Computationally efficient weighted least-squares design of FIR filters satisfying prescribed magnitude and phase specifications

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Abstract

An efficient method for the design of FIR filters satisfying prescribed magnitude and phase specifications is described. The method involves formulating an objective function in a quadratic form. The filter coefficients are obtained by solving a system of linear equations which involves either a Toeplitz matrix or a Toeplitz-plus-Hankel matrix. The computational complexity associated with such systems is only $O(N^2)$. © 1997 Elsevier Science B.V.

Zusammenfassung


Résumé

Nous décrivons dans cet article une méthode efficiente de conception de filtres FIR satisfaisant des spécifications d’amplitude et de phase. Cette méthode implique la formulation d’une fonction objectif de forme quadratique. Les coefficients du filtre sont obtenus par résolution d’un système d’équations linéaires correspondant à une matrice de type Toeplitz ou Toeplitz-plus-Hankel. La complexité en calcul associée à de tels systèmes est seulement $O(N^2)$. © 1997 Elsevier Science B.V.

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1. Introduction

For narrow transition-band specifications, linear-phase finite impulse response (FIR) filters produce large group delays. On the other hand, minimum-phase FIR filters can be designed to achieve a lower group-delay, but these do not provide a constant group-delay for the frequencies of interest. In such applications, there is a need to design filters having an approximately constant group-delay which is less than that produced by linear-phase filters. In many phase equalization problems, filters with constant magnitude and nonlinear phase characteristics are...
needed. In all these situations, FIR filters can be designed by approximating both magnitude and phase specifications.

Weighted least-squares (WLS) techniques have been successfully used in the design of FIR filters satisfying prescribed magnitude and phase specifications having equiripple magnitude characteristics \([1, 3]\).

In these techniques, a suitable frequency-dependent weighting function that yields an equiripple design is determined using iterative methods. In all these techniques, computational complexity involved in obtaining the filter coefficients is \(O(N^3)\).

In this correspondence, we present an efficient WLS method for the design of FIR filters meeting prescribed magnitude and phase specifications. It is shown that the filter coefficients can be obtained by solving a system of linear equations involving a Toeplitz-symmetric and positive-definite matrix. The solution of such a system of linear equations entails only \(O(N^2)\) complexity. In the design of allpass filters with a symmetric response about one-quarter the sampling frequency, the coefficients are obtained by solving a system of linear equations that involves a Toeplitz-plus-Hankel matrix. Again, the complexity of such a system of equations is \(O(N^2)\). Evidently, our method has an order of magnitude lower computational complexity than the techniques of \([1, 3]\). Our method, however, can be used in conjunction with the iterative methods of \([1, 3]\) to obtain the suitable frequency-dependent weighting function that yields an equiripple design.

2. Error function formulation and minimization

The frequency response of an FIR digital filter with \(N\) taps specified by a real-valued impulse response \(h(n)\) is given by

\[
H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-jn\omega}
\]

\[
= \sum_{n=0}^{N-1} h(n)\cos(n\omega) - j\sum_{n=0}^{N-1} h(n)\sin(n\omega)
\]

\[
= h^Tc(\omega) - jh^Ts(\omega),
\]

where

\[
h = [h(0), h(1), h(2), \ldots, h(N-1)]^T,
\]

\[
c(\omega) = [1, \cos(\omega), \cos(2\omega), \ldots, \cos((N-1)\omega)]^T,
\]

\[
s(\omega) = [0, \sin(\omega), \sin(2\omega), \ldots, \sin((N-1)\omega)]^T.
\]

The phase response of the filter is given by

\[
\phi(\omega) = -\tan^{-1}\left(\frac{h^T s(\omega)}{h^T c(\omega)}\right)\]

and the group-delay is given by

\[
\tau_\delta(\omega) = -\frac{d}{d\omega}\phi(\omega).
\]

The desired frequency response \(D(e^{j\omega})\) having an amplitude response \(M(\omega)\) and a phase response \(\rho(\omega)\) is given by

\[
D(e^{j\omega}) = M(\omega)e^{j\rho(\omega)} = M_R(\omega) + jM_I(\omega)
\]

and

\[
\tau_p(\omega) = -\frac{d\rho(\omega)}{d\omega}
\]

is the desired group-delay response.

The mean-square error between \(D(e^{j\omega})\) and \(H(e^{j\omega})\) can be expressed as

\[
E_{\text{mse}} = \sum_{l=1}^{M} W(\omega_l)[D(e^{j\omega_l}) - H(e^{j\omega_l})]^2
\]

\[
= \sum_{l=1}^{M} W(\omega_l)[(M_R(\omega_l) - h^Tc(\omega_l))^2
\]

\[
+ (M_I(\omega_l) + h^Ts(\omega_l))^2],
\]

where \(W(\omega)\) is a non-negative frequency-dependent weighting function and \(M\) is the number of points at which \(D(e^{j\omega})\) is sampled.

In minimizing \(E_{\text{mse}}\), we set \(\frac{\partial E_{\text{mse}}}{\partial h} = 0\) to obtain a system of linear equations \(Qh = d\), where

\[
Q = \sum_{l=1}^{M} W(\omega_l)(Q_c(\omega_l) + Q_s(\omega_l)),
\]

\[
d = \sum_{l=1}^{M} W(\omega_l)(M_R(\omega_l)c(\omega_l) - M_I(\omega_l)s(\omega_l)).
\]
where \( Q_r(\omega) = c(\omega)c^T(\omega) \) and \( Q_s(\omega) = s(\omega)s^T(\omega) \). Since \( M_r(\omega) \) and \( M_s(\omega) \) are both zero in the stopband of the desired characteristics, the above equation for \( d \) can be written as

\[
d = \sum_{l=1}^{M_p} W(\omega_l)(M_r(\omega_l)c(\omega_l) - M_s(\omega_l)s(\omega_l)),
\]

where \( M_p \) is the number of sample points in the passband of \( D(e^{j\omega}) \). The entries of \( Q \) and \( d \) are given as

\[
Q(n,m) = \sum_{l=1}^{M_r} W(\omega_l)\cos((n - m)\omega_l),
\]

\[
d(n) = \sum_{l=1}^{M_r} W(\omega_l)M(\omega_l)\cos(n\omega_l + \rho(\omega_l))
\]

for \( 0 \leq n, m \leq N - 1 \). It can be seen that \( Q \) is real, Toeplitz-symmetric and positive definite and hence only the first row (or column) of entries has to be evaluated. Consequently, the system of linear equations can be solved by the computationally efficient and robust Levinson method [4]. It can also be noted that the entries of \( Q \) are independent of \( D(e^{j\omega}) \).

For the case of allpass phase equalizers, the desired characteristic can be expressed as

\[
D(e^{j\omega}) = e^{j\rho(\omega)} = \cos(\rho(\omega)) + j\sin(\rho(\omega)),
\]

\( \omega \in [0, \pi] \)

where \( \rho(\omega) \) is the desired phase response. Again, a mean-square error \( E_{\text{mse}} \) can be minimized and system of linear equations can be obtained. If, however, \( \rho(\omega) \) is antisymmetric or symmetric with respect to \( \pi/2 \), the computational complexity of \( Q \) and \( d \) can be reduced significantly since only half the number of coefficients need be determined. Below, we shall consider the design of two allpass phase equalizers presented in [6] using our method.

### 2.1. Antisymmetric phase characteristics

The desired phase characteristic can be written as

\[
\rho(\omega) = -\frac{N - 1}{2} \omega + \hat{\rho}(\omega),
\]

where the first term on the right-hand side is the linear phase term and \( \hat{\rho}(\omega) \) is a function of \( \omega \) antisymmetric about \( \pi/2 \). It follows that [6]

\[
h\left(\frac{N - 1}{2} - n\right) = h\left(\frac{N - 1}{2} + n\right) = 0 \quad \text{when } n \text{ is odd.}
\]

It must be mentioned that the number of filter taps \( N \) is odd. In designing such allpass filters, \( Q \) and \( d \) are given as in (7) and (9) except that the rows and columns corresponding to the zero-valued coefficients are deleted.

### 2.2. Symmetric phase characteristics

When \( \hat{\rho}(\omega) \) in (13) is symmetric about \( \pi/2 \), the following conditions hold [6]:

\[
h\left(\frac{N - 1}{2} - n\right) = h\left(\frac{N - 1}{2} + n\right) \quad \text{when } n \text{ is even},
\]

\[
h\left(\frac{N - 1}{2} - n\right) = -h\left(\frac{N - 1}{2} + n\right) \quad \text{when } n \text{ is odd.}
\]

From (1), the allpass phase equalizer can be characterized by

\[
e^{j((N-1)/2)\omega}H(e^{j\omega})
\]

\[
= \sum_{n=0}^{U} a(n)\cos(2n\omega) + j\sum_{n=1}^{V} b(n)\sin((2n - 1)\omega),
\]

where

\[
U = \left\lfloor \frac{N - 1}{4} \right\rfloor, \quad V = \left\lfloor \frac{N + 1}{4} \right\rfloor,
\]

\[
a(n) = \begin{cases} h\left(\frac{N - 1}{2}\right), & n = 0, \\ 2h\left(\frac{N - 1}{2} - 2n\right), & n = 1, 2, \ldots, U, \end{cases}
\]

and

\[
b(n) = 2h\left(\frac{N - 1}{2} - 2n + 1\right), \quad n = 1, 2, \ldots, V.
\]

Consequently (16) can be written as

\[
e^{j((N-1)/2)\omega}H(e^{j\omega}) = a^T \hat{c}(\omega) + j b^T \hat{s}(\omega),
\]

where

\[
\hat{c}(\omega) = \left[ c(\omega_0), c(\omega_1), \ldots, c(\omega_{U-1}) \right]^T,
\]

\[
\hat{s}(\omega) = \left[ s(\omega_0), s(\omega_1), \ldots, s(\omega_{V-1}) \right]^T.
\]
where
\[ a = [a(0) a(1) \ldots a(U)]^T, \]
\[ b = [b(1) b(2) \ldots b(V)]^T, \]
\[ \hat{c}(\omega) = [\cos(2\omega) \ldots \cos(2U\omega)]^T, \]
\[ \hat{s}(\omega) = [\sin(\omega) \sin(3\omega) \ldots \sin((2V-1)\omega)]^T. \]

Since the coefficients associated with the real and imaginary parts of (16) differ in number, they are computed separately. The mean-square error associated with the real part can be written as
\[ E_a = \sum_{l=1}^{M} W(\omega_l)(\cos(\hat{\rho}(\omega_l)) - a^T \hat{c}(\omega_l))^2 d\omega. \quad (18) \]

By setting \( \partial E_a / \partial a = 0 \), we get \( \hat{Q}_a a = \hat{d}_a \), where
\[ \hat{Q}_a = \sum_{l=1}^{M} W(\omega_l)\hat{c}(\omega_l)\hat{c}^T(\omega_l) \quad (19) \]
and
\[ \hat{d}_a = \sum_{l=1}^{M} W(\omega_l)\cos(\hat{\rho}(\omega_l))\hat{c}(\omega_l). \quad (20) \]

Similarly, the mean-square error associated with the imaginary part can be written as
\[ E_b = \sum_{l=1}^{M} W(\omega_l)(\sin(\hat{\rho}(\omega_l)) - b^T \hat{s}(\omega_l)). \quad (21) \]

Again, by setting \( \partial E_b / \partial b = 0 \), we get \( \hat{Q}_b b = \hat{d}_b \), where \( \hat{Q}_b \) is given by
\[ \hat{Q}_b = \sum_{l=1}^{M} W(\omega_l)\hat{s}(\omega_l)\hat{s}^T(\omega_l) \quad (22) \]
and
\[ \hat{d}_b = \sum_{k=0}^{M} W(\omega_l)\sin(\hat{\rho}(\omega_l))\hat{s}(\omega_l). \quad (23) \]

It has been shown in [7] that matrices \( \hat{Q}_a \) and \( \hat{Q}_b \) can be decomposed into a sum of a Toeplitz and a Hankel matrix. Such a system of linear equations can be efficiently solved using the methods presented in [5, 2]. These methods have a computational complexity of \( O(N^2) \).

In order to design a filter having an equiripple magnitude response, appropriate weighting function, \( W(\omega) \), has to be used in the minimization of the mean-square error. Since there are no analytical methods to obtain the appropriate \( W(\omega) \), iterative techniques presented in [1, 3] can be used.

3. Conclusion

An efficient WLS method for the design of FIR filters satisfying prescribed magnitude and phase specifications is described. By incorporating symmetry and antisymmetry constraints of the impulse response, linear-phase FIR filters can easily be designed thus making our method general for the design of FIR filters.

References