strong chaos not only through the familiar period-doubling route but also via torus breakdown followed by period-doubling bifurcations [Thamilmaran et al., 2005].

In order to set up the actual experimental parameters, the numerical simulation is used to determine the correct parametric values for observing SNAs. The values of the diode conductance \( g_D \),

![Fig. 1. Circuit realization of a simple nonautonomous circuit. Here D is the p-n junction diode and \( g_D \) is a negative conductance. The external emfs are \( f_1(t) = E_{f1} \sin(\omega f_1 t) \) and \( f_2(t) = E_{f2} \text{sgn}[\sin(\omega f_2 t)] \). The values of the circuit elements are fixed as \( L = 50 \mu \text{H}, C = 10 \mu \text{F}, R = 1200 \Omega, E_{f1} = 400 \mu \text{V} \) and \( \omega f_1 = 17033 \text{Hz} \). The forcing amplitude \( E_{f1} \) and its frequency \( \omega f_1 \) are chosen as the control parameters. (b) \( i - v \) characteristics of the p-n junction diode and (c) two-segment piecewise-linear function mimicking the diode.](image)
the negative conductance $g_0$ and the break voltage $V$ are fixed as $1313\mu S, -0.45\, mS, R = 1900\, \Omega$ and $0.5\, V$, respectively. We have fixed the actual experimental values of the inductance, $L$, capacitance, $C$, external frequency $\omega_1$ and forcing strength $E_1$ of the square wave as $50.3\, mH, 10.35\, mF, 17033\, Hz$ and $400\, mV$, respectively, while we vary the amplitude $E_1$ and the frequency $\omega_1$ of the sinusoidal force as control parameters. The forcing functions $f_1(t)$ and $f_2(t)$ are obtained from two separate function generators of the type HP 33120A.

In order to study the dynamics of the circuit in detail, Eq. (1) can be converted into a convenient normalized form for numerical analysis by using the following rescaled variables and parameters
\[ t = t/\sqrt{LC}, \quad x = v/V, \quad y = (i_1/V)/(\sqrt{L/C}), \quad E_1 = E_1/V, \quad E_2 = E_2/V, \quad \omega_1 = \omega_1/\sqrt{LC}, \quad \omega_2 = \omega_2/\sqrt{LC}, \quad a = R\sqrt{C/L}, \quad b = g_0/\sqrt{L/C}, \quad \text{and} \quad \epsilon = g_0/\sqrt{LC}. \]

The normalized evolution equation so obtained is
\begin{align}
\dot{x} &= y + f(x), \quad (2a) \\
\dot{y} &= -x - ay + E_1 \sin(\phi) + E_2 \text{sgn}[\sin(\theta)], \quad (2b) \\
\dot{\phi} &= \omega_1, \quad (2c) \\
\dot{\theta} &= \omega_2. \quad (2d)
\end{align}

where
\[ f(x) = \begin{cases} (b - c)x + c, & x \geq 1, \\ bx, & x < 1. \end{cases} \quad (2e) \]

Here dot stands for differentiation with respect to $t$. Equation (2) is then numerically integrated using Runge–Kutta fourth order routine to identify the different dynamical scenarios corresponding to different values of the rescaled parameters. Various interesting dynamical transitions exhibited by Eq. (2), and that corresponding to the actual circuit (Fig. 1) for suitable experimental values, are demonstrated below.

### 3. Two Parameter Phase Diagram

To start with, we first demarcate the parameter space $(E_1, \omega_1)$ by numerically integrating Eq. (2) into quasiperiodic, strange nonchaotic and chaotic regimes, using the various quantification measures such as the Lyapunov exponents and its variance, spectral measures and distribution of finite time Lyapunov exponents. Further, different kinds of attractors can also be distinguished qualitatively by using the Poincaré surface of section plots in the $(\phi, x)$ plane (with $\phi$ modulo $2\pi$), which clearly indicate whether an attractor has a geometrically smooth or complicated structure. Finer distinction between the SNAs arising from different mechanisms can be made qualitatively by examining the Poincaré surface of section and quantitatively by the nature of distribution of finite time Lyapunov exponents.

The parameter space $(E_1, \omega_1)$ is scanned to distinguish the different dynamical behaviors exhibited by Eq. (2) in the range $E_1 \in (0.4, 1.1)$ and $\omega_1 \in (1.075, 1.275)$, which is shown in Fig. 2. It has also been confirmed that these dynamical behaviors are indeed exhibited by the circuit shown in Fig. 1 for the corresponding values of the parameters $E_1$ and $\omega_1$. Transitions from quasiperiodic attractor to SNA and subsequently to chaotic attractor can occur on increasing the value of the amplitude of the sinusoidal force $E_1$ for fixed value of its frequency $\omega_1$. SNAs created through different mechanisms are found to occur for different values of the frequency $\omega_1$ of the sinusoidal force. Now, we will outline the ranges of values of the frequency $\omega_1$ for which SNAs arise from quasiperiodic attractors through different mechanisms on increasing the value of the amplitude $E_1$ of the sinusoidal force.

First of all, we observed that SNAs created through the new route, namely the bubbling route,
is identified in the range of frequency $\omega_1 \in (1.085, 1.086)$, where bubbles appear in the strands of period-3 torus and then the bubbles get increasingly wrinkled in the range of the amplitude $E_1 \in (0.54, 0.55)$ of the sinusoidal forcing resulting in SNA, as pointed out in detail recently by as [Senthilkumar et al., 2008]. This phenomenon has been called the bubbling transition to SNA and it is denoted as BUB in Fig. 2. Further increase in the value of $E_1$ ends up with chaotic behavior indicated by C in Fig. 2. SNAs created through gradual fractalization ($F$) of period-3 (3T) quasiperiodic attractor are identified in the range of $\omega_1 \in (1.086, 1.111)$ as a function of the amplitude $E_1$. Intermittency route (INT) is found to be exhibited in the range of frequency $\omega_1 \in (1.111, 1.1268)$ and (1.1515, 1.2615). For $\omega_1 \in (1.1268, 1.1515)$, the gradual fractalization is followed by intermittency phenomenon on increasing the value of the external forcing $E_1$ (marked as (FINT) in Fig. 2). Torus doubling bifurcation from a period-3 torus (3T) to a period-6 (6T) torus and then to SNA via the Heagy–Hammel (HH) mechanism is found to occur in the range $\omega_1 \in (1.2501, 1.2615)$ on decreasing $E_1$ from $E_1 = 1.1$. In the following, we will describe all the four prominent routes in detail by both numerical simulation and experimental realization.

4. Fractalization Route to SNA

At first, we demonstrate the existence of gradual fractalization route, where a period-$k$ torus gets increasingly wrinkled and eventually forms a $k$-band SNA as a function of the system parameter $E_1$ for fixed values of $\omega_1$. The qualitative (geometric) structure of the attractor remains more or less the same during the process. Such a phenomenon has been observed in the present circuit in regions indicated as $F$ in Fig. 2 for the ranges of $\omega_1$ discussed earlier in Sec. 3.

4.1. Numerical analysis

To be specific we fix the value of the frequency $\omega_1$ of the sinusoidal forcing at $\omega_1 = 1.094$ and vary the amplitude $E_1$ of the sinusoidal forcing in the range $E_1 \in (0.5, 0.55)$. Three strands corresponding to period-3 torus (3T) for the value of $E_1 = 0.5$ are clearly seen in the Poincaré surface of section plot as shown in Fig. 3(a). The corresponding phase portrait and power spectrum are shown in Figs. 4(a)(i) and 4(a)(ii), respectively. As the amplitude $E_1$ of the sinusoidal forcing is increased from 0.5, the period-3 torus continuously deforms, gets increasingly wrinkled, and finally transits to SNA as shown in Fig. 3(b) for the value of $E_1 = 0.546$. The corresponding phase portrait and power spectrum are shown in Figs. 4(b)(i) and 4(b)(ii), respectively. Finally, to confirm that the SNA transits to a chaotic attractor beyond $E_1 = 0.55$, we have depicted the Poincaré surface of section of the latter in Fig. 3(c) while the corresponding attractor and power spectrum are shown in Fig. 4(c) for $E_1 = 0.57$.

The largest Lyapunov exponent ($\Lambda$) and its variance ($\mu$) defined as

$$\mu = \frac{1}{M} \sum_{i=1}^{M} (\Lambda - \lambda_i(N))^2 \quad (3)$$

are shown in Fig. 5 for a range $E_1 \in (0.5, 0.55)$ for the fixed value of $\omega_1 = 1.094$. The nature of the SNAs for the above values of the parameters is strange as confirmed by Figs. 3(b) and 4(b), while the largest Lyapunov exponent remains negative for the corresponding range of $E_1$ as seen in Fig. 5(a). It is very obvious from these figures that the period-3 torus with three smooth branches in the Poincaré map [Fig. 3(a)] gradually loses its smoothness and ultimately approaches a fractal behavior via SNA [Fig. 3(b)] before the onset of chaos as the amplitude of the sinusoidal forcing $E_1$ is increased further. Such a phenomenon is essentially a gradual fractalization of the torus as was shown by Nishikawa and Kaneko [1996] in their study of transition to chaos via SNA in the logistic map. In this route, there is no collision involved among the orbits and therefore, the Lyapunov exponent and its variance change only slowly.

In the present case, the transition from torus to SNA is clearly revealed by the largest Lyapunov exponent which varies smoothly in the torus region ($E_1 < E^*_1 = 0.5241$) while it varies irregularly in the SNA region ($E_1 > E^*_1$) as depicted in Fig. 5(a). This transition is also clearly indicated by the variance of the largest Lyapunov exponent in which the fluctuation is small in the torus region while it is large in the SNA region in Fig. 5(b). Finally, the transition of SNA into a chaotic attractor is confirmed by the change in the largest Lyapunov exponent from negative to positive values at $E_1 \approx 0.554$ as shown in the inset of Fig. 5(a).

To confirm further that the attractors depicted in Figs. 3 and 4 are quasiperiodic (a), strange
nonchaotic (b) and chaotic (c), we proceed to quantify the changes in their power spectrum. The spectral distribution function, defined as the number of peaks in the Fourier amplitude spectrum larger than some value $\sigma$, is used to distinguish between quasiperiodic attractor and SNA as well as SNA and chaotic attractor. The quasiperiodic attractors obey a scaling relationship $N(\sigma) \sim \log_{10}(1/\sigma)$, while the SNAs satisfy a scaling power-law relationship $N(\sigma) \sim \sigma^{-\beta}$, $1 < \beta < 2$ [Romeiras & Ott, 1987; Romeiras et al., 1987]. Similarly for the chaotic attractor, the scaling relation is $N(\sigma) \sim \sigma^{-\beta}$, $\beta > 2$. The spectral distribution function (filled circles) of the quasiperiodic attractor for the value of $E_1 = 0.5$ is shown in Fig. 6(a), which obeys the scaling relationship $N(\sigma) \sim \log_{10}(1/\sigma)$ as indicated by the solid line. Figure 6(b) shows the spectral distribution function for the SNA [Fig. 4(b)] for the value of $E_1 = 0.546$ satisfying a power-law relationship with the exponent $\beta = 1.9$, which is a characteristic feature of the SNA as pointed out above. For the chaotic attractor [Fig. 4(c)], the scaling exponent [Fig. 6(c)] turns out to be $\beta = 3.0$ as required. Again the solid lines in Figs. 6(b) and 6(c) represent the scaling law for SNA and chaos, respectively.

In addition to the qualitative picture of Poincaré surface of section in the ($\phi, x$) plane in distinguishing the type of mechanism through which SNA appears, it is also possible to distinguish them using the distribution of finite time Lyapunov exponents in a quantitative way. It has been shown [Prasad et al., 1997; Prasad et al., 1998] that a typical trajectory on an SNA actually possesses positive Lyapunov exponents in finite time intervals, although the asymptotic exponent is negative. As a consequence, it is possible to observe different characteristics of the SNA created through different mechanisms by a study of the differences in...
Consequently, we have calculated the distribution of finite time Lyapunov exponents for the attractors shown in Figs. 4(a)(i) and 4(b)(i) to confirm the nature of transition to SNA. Figure 7(a) illustrates the distribution of finite time Lyapunov exponents $P(N,\lambda)$ (solid line) for $N = 2000$ which is strongly peaked about the negative value of exponent when the attractor is torus whereas on the SNA the distribution [Fig. 7(b)] picks up a tail which extends into the local Lyapunov exponent.
Fig. 5. Transition from torus to SNA through gradual fractalization route for the same value of frequency as in Fig. 3 and in the range of amplitude $E_1 \in (0.5, 0.55)$ obtained numerically. (a) Largest Lyapunov exponent ($\Lambda$) and (b) its variance ($\mu$). Inset in (a) depicts transition from SNA to chaos for $E_1 \in (0.55, 0.57)$.

Fig. 6. Spectral distribution function (filled circles) calculated numerically. (a) Torus [Fig. 4(a)], (b) fractalized SNA [Fig. 4(b)] and (c) chaotic attractor [Fig. 4(c)] for the same values of frequency and amplitude as in Fig. 3. Solid line in (a) corresponds to the scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ and in (b) and (c) corresponds to the scaling relation $N(\sigma) \sim \sigma^{-\beta}$, with $\beta = 1.9$ and 3.0, respectively.
Fig. 7. Distribution of finite time Lyapunov exponent calculated from both numerical data (solid line) and experimental data (dashed line) of (a) torus [Fig. 4(a)] and (b) SNA [Fig. 4(b)] for the same values of frequency and amplitude as in Fig. 3.

Fig. 8. Snapshots of the experimental attractors and their power spectrum of the circuit shown in Fig. 1 for the corresponding values of the frequency $\omega_f$ and the amplitude $E_f$ of the sinusoidal forcing in Fig. 3. (a) Period-3 torus (3T), (b) fractalized SNA and (c) chaotic attractor: (i) phase portrait in the $(v_C, i_L)$ space; (ii) power spectrum.
$\lambda > 0$. This tail is directly correlated with the enhanced fluctuation in the Lyapunov exponent on SNAs as evidenced in Fig. 5(a). On the fractalized SNA the distribution is almost symmetric and the shape remains the same for torus regions as well as SNA regions as shown in Fig. 7. This confirms the existence of fractalization route to SNA in the present circuit.

4.2. Experimental confirmation

Further, to confirm that the above results hold good in the actual experimental circuit (Fig. 1), the snapshots of the experimental phase portraits and Fourier spectra of period-3 torus, fractalized SNA and chaotic attractor as seen in the oscilloscope are shown in Fig. 8. It is evident that the simulated results — the form of the attractors and their power spectra — shown in Fig. 4 and experimentally observed results shown in Fig. 8 are qualitatively similar to each other.

To verify further that the attractors depicted in Fig. 8 indeed correspond to torus, fractalized SNA and chaotic attractor, we proceed to quantify the experimental data of the corresponding attractors using the spectral distribution function and distribution of finite time Lyapunov exponents as discussed in the previous subsection.

The spectral distribution function (filled triangles) of the torus [Fig. 8(a)] is shown in Fig. 9(a) satisfying the scaling relationship $N(\sigma) \sim \log_{10} (1/\sigma)$ as indicated by the solid line which is the characteristic feature of quasiperiodic attractors. Figure 9(b) illustrates the spectral distribution function of SNA shown in Fig. 8(b) satisfying the power-law distribution with the value of the exponent being $\beta = 1.96$, a quantification measure for characterizing the existence of SNA. For the chaotic attractor [Fig. 8(c)],

Fig. 9. Spectral distribution function (filled triangles) calculated from the experimental data of (a) torus [Fig. 8(a)], (b) fractalized SNA [Fig. 8(b)] and (c) chaotic attractor [Fig. 8(c)]. Solid curve/line in (a) corresponds to the scaling relationship for the quasiperiodic attractors and in (b) and (c) corresponds to the scaling relations for the SNA and chaotic attractor, respectively.
the scaling exponent turns out to be $\beta = 4.7$ [Fig. 9(c)] as expected. This analysis confirms that the snapshots of the attractors shown in Fig. 8 indeed correspond to torus, SNA and chaos.

The distribution of finite time Lyapunov exponents calculated from the experimental data corresponding to the torus [Fig. 8(a)] and the SNA [Fig. 8(b)] are shown in Figs. 7(a) and 7(b) as dashed curves. They are almost symmetric and the shape remains the same for the torus as well as the SNA. This confirms the transition of the quasiperiodic attractor to SNA through the fractalization route in the present experimental circuit.

5. Fractalized Intermittency Route to SNA

The next route we have encountered is the fractalized intermittency route to strange nonchaotic attractor. Here, the period-3 torus is gradually fractalized and then intermittency phenomenon occurs before reaching chaotic behavior on increasing the value of the amplitude $E_1$ of the sinusoidal forcing for a fixed value of its frequency $\omega_1$. This sequence is observed in the range of frequency $\omega_1 \in (1.1268, 1.1512)$ as a function of the amplitude $E_1 \in (0.48, 0.54)$ indicated as F-INT in Fig. 2.

5.1. Numerical analysis

We now fix the value of the frequency at $\omega_1 = 1.1339$ and vary the amplitude in the range $E_1 \in (0.48, 0.53)$ to illustrate the transition from period-3 torus to SNA via fractalization followed by intermittency phenomenon. The three strands of the period-3 torus are plotted in Fig. 10(a) for the value $E_1 = 0.48$, whose phase portrait and power spectrum are shown in Figs. 11(a)(i) and 11(a)(ii),
respectively. Further increase in the value of $E_1$ results in gradual fractalization of period-3 torus as illustrated in Fig. 10(b) for the value $E_1 = 0.52$ and the corresponding phase portrait and power spectrum are depicted in Figs. 11(b)(i) and 11(b)(ii). On increasing the value of $E_1$ further, intermittency phenomenon results as shown in Fig. 10(c) for the value of $E_1 = 0.526$. The phase portrait of the SNA and its power spectrum are shown in Figs. 11(c)(i) and 11(c)(ii), respectively. Finally, to confirm that the SNA transits to a chaotic attractor beyond $E_1 = 0.535$, we have depicted the Poincaré surface of section of the latter in Fig. 10(d) with the corresponding attractor and power spectrum.

![Figure 10](image1.png)

![Figure 11](image2.png)

**Fig. 11.** Projection of the numerically simulated attractors and their power spectrum of Eq. (2) for the same values of the frequency $\omega_1$ and the amplitude $E_1$ of the sinusoidal forcing as in Fig. 10. (a) Period-3 torus (3T), (b) fractalized SNA, (c) intermittency SNA and (d) chaotic attractor: (i) phase portrait in the $(x,y)$ space; (ii) power spectrum.
in Figs. 11(d)(i) and 11(d)(ii), respectively, for $E_1 = 0.54$.

The transition from torus to SNA is also revealed in the spectrum of the largest Lyapunov exponent and its variance by smooth and irregular variation in their values below and above the critical value $E_c^1 = 0.5186$ as shown in Figs. 12(a) and 12(b), respectively. Finally, the transition of SNA into a chaotic attractor is confirmed by the change in the largest Lyapunov exponent from negative to positive values at $E_1 > 0.538$ as shown in the inset of Fig. 12(a).

The phase portrait of the attractor [Fig. 11(c)] looks indeed strange while the value of its Lyapunov exponent is negative as evidenced in Fig. 12(a). The torus [Fig. 11(a)], fractalized SNA [Fig. 11(b)], intermittency SNA [Fig. 11(c)] and chaotic attractor [Fig. 11(d)] are further characterized by their spectral distribution function (filled circles) as shown in Figs. 13(a)–13(d), respectively, satisfying the scaling relations as indicated by solid lines with the value of exponent $\beta = 1.6$ for fractalized SNA, $\beta = 1.92$ for intermittency SNA and $\beta = 2.6$ for chaotic attractor. The distribution of their finite time Lyapunov exponents (solid line) is also depicted in Figs. 14(a) and 14(b), respectively. The elongated tail on the positive side of the distribution of finite time Lyapunov exponents corresponding to the slow decay elucidates the intermittency phenomenon.

5.2. Experimental confirmation

The phase portrait and its power spectrum of the period-3 torus for the value of the amplitude $E_{f_1} = 0.24$ V are shown in Figs. 15(a)(i) and 15(a)(ii), respectively. Fractalized SNA and its power spectrum for the value of $E_{f_1} = 0.26$ V, corresponding to the numerical analysis in Fig. 11(b), are depicted in Fig. 15(b). The SNA and its power spectrum for

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12}
\caption{Transition from torus to SNA through fractalization followed by intermittency route for the same value of frequency as in Fig. 10 and in the range of amplitude $E_{f_1} \in (0.49, 0.53)$ obtained numerically. (a) Largest Lyapunov exponent (\(\Lambda\)) and (b) its variance (\(\mu\)). Inset in (a) depicts transition from SNA to chaos for $E_1 \in (0.52, 0.542)$.}
\end{figure}
Fig. 13. Spectral distribution function (filled circles) calculated numerically. (a) Torus [Fig. 11(a)], (b) fractalized SNA [Fig. 11(b)], (c) intermittency SNA [Fig. 11(c)] and (d) chaotic attractor [Fig. 11(d)]. Solid line in (a) corresponds to the scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ and in (b)-(d) corresponds to the scaling relation $N(\sigma) \sim \sigma^{-\beta}$, with $\beta = 1.6, 1.92$ and 2.6, respectively.

Fig. 14. Distribution of finite time Lyapunov exponent calculated from both numerical data (solid line) and experimental data (dashed line) of (a) torus [Fig. 11(a)] and (b) SNA [Fig. 11(c)].
Figure 15. Snapshots of the experimental attractors and their power spectrum of the circuit shown in Fig. 1 for the corresponding values of the frequency $\omega_{f1}$ and the amplitude $E_{f1}$ of the sinusoidal forcing in Fig. 10. (a) Period-3 torus (3T), (b) fractalized SNA, (c) intermittency SNA and (d) chaotic attractor: (i) phase portrait in the $(vC, iL)$ space; (ii) power spectrum.

The value of the amplitude $E_{f1} = 0.263$ V in the intermittency regime is shown in Fig. 15(c). The chaotic attractor and its power spectrum for the values of $E_{f1} = 0.27$ V, corresponding to the numerical results of Fig. 11(d), are depicted in Fig. 15(d). It has been confirmed that the simulation results and experimentally observed results in the phase-space, as well as the associated power spectra are qualitatively similar to each other.

The spectral distribution function (filled triangles) deduced from the experimental data of the attractors shown in Figs. 15(a)(i), 15(b)(i), 15(c)(i) and 15(d)(i) are plotted respectively in Figs. 16(a)–16(d) satisfying the scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ and power-law distribution with the exponent $\beta = 1.9$ for fractalized SNA, $\beta = 1.98$ for intermittency SNA and $\beta = 3.8$ for chaotic attractor, respectively, as indicated by solid lines. The distribution of their finite time Lyapunov exponents (dashed lines) are also shown in Figs. 14(a) and 14(b), respectively. Slowly decaying tail on the positive regime of the finite time Lyapunov exponents confirms the existence of intermittency route to SNA.
6. Intermittency Route to SNA

Now, we will demonstrate the birth of strange nonchaotic attractor from period-3 torus through intermittency phenomenon in the dynamical system [Eq. (2)] and its circuit (Fig. 1). It is observed in two specific ranges of the frequency, namely \( \omega_1 \in (1.1110, 1.1268) \) and \( (1.1512, 1.2615) \) of the sinusoidal force, as a function of its amplitude \( E_1 \in (0.52, 0.53) \) represented as INT in Fig. 2.

6.1. Numerical analysis

We have chosen the value of the frequency \( \omega_1 = 1.1182 \) specifically for illustration and vary the amplitude in the range \( E_1 \in (0.52, 0.53) \). The Poincaré surface of section plot of the period-3 torus is shown in Fig. 17(a) for the value of amplitude \( E_1 = 0.52 \) and its corresponding phase portrait and power spectrum are shown in Figs. 18(a)(i) and 18(a)(ii), respectively. The attractor depicted in Fig. 18(b)(i) is indeed strange but nonchaotic as the value of its Lyapunov exponent is negative [see Fig. 19(a)]. Finally, the transition from torus to SNA is also clearly revealed by smooth (\( E_1 < E_1^c = 0.5276 \)) and irregular (\( E_1 > E_1^c = 0.5276 \)) fluctuations in the values of the largest Lyapunov exponent and its variance, respectively, as evidenced in Figs. 19(a) and 19(b) as a function of amplitude in the range \( E_1 \in (0.52, 0.53) \). The attractor depicted in Fig. 18(b)(i) is indeed strange but nonchaotic as the value of its Lyapunov exponent is negative [see Fig. 19(a)]. Finally, the
transition of SNA into a chaotic attractor is confirmed by the change in the largest Lyapunov exponent from negative to positive values at $E_1 \approx 0.53$ as shown in the inset of Fig. 19(a). The spectral distribution function (filled circles) for the torus [Fig. 18(a)(i)], intermittency SNA [Fig. 18(b)(i)] and chaotic attractor [Fig. 18(c)(i)] are shown in Figs. 20(a)–20(c), respectively. They also obey the scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ for torus and power-law distribution with the exponents $\beta = 1.9$ and $\beta = 2.7$ as indicated by solid lines, respectively, as expected. Further, the distribution of the finite time Lyapunov exponents (solid line) shown in Fig. 21(a) confirms the existence of torus and the slowly decaying tail in its distribution in Fig. 21(b) confirms the birth of SNA through intermittency phenomenon.

6.2. Experimental confirmation

The snapshots of phase portrait and power spectrum of the period-3 torus exhibited by the circuit (Fig. 1) corresponding to the numerical analysis, Figs. 18(a)(i) and 18(a)(ii), for the value of $E_{f1} = 0.26 \text{V}$ are shown in Figs. 22(a)(i) and 22(a)(ii), respectively. Similarly, Figs. 22(b)(i) and 22(b)(ii) correspond to the phase portrait and power spectrum of SNA (corresponding to Figs. 18(b)(i) and 18(b)(ii) obtained numerically) for the value of the amplitude $E_{f1} = 0.265 \text{V}$. Finally, the chaotic attractor and its power spectrum for the value of $E_{f1} = 0.28 \text{V}$, corresponding to the numerical analysis, Fig. 18(c), are depicted in Fig. 22(c). The resemblance of the phase portraits of the torus, intermittency SNA and chaotic attractor of the experimental results with that of the numerical results confirms the transition from torus to SNA via intermittency phenomenon. Further, the experimental data corresponding to the torus, intermittency SNA and chaotic attractor are quantified in terms of the power spectral distribution (filled triangles) as shown in Figs. 23(a)–23(c), respectively, satisfying scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ by torus and power-law distribution with the exponent...
Fig. 18. Projection of the numerically simulated attractors and their power spectrum of Eq. (2) for the same values of the frequency $\omega_1$ and the amplitude $E_1$ of the sinusoidal forcing as in Fig. 17. (a) Period-3 torus (3T), (b) intermittency SNA and (c) chaotic attractor: (i) phase portrait in the $(x,y)$ space; (ii) power spectrum.
Fig. 19. Transition from torus to SNA through intermittency route for the same value of frequency as in Fig. 17 and in the range of amplitude \( E_1 \in (0.52, 0.532) \) obtained numerically. (a) Largest Lyapunov exponent (\( \Lambda \)) and (b) its variance (\( \mu \)). Inset in (a) depicts transition from SNA to chaos for \( E_1 \in (0.536, 0.56) \).

Fig. 20. Spectral distribution function (filled circles) calculated numerically. (a) Torus [Fig. 18(a)], (b) intermittency SNA [Fig. 18(b)] and (c) chaotic attractor [Fig. 18(c)]. Solid line in (a) corresponds to the scaling relation \( N(\sigma) \sim \log_{10}(1/\sigma) \) and in (b) and (c) corresponds to the scaling relation \( N(\sigma) \sim \sigma^{-\beta} \), with \( \beta = 1.9 \) and 2.7, respectively.
Fig. 21. Distribution of finite time Lyapunov exponent calculated from both numerical data (solid line) and experimental data (dashed line) of (a) torus [Fig. 18(a)] and (b) SNA [Fig. 18(b)].

Fig. 22. Snapshots of the experimental attractors and their power spectrum of the circuit shown in Fig. 1 for the corresponding values of the frequency $\omega_f$ and the amplitude $E_f$ of the sinusoidal forcing in Fig. 17. (a) Period-3 torus (3T), (b) intermittency SNA and (c) chaotic attractor: (i) phase portrait in the $(v_C, i_L)$ space; (ii) power spectrum.
7.1. Numerical analysis

In particular, we fix the value of the frequency of the sinusoidal forcing at $\omega_1 = 1.2515$ and investigate the birth of SNA via HH route on decreasing the value of the amplitude from $E_1 = 1.09$ in the range $E_1 \in (1.09, 1.06)$. The Poincaré surface of section of the period-3 torus is illustrated in Fig. 24(a) for $E_1 = 1.09$. The corresponding phase portrait and its power spectrum are shown in Figs. 25(a)(i) and 25(a)(ii), respectively. The period doubled torus (6T), is depicted in Fig. 24(b) for the value of $E_1 = 1.072$, whose phase portrait and power spectrum are shown in Figs. 25(b)(i) and 25(b)(ii). On decreasing the value of the amplitude, further doubling does not take place. Instead, the period-6 torus becomes wrinkled by colliding with its unstable period-3 torus and finally the period-6 torus merges to give birth to SNA as seen in Fig. 24(c) for $E_1 = 1.06$, whose phase portrait and power
Fig. 24. Projection of the numerically simulated Poincaré surface section plots of the attractors of Eq. (2) in the $(\phi, x)$ plane for the fixed value of the frequency of the sinusoidal forcing $\omega_1 = 1.2515$ as a function of its amplitude $E_1$ indicating the transition from quasiperiodic attractor to SNA through Heagy–Hammel route: (a) period-3 torus (3T) for $E_1 = 1.09$, (b) period-6 torus (6T) for $E_1 = 1.072$, (c) Heagy–Hammel SNA for $E_1 = 1.06$ and (d) chaotic attractor for $E_1 = 1.05$.

Fig. 25. Projection of the numerically simulated attractors and their power spectrum of Eq. (2) for the same values of the frequency $\omega_1$ and the amplitude $E_1$ of the sinusoidal forcing as in Fig. 24. (a) Period-3 torus (3T), (b) period-6 torus (6T), (c) Heagy–Hammel SNA and (d) chaotic attractor: (i) phase portrait in the $(x, y)$ space; (ii) power spectrum.
spectrum are given in Figs. 25(c)(i) and 25(c)(ii), respectively. Finally, to confirm that the SNA transits to a chaotic attractor before $E_1 = 1.055$, we have depicted the Poincaré surface of section of the latter in Fig. 24(d) with the corresponding attractor and power spectrum in Fig. 25(d) for $E_1 = 1.05$. The corresponding largest Lyapunov exponent and its variance are shown in Figs. 26(a) and 26(b), respectively. The transition from torus to SNA is also indicated by the smooth ($E_1 < E_1^* = 1.0744$) and irregular ($E_1 > E_1^* = 1.0744$) variation in the values of the largest Lyapunov exponent and its variance as depicted in Fig. 26. The transition of SNA into a chaotic attractor is confirmed by the change in the largest Lyapunov exponent from negative to positive values at $E_1 > 1.054$ as shown in the inset of Fig. 26(a). The spectral distributions (filled circles) satisfy scaling relationship
Fig. 26. Transition from torus to SNA through Heagy–Hammel route for the same value of frequency as in Fig. 24 and in the range of amplitude $E_1 \in (1.09, 1.06)$ obtained numerically. (a) Largest Lyapunov exponent ($\Lambda$) and (b) its variance ($\mu$). Inset in (a) depicts transition from SNA to chaos for $E_1 \in (1.07, 1.05)$.

Fig. 27. Spectral distribution function (filled circles) calculated numerically. (a) Period-3 torus [Fig. 25(a)], (b) period-6 torus [Fig. 25(b)], (c) SNA [Fig. 25(c)] and (d) chaotic attractor [Fig. 25(d)]. Solid line in (a) and (b) corresponds to the scaling relation $N(\sigma) \sim \log_{10}(1/\sigma)$ and in (c) and (d) corresponds to the scaling relation $N(\sigma) \sim \sigma^{-\beta}$, with $\beta = 1.86$ and 2.8, respectively.
Fig. 28. Distribution of finite time Lyapunov exponent calculated from both numerical data (solid line) and experimental data (dashed line) of (a) torus [Fig. 25(a)] and (b) SNA [Fig. 25(c)].

Fig. 29. Snapshots of the experimental attractors and their power spectrum of the circuit shown in Fig. 1 for the corresponding values of the frequency $\omega_f$ and the amplitude $E_f$ of the sinusoidal forcing in Fig. 25. (a) Period-3 torus (3T), (b) period-6 torus (6T), (c) Heagy–Hammel SNA and (d) chaotic attractor: (i) phase portrait in the $(v_C,i_L)$ space; (ii) power spectrum.
N(σ) \sim \log_{10}(1/σ) \text{ for } 3T \text{ torus and } 6T \text{ torus and power-law distribution with the exponent } \beta = 1.86 \text{ for SNA and } \beta = 2.8 \text{ for chaotic attractor as indicated by solid lines in Fig. 27. The distribution of the finite time Lyapunov exponents (solid line) is also shown in Figs. 28(a) and 28(b) for both the torus and SNA, respectively. The sudden decay in the distribution of finite time Lyapunov exponent for the SNA [Fig. 28(b)] confirms the birth of SNA via Heagy–Hammel route.}

7.2. Experimental confirmation

The snapshot of the phase portrait and power spectrum of the period-3 torus are presented in Figs. 29(a)(i) and 29(a)(ii) for \( E_{f1} = 0.545 \text{ V} \), respectively. Similarly, the period-6 torus and its power spectrum for the value of the amplitude \( E_{f1} = 0.536 \text{ V} \) are depicted in Figs. 29(b)(i) and 29(b)(ii), respectively. The SNA and its power spectrum corresponding to the value \( E_{f1} = 0.53 \text{ V} \) are shown in Figs. 29(c)(i) and 29(c)(ii). Finally, the chaotic attractor and its power spectrum for the values of \( E_{f1} = 0.525 \text{ V} \), are depicted in Figs. 29(d)(i) and 29(d)(ii). The experimental data of the period-3 torus [Fig. 29(a)(i)], period-6 torus [Fig. 29(b)(i)], HH SNA [Fig. 29(c)(i)] and the chaotic attractor [Fig. 29(d)(i)] are further characterized by their spectral distribution functions (filled triangles) as shown in Figs. 30(a)–30(d), respectively. The 3T torus and 6T torus satisfy the scaling relation \( N(σ) \sim \log_{10}(1/σ) \), while power-law distribution with the exponent \( \beta = 1.96 \) for HH SNA and \( \beta = 4.7 \) for chaotic attractor, respectively, are identified in Fig. 30 and are indicated by solid lines. The distribution of finite time Lyapunov exponents of both the torus and the SNA calculated from their experimental data are plotted in Figs. 28(a) and 28(b) as dashed line. The sudden decay in the tail of the distribution for SNA confirms the birth of SNA via Heagy–Hammel route.
8. Bubbling Route to SNA
Apart from the above studied routes, we have identified a new route called bubbling route to SNA [Senthilkumar et al., 2008]. In this new route, bubbles appear in the strands of the torus as the value of the amplitude $E_1$ of the sinusoidal forcing is increased for a fixed value of its frequency $\omega_1$. The sizes of the bubbles increase further on increasing $E_1$.

Fig. 31. Projection of the numerically simulated Poincaré surface of section of the attractors of Eq. (2) in the $(\phi, x)$ plane for a fixed value of the frequency of the sinusoidal forcing, $\omega_1 = 1.0852$, as a function of its amplitude $E_1$ indicating the transition from quasiperiodic attractor to SNA through bubbling route: (a) period-3 torus (3T) for $E_1 = 0.5$, (b) bubbled strands of period-3 torus (3T) for $E_1 = 0.52$, (c) enlarged bubbles in the strands of period-3 torus (3T) for $E_1 = 0.54$, (d) fractalized bubbles for $E_1 = 0.546$ with the remaining parts (away from bubble) of the strands unaffected and (e) chaotic attractor (widely interspersed bubbles) for $E_1 = 0.56$. 
the amplitude $E_1$ and the bubbles increasingly get wrinkled (while the rest of the strands of the torus outside the bubbles remain largely unaffected) resulting in the birth of an SNA. This bubbling route is observed in the rather narrow range of frequency $\omega_1 \in (1.085, 1.086)$ as a function of the amplitude of the sinusoidal forcing $E_1 \in (0.5, 0.55)$ indicated as BUB in Fig. 2. It is to be noted that this route is significantly different from the well-known fractalization route [Nishikawa & Kaneko, 1996], where the entire strands of the $n$-period torus continuously deform and get extremely wrinkled as a function of the control parameter. The formation of SNAs through this novel bubbling route has been identified in the literature for the first time to the best of our knowledge.

8.1. Numerical analysis

We have fixed the value of the frequency of the sinusoidal forcing at $\omega_1 = 1.0852$ for illustration and varied its amplitude in the range $E_1 \in (0.5, 0.55)$ to elucidate the emergence of bubbling route to SNA in the present system (2). The Poincaré surface of section plot of the three strands corresponding to a period-3 torus for the value of $E_1 = 0.5$ is shown in Fig. 31(a). The corresponding phase portrait and power spectrum are depicted in Figs. 32(a)(i) and 32(a)(ii), respectively. As the value of the amplitude $E_1$ is increased further, bubbles start to appear in all the three strands starting from $E_1 = 0.516$. These are shown in Fig. 31(b) for $E_1 = 0.52$ and the corresponding phase portrait and power spectrum are shown in Figs. 32(b)(i) and 32(b)(ii), respectively. Further increase in the value of $E_1$ results in an increase in the size of the bubbles as shown in Fig. 31(c) for the value of $E_1 = 0.54$, whose phase portrait and power spectrum are shown in Figs. 32(c)(i) and 32(c)(ii), respectively. Beyond the value of $E_1 = 0.54$, the strands of bubbles deform and get increasingly wrinkled (while the other parts of the strands of period-3 torus outside...
the bubbles remain unaltered) leading to the formation of the SNA as depicted in Fig. 31(d) for the value of $E_1 = 0.546$. The phase portrait and power spectrum for this value of $E_1$ are shown in Figs. 32(d)(i) and 32(d)(ii), respectively. Finally, to confirm that the SNA transits to a chaotic attractor beyond $E_1 = 0.55$, we have depicted the Poincaré surface of section of the latter in Fig. 31(e) with the corresponding attractor and power spectrum in Fig. 32(e) for $E_1 = 0.56$.

The mechanism for the bubbling route is that the quasiperiodic orbit becomes increasingly unstable in its transverse direction as a function of the control parameter ($E_1$), resulting in the formation of the doubled strands (bubbles), as seen in Fig. 31(b), in certain parts of the main strands. This instability of the quasiperiodic attractor arises due to the presence of the square wave pulse (finite amplitude for finite durations). Further increase in the value of the amplitude of the forcing ($E_1$) results
Fig. 33. Snapshots of the experimental attractors and their power spectrum of the circuit shown in Fig. 1 for the corresponding values of the frequency $\omega_{f_1}$ and the amplitude $E_{f_1}$ of the sinusoidal forcing in Fig. 31. (a) Period-3 torus (3T), (b) bubbled period-3 torus, (c) period-3 torus with enlarged bubbles, (d) fractalized bubbles (SNA) and (e) chaotic attractor: (i) phase portrait in the $(v_C; i_L)$ space; (ii) power spectrum.
in an increase in the size of the doubled strands (bubbles) as shown in Fig. 31(c), and then the doubled strands become extremely wrinkled (without a complete doubling of the entire main strand) resulting in the SNA as depicted in Fig. 31(d).

We have presented the quantitative confirmation of the above results to distinguish between torus and SNA, and SNA and chaos in Senthilkumar et al. [2008].

8.2. Experimental confirmation

As a next step, in order to confirm the results of our numerical simulation in the experimental circuit shown in Fig. 1, a snapshot of the dynamical behavior for the corresponding values of the experimental parameters is obtained (as mentioned in Sec. 2) and compared with that of the numerical results. Further, the corresponding experimental data are analyzed using various quantification measures mentioned in the previous section to confirm the nature of the dynamical behavior.

We have depicted the snapshots of the phase portraits and the corresponding power spectra of the attractors as seen in the oscilloscope (which is connected to the circuit shown in Fig. 1) in Fig. 33 for the corresponding values of the parameters of numerical simulation. The experimental period-3 torus and its power spectrum corresponding to the numerical results, Fig. 32(a), are shown in Figs. 33(a)(i) and 33(a)(ii). The attractors in the bubbling regime for the values of the amplitude of the sinusoidal forcing $E_1 = 0.26$ V and 0.27 V are shown in Figs. 33(b)(i) and 33(c)(i), respectively. The corresponding power spectra are shown in Figs. 33(b)(ii) and 33(c)(ii), respectively. The experimental phase portrait of the strange nonchaotic attractor and its power spectrum for the value of $E_1 = 0.273$ V are depicted in Figs. 33(d)(i) and 33(d)(ii), respectively. It is also seen that the spectra of the quasiperiodic attractors are concentrated at a small discrete set of frequencies while the spectrum of the SNA has a much richer set of harmonics.

Further, the resemblance of the attractors illustrated in Fig. 33(i) with that of the attractors in Fig. 32(i) confirms the existence of the bubbling transition to the SNA in this negative conductance series LCR circuit with a diode having both sinusoidal and nonsinusoidal forces as quasiperiodic forcings. Finally, the chaotic attractor for $E_1 = 0.28$ V and its power spectrum are shown in Fig. 33(e). For details on the quantitative analysis of these experimental results, we may refer to [Senthilkumar et al., 2008].

9. Conclusion

In this paper, we have reported the birth of SNAs through various routes namely, fractalization, fractalization followed by intermittency, intermittency, Heagy–Hammel and bubbling routes in the series LCR circuit with one of the forcings as a non-sinusoidal forcing. At first, we have presented the numerical analysis of the dynamical system represented by Eq. (1c) of the circuit (Fig. 1) for suitable values of the amplitude $E_1$ and the frequency $\omega_1$ of the sinusoidal force while the other parameters are held fixed. Then, we have experimentally confirmed the existence of SNAs by the snapshots of the phase portraits of the various attractors for the corresponding experimental values of the parameters. Further, the corresponding experimental data are analyzed using various quantification measures attributing to the existence of torus, SNA and birth of SNA through different routes and transition to chaos. We have characterized the quasiperiodic attractors, SNAs and chaotic attractors using maximal Lyapunov exponent and its variance, Poincaré maps, Fourier amplitude spectra, spectral distribution function and distribution of finite time Lyapunov exponents. We have also examined the distribution of local Lyapunov exponents and found that they take on different characteristics for SNAs created through different mechanisms. The experimental observations, numerical simulations and characteristic analysis showed that the simple dissipative negative conductance series LCR circuit even with nonsinusoidal (square wave) force as one of the quasiperiodic force does indeed have strange nonchaotic behaviors of different types.

Acknowledgments

This work has been supported by the Department of Science and Technology, Government of India sponsored IRHPA research project. D. V. Senthilkumar has been supported by Alexander von Humboldt Foundation and M. Lakshmanan has been supported by a DST Ramanna Fellowship program research grant.

References


