Bi-cooperative Games with fuzzy bi-coalitions

Surajit Borkotokey∗ Pranjal Sarmah †

Abstract

In this paper, we introduce the notion of a bi-cooperative game with fuzzy bi-coalitions and discuss the related properties. In real game theoretic decision making problems, many criteria concerning the formation of coalitions have bipolar motives. Our model tries to explore such bipolarity in fuzzy environment. The corresponding Shapley axioms are proposed. An explicit form of the Shapley value as a possible solution concept to a particular class of such games is also obtained. Our study is supplemented with an illustrative example.

Keywords: Fuzzy sets, bi-cooperative Games, bi-coalitions, Shapley function.

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∗Department of Mathematics, Dibrugarh University, Dibrugarh, India-786004
†Department of Mathematics, Gauhati University, Guwahati, India-781014
1 Introduction

Given a finite set \( N \) of \( n \) players, and \( \mathcal{Q}(N) \), the set of all pairs \((S, T)\) with \( S, T \subseteq N \), and \( S \cap T = \emptyset \), a bi-cooperative game is defined as a function \( b : \mathcal{Q}(N) \to \mathbb{R} \) satisfying \( b(\emptyset, \emptyset) = 0 \). The pairs \((S, T)\) are called bi-coalitions. Players in \( S \) are usually called positive contributors while those of \( T \) are negative contributors to the game and the members of \( N \setminus (S \cup T) \) are the neutral players. Thus for each \((S, T) \in \mathcal{Q}(N)\), \( b(S, T) \) represents the worth when players in \( S \) contribute positively, those in \( T \) contribute negatively and the remaining players do not participate. Note that here, worths may be both positive and negative. Bi-cooperative games have been introduced by Bilbao et al. [4] as a generalization of TU cooperative games. In the literature on game theory, however, the first works on the introduction of several levels of cooperations are those of multi-choice games [20]. The major domain of application of games with several “levels of participations” is the “Cost Allocation Problems” [32]. The idea of choosing between two mutually exclusive roles by the players in a cooperative situation was originated from Felsenthal and Machover [13] who defined ternary voting games.

A one point solution concept for bi-cooperative games is a function, which assigns to every game an \( n \)-dimensional real vector that represents a payoff distribution over the players. In [23], Labreuche et al. have axiomatized a value intuitively. A value for bi-cooperative games has also been defined in [4] (and similarly in [17, 18]), however the two approaches differ by principle. While the later is isomorphic to multi-choice games with three levels of participations, the former is a bipolar extension of the power set of \( N \).

Theory of fuzzy sets since its inception in 1965, by L.A. Zadeh [41], has become an important tool to model human behaviour in decision sciences. A large amount of research works has been carried out in the theory of cooperative games in fuzzy environment, see for example [2, 3, 7, 8, 9, 24, 25, 34]. When players partially participate in a coalition with some membership degrees in the interval \([0, 1]\), we call it a fuzzy coalition. In a similar fashion, when players participate partially in a bi-coalition (i.e in either of the mutually exclusive roles), we can call it a fuzzy bi-coalition. As shown in the following example, there are situations where the notion of a fuzzy bi-coalition essentially makes sense.

Example (The Issue of Mega Dam Construction) The issue of constructing mega dams over some of the rivers in India has been creating mass concern both in support and against. Mega dams are built mainly for generating hydro-electricity. The bigger is the dam, the more electricity is produced. Environmentalists and other pressure groups usually oppose such drives citing to adverse ecological impact on the downstream areas and rather advocate for small ones. A substantial amount of water stored in those dams can be used for irrigation, for maintaining a steady flow in the river to control flood and other welfare projects. A reduction of the height of dam walls
would lower the possibility of large mass destruction during disaster. The agencies, involved in
construction and functioning of the dams, on the other hand, benefit more from mega projects,
as the production and maintenance costs for the small ones and those involving other welfare
components are comparatively high. Meanwhile, there are also agencies, who would mobilize the
common illiterate people against such projects for their own political motives. Thus, the issue of
mega dam creates three disjoint groups of activists: the supporters’ group, the opposition group
and the neutral group whose members are indifferent to the issue. However, in this case, members
of either groups (except the neutral group) will not only rest by supporting or opposing the move
and rather would try to exert pressure over the other group in support of their own stand by
means of scientific expertise, mass awareness and mobilization, shouldering financial responsibili-
ties, forming political pressure groups and so on to name a few. If we can measure the levels of
the contributions of each individual in a group by some suitable mean, then this situation, where
participation of the players are bipolar yet layered in the sense that each member provides her
contribution (ranging between 0 and 1) in running the project or to scrap it or remains indifferent
(by contributing nothing to either of the groups) would require a rather generalized game theoretic
framework. The game representing such situations would be called a bi-cooperative game with
fuzzy bi-coalitions (or fuzzy bi-cooperative game in short). Investment problems in Share markets,
coalition formation among political parties, cost allocations involved in sharing of technology and
so on. are some other pertinent examples. In the literature so far, no such study on the use of
fuzzyness for describing a bi-cooperative game is noticed.

In this paper, we introduce the notion of a bi-cooperative game with fuzzy bi-coalitions which
has significant analogy with cooperative games with fuzzy coalitions discussed by Aubin [3], But-
nariu [8], Tsurumi et al. [34], Butnariu and Kroupa [9], Zhang et al. [24] and so on. The fuzzy games
hitherto found in the literature do not address the problem of putting memberships in either of two
mutually exclusive roles. It is indeed interesting to see that the existing models [3, 8, 9, 24, 34] do
not address the effects of players participating in such roles, rather investigate how to distribute
the gain among the members according to their rates of participation in a single fuzzy coalition.
Our study is based on a generalization of the bi-cooperative games into its fuzzy counterpart. We
propose the axioms of a Shapley value as a possible solution concept to a bi-cooperative game
with fuzzy bi-coalitions. These axioms are applicable to any class of bi-cooperative games with
fuzzy bi-coalitions. We further introduce a new family of fuzzy bi-cooperative games similar to
ones in cooperative settings, however, they differ from their counterparts by the formulation of the
problem domain. This class, whose members will be called fuzzy bi-cooperative game in Choquet
Integral form, is an extension of the one given by Tsurumi et al. [34]. Grabisch [16] gave a natural
justification of using Choquet Integrals as a linear interpolator between vertices of the \([-1,1]^n\)
hypercube using the least possible number of vertices, in the context of multi-criteria decision
making. Since a fuzzy game can be seen as an extension of a crisp game (defined on the vertices only), to the entire hypercube, and the Choquet Integral in both cooperative and bi-cooperative format is the best linear interpolator, it becomes a natural choice for extending a crisp game to its fuzzy counterpart in this way. This class also possesses monotonicity (whenever the corresponding crisp game is monotone) and continuity with respect to players’ memberships in both positive and negative roles. An explicit form of the Shapley function for this class of games has been obtained and finally an illustrative example is provided to supplement our findings. We have used the properties of a signed choquet integral discussed in [10, 11, 14, 15, 29] in obtaining the Shapley value of this class.

This paper is organized as follows: In section 2, we discuss the relevant properties of a bi-cooperative game and the corresponding solution concepts. In section 3, we introduce the notion of a bi-cooperative game with fuzzy bi-coalitions and propose the Shapley axioms for this game. In section 4, a special class of bi-cooperative games with fuzzy bi-coalitions is defined and corresponding properties are investigated. A possible Shapley function for this class has been investigated and characterized with the Shapley axioms. Section 5 and 6, respectively include an illustrative example and the concluding remarks. Throughout the paper, we denote by \( 2^N \), the set of coalitions of \( N \). We will often omit braces for singletons, e.g., writing \( S \cup i \) and \( S \setminus i \) instead of \( S \cup \{i\} \) and \( S \setminus \{i\} \) respectively whenever \( S \subseteq N \). Moreover cardinality of a subset \( S \) of \( N \) will be denoted by corresponding lower case letter, i.e. \(|S| =: s\).

2 Bi-cooperative games and a solution concept

Let us denote by \( \mathcal{BG}^N \) the class of all bi-cooperative games on \( N \). While in standard cooperative games, each coalition \( S \in 2^N \) can be identified with a \( \{0,1\} \)-vector, in a bi-cooperative game, each bi-coalition \((S,T)\) can be identified with the \( \{-1,0,1\} \)-vector \( 1_{S,T} \) defined, for all \( i \in N \), by

\[
1_{S,T}(i) = \begin{cases} 
1, & \text{if } i \in S \\
-1, & \text{if } i \in T \\
0, & \text{otherwise}
\end{cases}
\]

It is indeed interesting to note that the bi-cooperative games given in [4, 17, 18] are isomorphic to multi-choice games given in [20] with three levels of participation. This is because the order relation denoted by \( \sqsubseteq \) in \( Q(N) \) is the one implied by monotonicity (i.e. for \((S,T),(S',T') \in Q(N), (S,T) \sqsubseteq (S',T') \iff S \subseteq S' \text{ and } T \subseteq T' \)). This makes the two elements \((\emptyset,N)\) and \((N,\emptyset)\) as the bottom and top elements of \( Q(N) \). Nevertheless, in [23], the order relation is taken as the product order, i.e. for \((S,T),(S',T') \in Q(N), (S,T) \sqsubseteq (S',T') \iff S \subseteq S' \text{ and } T \subseteq T' \) so that \((\emptyset,\emptyset)\) becomes the bottom element and all \((S,N \setminus S), S \subseteq N\), the top elements. This idea adheres to the notion of bi-cooperative games as it incorporates bipolarity, a crucial concept for such games.
Therefore, in this paper, we shall adopt this later approach. In [23], a value on $BG^N$ is defined as a function $\Phi : BG^N \rightarrow (\mathbb{R}^n)^Q(N)$ which associates each bi-cooperative game $b$ a vector $(\Phi_1(b), \Phi_2(b), ..., \Phi_n(b))$ representing a payoff distribution to the players in the game. Note that the elements $\Phi_i(b)$ belong to $\mathbb{R}^Q(N)$ and hence for $(S,T) \in Q(N), \Phi_i(b)(S,T) \in \mathbb{R}, \forall i \in N$. It is further interesting to note that there are two more definitions of a value found in the literature (see [23]). In [4, 17, 18], the value is defined as a function $\Phi : BG^N \rightarrow \mathbb{R}^n$. In the same references, a refined value is also defined as a function $\Phi : BG^N \rightarrow (\mathbb{R}^n)^2$. However, we shall adopt here, the definition given in [23] as it endorses bipolarity of a bi-cooperative game in a more natural way. There are different proposals of solution concepts hitherto found in the literature as already mentioned in the introduction. Among them the first proposal was put forward by Felsenthal and Machover [13] especially for ternary voting games. Their notion applies mainly to voting games and the corresponding expressions seem to be too technical. In proposals made by Labreuche and Grabisch [17, 18] and Bilbao et al. [4], only the largest and least elements of the lattice $(Q(N), \sqsubseteq)$ are of primary concerns. This would require all players to agree to a certain proposal or oppose it to form the bi-coalitions $(N, \emptyset)$ and $(\emptyset, N)$ respectively. However this requirement is too harsh to impose in most situations and as already mentioned, in particular, does not comply with the notion of bipolarity present in a bi-cooperative game. Following [23], we have the next definition.

**Definition 2.1.** Let $b \in BG^N$. A player $i$ is called positively monotone with respect to $b$ if

$$\forall (S,T) \in Q(N \setminus i), \ b(S \cup i, T) \geq b(S, T).$$

A player $i$ is negatively monotone with respect to $b$ if

$$\forall (S,T) \in Q(N \setminus i), \ b(S, T \cup i) \leq b(S, T).$$

The bi-cooperative game $b$ is monotone if all players are positively and negatively monotone with respect to $b$.

Let $BG_M(N)$ denote the class of monotone games over $Q(N)$.

**Remark 2.2.** The expression $b(S \cup i, T) - b(S, T)$ (respectively $b(S, T) - b(S, T \cup i)$) may be called the marginal contribution of player $i$ with respect to $(S,T) \in Q(N \setminus i)$ when she is a positive contributor (respectively a negative contributor).

Now, for $(S,T) \in Q(N)$, set,

$$Q_{(S,T)}(N) = \{(S', T') \in Q(N), S' \subseteq S, T' \subseteq T\} \quad (2.1)$$

Prior to defining the value in a bi-cooperative game we define the following:
Definition 2.3. Let \((S,T) \in Q(N)\) and \(b \in \mathcal{BG}^N\). Player \(i \in N\) is a null player in \((S,T)\) for \(b\), if it satisfies
\[
b(S' \cup i, T') = b(S', T') = b(S', T' \cup i)
\]
for every \((S', T') \in Q(S,T)(N \setminus i)\).

Note that the above definition of a null player is specified not only by the game, but also by a bi-coalition and is a slight variation of the one given in [23]. However it will not change the formulation of a value given there. This will be evidenced as a simple consequence of theorem 2.5, that the only terms of a bi-cooperative game \(b\) that are used to determine the value belong to the set \(Q(S,T)(N)\), defined in (4.1). Thus keeping this in mind, the following definition of a value on \(\mathcal{BG}^N\) and its corresponding characterization are derived from that given in [23].

Definition 2.4. A function \(\Phi' : \mathcal{BG}^N \to (\mathbb{R}^n)^{Q(N)}\) defines a value due to Labreuche and Grabisch, for every \((S,T) \in Q(N)\), if it satisfies the following axioms:

Axiom b1 (Efficiency) : If \(b \in \mathcal{BG}^N\), it holds,
\[
\sum_{i \in N} \Phi'_i(b)(S,T) = b(S,T)
\]

Axiom b2 (Linearity) : For all \(\alpha, \beta \in \mathbb{R}\) and \(b, v \in \mathcal{BG}^N\),
\[
\Phi'_i(\alpha b + \beta v) = \alpha \Phi'_i(b) + \beta \Phi'_i(v).
\]

Axiom b3 (Null Player Axiom) : If player \(i\) is null for \(b \in \mathcal{BG}^N\) in \((S,T)\), then,
\[
\Phi'_i(b)(S,T) = 0.
\]

Axiom b4 (Intra-Coalition Symmetry): If \(b \in \mathcal{BG}^N\) and a permutation \(\pi\) is defined on \(N\), such that \(\pi S = S\) and \(\pi T = T\), then it holds that, for all \(i \in N\), \(\Phi'_{\pi_i}(\pi b)(\pi S, \pi T) = \Phi'_i(b)(S,T)\)

Where \(\pi b(\pi S, \pi T) = b(S,T)\) and \(\pi S = \{\pi i : i \in S\}\).

Axiom b5 (Inter-Coalition Symmetry) : Let \(i \in S\) and \(j \in T\), and \(b_i, b_j\) be two bi-cooperative games such that for all \((S', T') \in Q(S,T)(N \setminus \{i,j\})\),
\[
b_i(S' \cup i, T') - b_i(S', T') = b_j(S', T' \cup j)
\]
\[
b_i(S' \cup i, T' \cup j) - b_i(S', T' \cup j) = b_j(S' \cup i, T') - b_j(S' \cup i, T' \cup j)
\]

Then,
\[
\Phi'_i(b_i)(S,T) = -\Phi'_j(b_j)(S,T).
\]

Axiom b6 (Monotonicity): Given \(b, b' \in \mathcal{BG}^N\) such that \(\exists i \in N\) with
\[
b'(S', T') = b(S', T')
\]
\[
b'(S' \cup i, T') \geq b(S' \cup i, T')
\]
\[
b'(S', T' \cup i) \geq b(S', T' \cup i)
\]

for all \((S', T') \in Q(S,T)(N \setminus i)\), then \(\Phi'_i(b')(S,T) \geq \Phi'_i(b)(S,T)\).
Note that the value defined above is specified not only by the game but also by the bi-coalition \((S, T)\).

**Theorem 2.5.** Let \(\Phi'\) be a value on \(BG^N\) for \((S, T) \in Q(N)\). The value \(\Phi'\) satisfies Axiom \((b1)-(b6)\) if and only if it has for all \(i\),

\[
\Phi'_i(b)(S, T) = \sum_{K \subseteq (S \cup T) \setminus i} \frac{k!(s + t - k - 1)!}{(s + t)!} [V(K \cup i) - V(K)]
\]

(2.7)

where for \(K \subseteq S \cup T\), \(V(K) := b(S \cap K, T \cap K)\).

The following result as a corollary to Theorem 2.5 is also important for our findings:

**Result 2.6.** We have,

\[
\forall i \in N \setminus (S \cup T), \quad \Phi'_i(b)(S, T) = 0
\]

(2.8)

\[
\forall i \in S, \text{ with } i \text{ positively monotone}, \quad \Phi'_i(b)(S, T) \geq 0
\]

(2.9)

\[
\forall i \in T, \text{ with } i \text{ negatively monotone}, \quad \Phi'_i(b)(S, T) \leq 0
\]

(2.10)

### 3 Bi-cooperative game with fuzzy bi-coalitions and Shapley value

In this section, we define a bi-cooperative game with fuzzy bi-coalitions and a corresponding Shapley function as a solution concept. Let us now define a fuzzy bi-coalition as follows:

**Definition 3.1.** Let \(N = \{1, 2, ..., n\}\) be as usual the players’ set. A fuzzy bi-coalition is an expression \(A\) of \(N\) given by

\[
A = \{< i, \mu_A(i), \nu_A(i) > \mid i \in N, \min_{i \in N} (\mu_A(i), \nu_A(i)) = 0\}
\]

where, \(\mu_A^N : N \to [0, 1], \nu_A^N : N \to [0, 1]\) represent respectively, the membership functions over \(N\) of the fuzzy sets of positive and negative contributors of \(A\).

**Remark 3.2.** Admission of the minimum condition in the above definition is justified by the fact that the two options (positive and negative contributions) being mutually exclusive, one can not choose a little bit of both options simultaneously.

Thus, it is apparent from Definition 3.1 that a fuzzy bi-coalition \(A\) of \(N\) is fully characterized by the functions \(\mu_A^N\) and \(\nu_A^N\). As \(N\) is fixed here, now onwards we omit \(N\) from \(\mu_A^N\) and \(\nu_A^N\) and simply use \(\mu_A\) and \(\nu_A\). We call \(i\) a positive contributor in \(A\) if \(\mu_A(i) > 0\) and a negative contributor if \(\nu_A(i) > 0\). Let \(FB(N)\) denote the set of all fuzzy bi-coalitions on \(N\). Moreover, every crisp bi-coalition can be considered as a fuzzy bi-coalition with memberships either 0 or 1.
Thus with an abuse of notations, we write \( Q(N) \subseteq F_B(N) \).

For comparing two \( A, B \in F_B(N) \), we adopt the following operations and relations in accordance with their crisp counterparts given in [23]:

\[
A \leq B \iff \mu_A(i) \leq \mu_B(i) \text{ and } \nu_A(i) \leq \nu_B(i) \forall i \in N. \\
A = B \iff \mu_A(i) = \mu_B(i) \text{ and } \nu_A(i) = \nu_B(i) \forall i \in N.
\]

For any \( A \in F_B(N) \), we denote by \( F_B(A) \), the set of all fuzzy bi-coalitions \( B \) such that \( B \leq A \). The union and intersection of two fuzzy bi-coalitions \( A \) and \( B \) are obtained using the maximum and minimum operators \('\lor'\) and \('\land'\) respectively. Formally we have :

\[
A \cup B = \{<i, \mu_A(i) \lor \mu_B(i), \nu_A(i) \lor \nu_B(i) > | i \in N\} \\
A \cap B = \{<i, \mu_A(i) \land \mu_B(i), \nu_A(i) \land \nu_B(i) > | i \in N\}.
\]

The Support of a fuzzy bi-coalition \( A \), denoted by \( \text{Supp}(A) \) is given by

\[
\text{Supp}(A) = \{\{i \in N | \mu_A(i) > 0\}, \{i \in N | \nu_A(i) > 0\}\}
\]

**Definition 3.3.** The null fuzzy bi-coalition \( \emptyset_B \) is given by

\[
\emptyset_B = \{<i, \mu_{\emptyset_B}(i), \nu_{\emptyset_B}(i) > | i \in N\}
\]

where \( \mu_{\emptyset_B}(i) = 0 \), and \( \nu_{\emptyset_B}(i) = 0 \) \( \forall i \in N \).

Let us define a bi-cooperative game with fuzzy bi-coalitions as follows:

**Definition 3.4.** A bi-cooperative game with fuzzy bi-coalitions is a function \( w : F_B(N) \rightarrow \mathbb{R} \) with \( w(\emptyset_B) = 0 \). We call the value \( w(A) \), the worth of \( A \) due to the fuzzy or partial contributions by the members of \( N \).

The worth \( w(A) \) for every \( A \in F_B(N) \) may be interpreted as the gain (whenever \( w(A) > 0 \)) or loss (whenever \( w(A) < 0 \)) that \( A \) can receive when the players can participate in it in two distinct capacities. Now onwards, we shall call a “bi-cooperative game with fuzzy bi-coalitions” a “fuzzy bi-cooperative game” in short. Let us denote by \( \mathcal{G}_{FB}(N) \) the class of all fuzzy bi-cooperative games. It is apparent from the above definition that the class \( \mathcal{BG}^N \), of crisp bi-cooperative games is a subclass of the class \( \mathcal{G}_{FB}(N) \) of fuzzy bi-cooperative games.

**Definition 3.5.** Let \( w \in \mathcal{G}_{FB}(N) \). Player \( i \) is called positively monotone in fuzzy sense with respect to \( w \) if for every \( A \in F_B(N) \) with \( \mu_A(i) = 0 = \nu_A(i) \), and all \( I \in F_B(N) \) such that \( \mu_I(i) > 0 \) and \( \mu_I(j) = 0 = \nu_I(j) \) for \( j \neq i \), we have \( w(A \cup I) \geq w(A) \). Similarly, player \( i \) is negatively monotone in fuzzy sense with respect to \( w \) if for every \( A \in F_B(N) \) with \( \mu_A(i) = 0 = \nu_A(i) \), and all \( I \in F_B(N) \) such that \( \nu_I(i) > 0 \) and \( \mu_I(j) = 0 = \nu_I(j) \) for \( j \neq i \), we have \( w(A \cup I) \leq w(A) \).

The game \( w \in \mathcal{G}_{FB}(N) \) is monotone in fuzzy sense if every player is both positively and negatively monotone in fuzzy sense.
Preparatory to the definition of the Shapley axioms for a fuzzy bi-cooperative game, we describe the following:

**Definition 3.6.** If \( A \in \mathcal{F}_B(N) \), and \( w \in \mathcal{G}_{FB}(N) \), the player \( i \in N \) is said to be null for \( w \) in \( A \) if \( w(B \cup I) = w(B) \) for all \( B \in \mathcal{F}_B(A) \) with \( \mu_B(i) = \nu_B(i) = 0 \) and all \( I \in \mathcal{F}_B(N) \) such that \( \mu_I(j) = \nu_I(j) = 0 \) when \( j \neq i \), where \( \cup \) is defined in relation 3.1.

**Definition 3.7.** Let \( A \in \mathcal{F}_B(N) \), for any permutation \( \pi \) on \( N \), define the fuzzy bi-coalition \( \pi A \) by

\[
\mu_{\pi A}(i) = \mu_A(\pi^{-1}i) \tag{3.4}
\]

\[
\nu_{\pi A}(i) = \nu_A(\pi^{-1}i) \tag{3.5}
\]

Then \( \pi A \) is called a permutation of the fuzzy bi-coalition \( A \).

Now we define the Shapley function for bi-cooperative games with fuzzy bi-coalitions as follows:

**Definition 3.8.** A function \( \Phi : \mathcal{G}_{FB}(N) \rightarrow (\mathbb{R}^n)^\mathcal{F}_B(N) \) is said to be a Shapley function on \( \mathcal{G}_{FB}(N) \) if it satisfies the following six axioms:

**Axiom f1** (Efficiency): If \( w \in \mathcal{G}_{FB}(N) \) and \( A \in \mathcal{F}_B(N) \), then

\[
\sum_{i \in N} \Phi_i(w)(A) = w(A).
\]

**Axiom f2** (Linearity): For \( \alpha, \beta \in \mathbb{R} \) and \( w, w' \in \mathcal{G}_{FB}(N) \) we must have

\[
\Phi(\alpha w + \beta w') = \alpha \Phi(w) + \beta \Phi(w'),
\]

**Axiom f3** (Null Player Axiom): If player \( i \in N \) is a null player for \( w \in \mathcal{G}_{FB}(N) \), in \( A \in \mathcal{F}_B(N) \), then \( \Phi_i(w)(A) = 0 \).

**Axiom f4** (Intra coalition Symmetry): For any \( w \in \mathcal{G}_{FB}(N) \), and a permutation \( \pi \), defined on \( N \), it holds for all \( i \in N \),

\[
\Phi_i(w)(A) = \Phi_{\pi i}(\pi w)(\pi A) \tag{3.6}
\]

Where \( \pi w \in \mathcal{G}_{FB}(N) \) is defined by \( \pi w(\pi A) = w(A) \), with \( \pi A \) defined as in Definition 3.7.

**Axiom f5** (Inter coalition Symmetry): Given \( A \in \mathcal{F}_B(N) \) and \( i, j \in N \), if \( w_i \) and \( w_j \) are two bi-cooperative games with fuzzy bi-coalitions such that for every \( B \in \mathcal{F}_B(A) \) with \( i, j \notin \text{Supp}(B) \) (i.e. \( \mu_B(i) = 0 = \mu_B(j) \) and \( \nu_B(i) = 0 = \nu_B(j) \)), and every pair of \( I, J \in \mathcal{F}_B(N) \), such that \( \mu_I(i) = \nu_I(j) > 0 \) or \( \mu_J(i) = \nu_J(i) > 0 \) and \( \mu_I(k) = \mu_J(k) = \nu_I(k) = \nu_J(k) \forall k \in N \setminus \{i, j\} \), it holds that

\[
w_i(B \cup I) - w_i(B) = w_j(B) - w_j(B \cup J)
\]

\[
w_i(B \cup I \cup J) - w_i(B \cup J) = w_j(B \cup I) - w_j(B \cup I \cup J)
\]
then, \( \Phi_i(w)(A) = -\Phi_j(w)(A) \).

**Axiom f6 (Monotonicity):** Let \( w \) and \( w' \) be two bi-cooperative games with fuzzy bi-coalitions and \( A \in \mathcal{F}_B(N) \). Let further that there exists an \( i \in N \) such that for every \( I \in \mathcal{F}_B(N) \) with \( \mu_I(i) > 0 \) or \( \nu_I(i) > 0 \), \( \mu_I(j) = \nu_I(j) = 0 \), \( \forall j \neq i \), and for all \( B \in \mathcal{F}_B(A) \) such that \( \mu_B(i) = \nu_B(i) = 0 \), it holds that

\[
\begin{align*}
\Phi_i(w')(I) & \geq \Phi_i(w)(I) \\
w'(B) & = w(B) \\
w'(B \cup I) & \geq w(B \cup I)
\end{align*}
\]

Then, \( \Phi_i(w')(A) \geq \Phi_i(w)(A) \).

**Remark 3.9.** It is easy to see that if \( \Phi \) satisfies Axioms f1-f6 then \( \Phi|_{BG} \) (restriction of \( \Phi \) to the class of crisp bi-cooperative games) satisfies Axioms b1-b6. Thus we can recover the crisp value from its fuzzy counterpart under restriction of its domain. As a matter of fact all the above axioms are obtained intuitively from their crisp analogues by generalizing the idea of participation of players. For example, *Axiom f5* (an analogue to *Axiom b5*), says that when contribution of a player \( i \) to a game \( w_i \), is exactly opposite of that of player \( j \), to a game \( w_j \), (\( i \) and \( j \) having equal rates of participations) then \( i \)'s payoff will be exactly opposite to the one for \( j \). Similarly, *Axiom f6* implies that if \( i \) provides some positive contribution to \( B \in \mathcal{F}_B(A) \), and the added value for \( w' \) is greater than that for \( w \) or if \( i \) provides some negative contribution to \( B \in \mathcal{F}_B(A) \), and the negative added value for \( w' \) is lesser than that for \( w \) in absolute value, then its payoff due to \( w' \) can not be lesser than the one due to \( w \). This establishes a well deserved link between the crisp and fuzzy frameworks pertaining to bi-cooperative games. Furthermore, the definition above can be adopted for any class of bi-cooperative games with fuzzy bi-coalitions. In the next section, we propose a new family of bi-cooperative games with fuzzy bi-coalitions and discuss the corresponding properties.

### 4 Fuzzy bi-cooperative games in *Choquet Integral* form

Here, we define the class of games in *Choquet Integral* form and study their properties. Note that any game we consider here, associates a bi-cooperative game as its crisp counterpart. We begin with the following definitions:

**Definition 4.1.** For \( A \in \mathcal{F}_B(N) \) and \( \alpha \in [0,1] \), the expression \([A]^{\alpha}, [A]_{\alpha}\) is called the \( \alpha \)-level set of \( A \) where

\[
\begin{align*}
[A]^{\alpha} & = \{ i \in N | \mu_A(i) \geq \alpha \} \\
[A]_{\alpha} & = \{ i \in N | \nu_A(i) \geq \alpha \}.
\end{align*}
\]
Remark 4.2. It is evident from definition 4.1 that,

(a) Every $\alpha$-level set $([A]^{h}, [A]_{h}); (\alpha \in [0, 1])$ of a fuzzy bi-coalition $A$, is indeed a crisp bi-coalition (associated with $A$) where all players in positive or negative roles have participations level not below $\alpha$. Thus for $h \in (0, 1]$, the symbol $\mathcal{Q}_{([A]^{h}, [A]_{h})}(N)$ will represent as usual the set:

$$\mathcal{Q}_{([A]^{h}, [A]_{h})}(N) = \{(S', T') \in \mathcal{Q}(N), S' \subseteq [A]^{h}, T' \subseteq [A]_{h}\}$$

(4.1)

(b) Every $(S', T') \in \mathcal{Q}_{([A]^{h}, [A]_{h})}(N), h \in (0, 1]$ has a representation $([B]^{h}, [B]_{h})$ for some $B \in \mathcal{F}_{B}(A)$ given by $\mu_{B}(i) = \mu_{A}(i)$ when $i \in S'$ with $\nu_{B}(i) = 0$ otherwise and $\nu_{B}(i) = \nu_{A}(i)$ when $i \in T'$ with $\nu_{B}(i) = 0$ otherwise. Moreover, for every $A \in \mathcal{F}_{B}(N), ([A]^{h}, [A]_{h}) \subseteq ([A]^{k}, [A]_{k})$, whenever $h, k \in (0, 1]$ and $h \geq k$.

In what follows next, for ready reference, we shall mention here the definition and some pertinent properties of a signed Choquet integral which is a generalization of what Choquet [10] proposed in 1955. The rationale behind defining one is to relax the monotonicity of a regular Choquet integral so that the belonging fuzzy measures are not necessarily monotone and may even take negative values. If no ambiguity arises, we call a signed Choquet integral simply a Choquet integral. For details we recommend [36, 37, 38, 39, 31, 35, 29].

Definition 4.3. Let $v$ be an arbitrary set function. For any $x \in \mathbb{R}^{n}$, we define the signed Choquet integral $\int x \, dv$, analogously to the regular Choquet integral (refer to [16, 17, 18, 19]), as,

$$\int_{0}^{\infty} [v(\{j: x_{j} \geq t\}) - v(\emptyset)] \, dt + \int_{-\infty}^{0} [v(\{j: x_{j} \geq t\}) - v(\{1, \cdots, n\})] \, dt.$$  

(4.2)

It follows that we can compute the Choquet integral of $x$ with respect to $v$ in following three steps:

(i) Take a permutation $\sigma$ on $\{1, \cdots, n\}$ that is compatible with $x$, i.e. $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$.

(ii) Define $\pi_{\sigma(j)} := v(\{\sigma(1), \cdots, \sigma(j)\}) - v(\{\sigma(1), \cdots, \sigma(j-1)\}) \ \forall j$.

(iii) Then $\int x \, dv = \sum_{j=1}^{n} \pi_{j}x_{j}$ is the signed Choquet integral of $x$ with respect to $v$. We denote it by $C_{v}(x)$.

The numbers $\pi_{j}$ are called decision weights. In general, they can be well negative. They are all nonnegative if and only if $v$ is monotonic. Let $x, x' \in \mathbb{R}^{n}$. We say that $x, x'$ are comonotonic if there exists a permutation $\sigma$ on $\{1, \cdots, n\}$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \cdots \leq x_{\sigma(n)}$ and $x'_{\sigma(1)} \leq x'_{\sigma(2)} \cdots \leq x'_{\sigma(n)}$. The characteristic property of the signed Choquet integral is comonotonic additivity, which means that $C_{v}(x+y) = C_{v}(x) + C_{v}(y)$ whenever $x$ and $y$ are comonotonic. Comonotonic additivity leads to Cauchy equation. Moreover any such signed Choquet integral is positively homogeneous i.e. $C_{v}(\lambda x) = \lambda C_{v}(x), \ \forall \lambda \geq 0$. The following result due to Schmeidler [31] characterizes a signed Choquet integral:
Theorem 4.4. \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) is a signed Choquet integral if and only if it is continuous and satisfies comonotonic additivity.

We define the following:

**Definition 4.5.** Given \( A \in \mathcal{F}_B(N) \), let \( Q(A) = Q_1(A) \cup Q_2(A) \) where \( Q_1(A) = \{ \mu_A(i) | \mu_A(i) > 0, i \in N \} \) and \( Q_2(A) = \{ \nu_A(i) | \nu_A(i) > 0, i \in N \} \) and let \( q(A) \) be the cardinality of \( Q(A) \). Let us write the elements of \( Q(A) \) in the increasing order as \( h_1 < ... < h_{q(A)} \). Then given \( b \in \mathcal{BG}^N \), a game \( w \in \mathcal{G}_{FB}(N) \) is said to be a fuzzy bi-cooperative game in Choquet Integral form over \( \mathcal{F}_B(N) \) if it holds that:

\[
 w(A) = \sum_{l=1}^{q(A)} b([A]_{h_l}, [A]_{h_l})(h_l - h_{l-1}) \tag{4.3}
\]

whence, \( h_0 = 0 \). We denote by \( \mathcal{G}_{FB}(N) \) the class of all bi-cooperative games with fuzzy bi-coalitions in Choquet Integral form over \( \mathcal{BG}^N \).

For \( A \in \mathcal{F}_B(N) \), define \( x^A \in [-1, 1]^N \), by,

\[
 x_i^A = \begin{cases} 
 \mu_A(i), & \text{if } \mu_A(i) > 0 \\
 -\nu_A(i), & \text{if } \nu_A(i) > 0 \\
 0, & \text{else}
\end{cases}
\]

then, the expression (4.3) for \( w(A) \) is exactly the Choquet integral \( C_b(x^A) \) of \( x^A \) with respect to \( b \). Again, if we construct \( F_{S,T} \in \mathcal{F}_B(N) \) by

\[
 \mu_{F_{S,T}}(i) = \begin{cases} 
 1, & \text{if } i \in S \\
 0, & \text{else}
\end{cases}
\]

\[
 \nu_{F_{S,T}}(i) = \begin{cases} 
 1, & \text{if } i \in T \\
 0, & \text{else}
\end{cases}
\]

then we get,

\[
 w(F_{S,T}) = C_b(1_S, -1_T, 0_{N \setminus (S \cup T)}) = C_b(1_{S,T}) = b(S, T). \tag{4.4}
\]

What follows, is simply an analogue of the “properly weighted with respect to bi-capacities (PWBC)” property given by Grabisch et al. [18].

Furthermore, there is a one to one correspondence between the members of \( \mathcal{BG}^N \) and \( \mathcal{G}_{FB}(N) \).

We call the crisp game corresponding to the game with fuzzy bi-coalitions in Choquet Integral form, the associated crisp game and the game with fuzzy bi-coalitions in Choquet Integral form corresponding to a crisp game, the associated fuzzy game.
Remark 4.6. Let \( w \in G_{FB}^c(N) \). Given \( A \in F_B(N) \), consider a set \( \{r_1, \ldots, r_m\} \supseteq Q(A) \), such that \( 0 \leq r_1 \leq \ldots \leq r_m \leq 1 \). Then the following holds:

\[
w(A) = \sum_{l=1}^{m} b([A]_{r_l}, [A]_{r_{l-1}})(r_l - r_{l-1})
\]

where \( r_0 = 0 \).

The following theorem is important:

**Theorem 4.7.** Define a metric \( d \) on \( F_B(N) \) by

\[
d(A, B) = \sum_{i \in N} |\mu_A(i) - \mu_B(i)| + |\nu_A(i) - \nu_B(i)|
\]

Then \( w \in G_{FB}^c(N) \) is continuous with respect to a fuzzy bi-coalition under the usual metric on \( \mathbb{R} \).

*Proof.* The proof follows directly from the continuity of the Choquet Integral with respect to its integrand given by theorem 4.4. \( \square \)

**Theorem 4.8.** Every \( w \in G_{FB}^c(N) \) is monotone in fuzzy sense whenever \( b \in BG^{M}(N) \).

*Proof.* The result follows from the non-decreasing monotonicity property proposed as a characterization of Choquet integrals by Grabich et al. [19]. \( \square \)

### A Shapley Function on \( G_{FB}^c(N) \)

We now provide a Shapley function as a solution concept to the class \( G_{FB}^c(N) \). Prior to that, we state and prove an important result as follows:

**Lemma 4.9.** Given \( w \in G_{FB}^c(N) \), a permutation \( \pi \) on \( N \), a permutation \( \pi A \) of \( A \in F_B(N) \) and the bi-cooperative game \( \pi b \), define \( \pi w \in G_{FB}^c(N) \) by

\[
\pi w(\pi A) = w(A)
\]

Then,

\[
\pi w(\pi A) = \sum_{l=1}^{q(\pi A)} \pi b([\pi A]_{h_l}, [\pi A]_{h_{l-1}})(h_l - h_{l-1})
\]

*Proof.* The proof follows directly from the facts that \( \pi [A]^{\alpha} = [\pi A]^{\alpha} \), \( \pi [A]_{\alpha} = [\pi A]_{\alpha} \) for any \( \alpha \in (0, 1] \) where \( Q(A) = Q(\pi A) \). \( \square \)

Given \( b \in BG^N \), let \( \Phi'(b)(S,T) \) denote the value of \( b \) for \( (S,T) \in Q(N) \), due to Labreuche and Grabisch given by Theorem 2.5

Define \( \Phi : G_{FB}^c(N) \rightarrow (\mathbb{R}^n)^{F_B(N)} \) by:

\[
\Phi_i(w)(A) = \sum_{l=1}^{q(A)} [\Phi'_i(b)_{l}]([A]_{h_l}, [A]_{h_{l-1}})(h_l - h_{l-1})
\]

In the next theorem, we show that \( \Phi \) is a Shapley function on \( G_{FB}^c(N) \).
Theorem 4.10. The function $\Phi : \mathcal{G}_{FB}(N) \to (\mathbb{R}^n)^{F_B(N)}$ given by equation (4.7) is a Shapley function on $\mathcal{G}_{FB}(N)$.

Proof. We require to prove that $\Phi$ satisfies Axioms $f1-f6$ over $\mathcal{G}_{FB}(N)$ and all fuzzy bi-coalitions in $F_B(N)$. However, by the properties of linearity, comonotonicity and positive homogeneity of Choquet integrals, it suffices to take for any members $A \in F_B(N)$, elements of the form $F_{S,T} \in F_B(N)$ with $(S,T) \in F_B(N)$. By virtue of expressions (4.3) and (4.4), it is easy to see that when $\Phi$ satisfies Axioms $f1-f6$, $\Phi'$ of (4.7) satisfies Axioms $b1-b6$ and their conclusions. Therefore, as a consequence, reverting back to $\Phi$ from $\Phi'$, we obtain the conclusions of Axioms $f1-f6$ as well.

Our next goal would be to axiomatize the Shapley value so obtained.

Theorem 4.11. If a function $\Phi : \mathcal{G}_{FB}(N) \to (\mathbb{R}^n)^{F_B(N)}$ satisfies to axioms of Efficiency, Linearity, Null player, Intra-coalition Symmetry, Inter-coalition Symmetry and Monotonicity (Axiom $f1-f6$), and $\Phi'$ given by equation (2.7), represents the value for every crisp game associated to the corresponding member of $\mathcal{G}_{FB}(N)$, then $\Phi$ must be given by equation (4.7).

Proof. For $(K,L) \in Q(N)$, define, $U_{K,L} \in B\mathcal{G}^N$ by

$$U_{K,L}(S',T') = \begin{cases} 1, & \text{if } S' = K \text{ and } T' = L \\ 0, & \text{otherwise} \end{cases}$$

Then every $b \in B\mathcal{G}^N$ is uniquely represented as $b = \sum_{(K,L) \in Q(N)} b(K,L)U_{K,L}$. For $A \in F_B(N)$, we have from 4.3,

$$w(A) = \sum_{l=1}^{q(A)} \left\{ \sum_{(K,L) \in Q(N)} b(K,L)U_{K,L}([A]_{h_l}, [A]_{h_l}) \right\} (h_l - h_{l-1})$$

Let $\Phi|_{B\mathcal{G}^N(N)} = \Phi'$ (Remark 3.9). Also from definition, we have $w|_{Q(N)} = b$. Thus if $\Phi$ satisfies Axiom $f1-f6$, then for every $A \in F_B(N)$, $\Phi'$ will satisfy Axiom $b1-b6$ for $([A]_h, [A]_h) \forall h \in (0,1]$. Thus $\Phi'$ defines the value given by Theorem 2.5 for $([A]_h, [A]_h) \forall h \in (0,1]$. By linearity,

$$\Phi_i(w)(A) = \sum_{l=1}^{q(A)} \sum_{(K,L) \in Q(N)} b(K,L)\Phi_i(U_{K,L})([A]_{h_l}, [A]_{h_l})(h_l - h_{l-1}) \quad (4.8)$$

$$= \sum_{l=1}^{q(A)} \left\{ \sum_{(K,L) \in Q(N)} b(K,L)\Phi_i(U_{K,L})([A]_{h_l}, [A]_{h_l}) \right\} (h_l - h_{l-1}) \quad (4.9)$$

$$= \sum_{l=1}^{q(A)} \Phi'_i([A]_{h_l}, [A]_{h_l})(h_l - h_{l-1}) \quad (4.10)$$

This completes the proof.
5 An Illustrative Example

Let us now make use of our model in explaining the mega dam issue discussed in the introduction. Let us consider a hypothetical model with three players 1, 2, and 3, of which 1 and 3 put fractions of their resources for running the project and 2 provides her resource to stop it. One can interpret these players as the Government (Player 1), Group of Environmentalists (Player 2), and the Agency in charge of dam construction and production of hydro-electricity (Player 3). Assume that in the process, we get a fuzzy bi-coalition $A = \{< 1, 0.3, 0 >, < 2, 0, 0.5 >, < 3, 0.4, 0 >\}$. Let the associated bi-cooperative game $b$ in this case, represent the strength of the supporting group over its opponents in a bi-coalition towards the decision to construct the mega dam. The values of $b$ at different bi-coalitions are given as follows:

$b(\emptyset, \emptyset) = 0$, $b(\emptyset, 2) = 0$, $b(1, \emptyset) = 4$, $b(3, \emptyset) = 2$, $b(3, 2) = 1$, $b(\{1, 3\}, 2) = 6$, $b(\{1, 3\}, \emptyset) = 11$, $b(1, 2) = 2$.

Using equation 4.3, we obtain $w(A) = 1.9$. Here, $w$ represents physically, the expected strength of the decision to support the mega dam issue in regards to the resources provided by the constituent agencies. Consequently, the components of the Shapley value for $w$, with respect to a fuzzy bi-coalition would represent the individual contributions of the players in attaining that strength. Thus,

$$\Phi'_1(b)(S, T)$$

<table>
<thead>
<tr>
<th>$\Phi'_1(b)(S, T)$</th>
<th>${{1, 3}, {2}}$</th>
<th>${{3}, {2}}$</th>
<th>$\emptyset, {2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi'_1(b)(S, T)$</td>
<td>4.83</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi'_2(b)(S, T)$</td>
<td>-2.1</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi'_3(b)(S, T)$</td>
<td>3.333</td>
<td>1.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Computation of the Values of the associated crisp game at different bi-coalitions

After computations, the components of the Shapley value are found to be:

$$\Phi_1(w)(A) \approx 1.45$$
$$\Phi_2(w)(A) \approx -0.68$$
$$\Phi_3(w)(A) \approx 1.15$$

This example shows that although membership of player 1 is less than that of 3, yet she receives more payoff. This is justified because player 1’s marginal contribution in a positive role is more than that of player 3 in situations when, player 2 puts her negative contribution or remains indifferent.
6 Conclusion

In this study, we have developed a framework for a bi-cooperative game with fuzzy bi-coalitions. Any member of this class incorporates the contributions by the players in two mutually exclusive roles so that the set of players at each level of participation can be divided into three disjoint groups: group of positive contributors, negative contributors and the group of indifferent players. A hypothetical yet realistic example is formulated to illustrate our findings. Here, we have maintained a strict bi-polarity of the games, thus defined, however it seems to be possible to introduce a similar notion for multichoice games as well. In our next study, we propose to highlight on this aspect. Further, we have defined here the class of fuzzy bi-cooperative games in Choquet Integral form. It has been further shown that the members of this class satisfy both continuity and monotonicity. A solution concept namely the Shapley function to this class has been proposed. There may exist other fuzzy bi-coalitions whose payoff can not be expressed by crisp bi-coalitional values and participation levels. In our future work, we plan to introduce more of such games and study their properties in this context.

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References


