General Observers for Nonlinear Systems

V. SUNDARAPANDIAN
Department of Mathematics
Indian Institute of Technology
Kanpur-208 016, Uttar Pradesh, India
vsundara@iitk.ac.in

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Abstract—This paper is a geometric study of finding general exponential observers for nonlinear systems. Using center manifold theory, we derive necessary and sufficient conditions for general exponential observers for Lyapunov stable nonlinear systems. As an application of our characterization of general exponential observers, we give a construction procedure for identity exponential observers for nonlinear systems. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The problem of designing observers for nonlinear systems was introduced by Thau [1]. Over the past three decades, considerable attention has been paid in the literature to the construction of observers for nonlinear systems [2–6]. In recent years, there has been some interest in designing general observers for nonlinear systems [7]. Explicitly, in [7], Kazantzis and Kravaris have used Lyapunov's auxiliary theorem to derive sufficient conditions for solving the general observer design problem proposed by them. One of the main assumptions made in [7] is that the linearization matrix A of the nonlinear plant dynamics \( \dot{x} = f(x) \) has eigenvalues \( \lambda_i \ (i = 1, 2, \ldots, n) \) such that

\[
0 \notin \text{Convex-Hull} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}. \tag{1}
\]

It follows trivially from (1) that eigenvalues of the linearization matrix A cannot lie on the imaginary axis of the complex plane. That is, the paper [7] does not address the important critical case of the nonlinear systems.

In this paper, we extend the definition of observers for linear systems [8] in a natural way to nonlinear systems. Using center manifold theory [9], we derive necessary and sufficient conditions for general exponential observers for Lyapunov stable nonlinear systems. As an application of our characterization theorem for general observers, we derive a construction procedure for identity exponential observers. Finally, we illustrate our results with some examples.

This paper is organized as follows. In Section 2, we give a basic definition of general exponential observers for nonlinear systems, and discuss the basic properties of these observers. In Section 3,
we present necessary and sufficient conditions for general exponential observers of Lyapunov stable nonlinear systems. As an application of this characterization result, we present a construction procedure for finding identity exponential observers for Lyapunov stable nonlinear systems. In Section 4, we give some examples illustrating the main results presented in this paper.

2. DEFINITION AND PROPERTIES OF GENERAL EXPONENTIAL OBSERVERS

In this paper, we consider a $C^1$ nonlinear plant of the form

$$\begin{align*}
x &= f(x), \\
y &= h(x),
\end{align*} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^p$ the output of plant (2). The state $x$ belongs to an open neighborhood $X$ of the origin of $\mathbb{R}^n$. We assume that $f : X \to \mathbb{R}^n$ is a $C^1$ vector field and also that $f(0) = 0$. We also assume that the output mapping $h : X \to \mathbb{R}^p$ is a $C^1$ map, and also that $h(0) = 0$. Let $Y \triangleq h(X)$.

The general observer design problem is to estimate

$$w = q(x), \quad w \in \mathbb{R}^m, \tag{3}$$

where $1 \leq m \leq n$ and $q : X \to \mathbb{R}^m$ is a $C^2$ mapping with $q(0) = 0$.

Explicitly, we have the following definition.

**Definition 1.** A $C^1$ dynamical system described by

$$\begin{align*}
z &= g(z, y), \\
z &= g(z, y), \quad z \in \mathbb{R}^m, \tag{4}
\end{align*}$$

where $z$ is defined in a neighborhood $Z$ of the origin of $\mathbb{R}^m$ and $g : Z \times Y \to \mathbb{R}^m$ is a $C^1$ map with $g(0, 0) = 0$, is called a general exponential observer for plant (2) corresponding to (3) if the composite system (2)--(4) satisfies the following two requirements.

(O1) If $w(0) = z(0)$, then $w(t) = z(t), \forall t \geq 0$.

(O2) There exists a neighborhood $V$ of the origin of $\mathbb{R}^m$ such that for all $w(0) - z(0) \in V$, the estimation error $w(t) - z(t)$ tends to zero exponentially as $t \to \infty$.

If a general exponential observer (4) satisfies the additional properties that $m = n$ and $q$ is a $C^2$ diffeomorphism, then it is called a full-order general exponential observer. A full-order general exponential observer (4) with the additional property that $q = \text{id}_X$ is called an identity exponential observer which is the same as the standard definition of local exponential observers for nonlinear systems [4,10].

The estimation error $e$ is defined by

$$e \triangleq z - q(x).$$

Then $e$ satisfies the differential equation

$$\dot{e} = g(q(x) + e, h(x)) - Dq(x)f(x). \tag{5}$$

Now, we consider the composite system

$$\begin{align*}
\dot{x} &= f(x), \\
\dot{e} &= g(q(x) + e, h(x)) - Dq(x)f(x). \tag{6}
\end{align*}$$

Next, we state and prove a simple lemma which provides a geometric characterization of Condition I in Definition 1.
Lemma 1. The following statements are equivalent.

(a) Condition I in Definition 1 holds for the composite system (2)-(4).
(b) \( g(q(x), h(x)) = Dq(x)f(x), \forall x \in X \).
(c) The submanifold defined via \( e = 0 \) is invariant under the flow of the composite system (6).

Proof. We prove this lemma by showing that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (a).

(a) \( \Rightarrow \) (b). Suppose that (a) holds; i.e., suppose that Condition (O1) in Definition 1 holds for the composite system (2)-(4). Then, by definition, it follows that

\[ q(x(0)) = z(0) \Rightarrow q(x(t)) = z(t), \quad \forall t \geq 0. \]

In particular, if \( q(x(0)) = z(0) \), we must have

\[ \frac{d}{dt} q(x(t)) = \frac{d}{dt} z(t), \quad \forall t \geq 0, \]

which implies that

\[ Dq(x(t)) f(x(t)) = g(z(t), h(x(t))) = g(q(x(t)), h(x(t))), \quad \forall t \geq 0. \]

(7)

Taking \( t = 0 \) in (7), we get

\[ Dq(x(0)) f(x(0)) = g(q(x(0)), h(x(0))). \]

Since \( x(0) \in X \) is arbitrary, it is immediate that we must have

\[ Dq(x) f(x) = g(q(x), h(x)), \quad \forall x \in X. \]

Thus, we have shown that (a) \( \Rightarrow \) (b).

(b) \( \Rightarrow \) (c). Suppose that (b) holds; i.e., suppose that

\[ g(q(x), h(x)) = Dq(x) f(x), \quad \forall x \in X. \]

(8)

We know that \( e(t) \) satisfies the differential equation

\[ \dot{e} = g(q(x) + e, h(x)) - Dq(x) f(x). \]

(9)

Setting \( e = 0 \), the dynamics in (9) reduces to

\[ \dot{e} = g(q(x), h(x)) - Dq(x) f(x) = 0, \quad \text{[using (8)]}, \]

showing that the submanifold defined via \( e = 0 \) is invariant under the flow of the composite system (6). Thus, we have shown that (b) \( \Rightarrow \) (c).

(c) \( \Rightarrow \) (a). Suppose that (c) holds; i.e., suppose that the submanifold defined via \( e = 0 \) is invariant under the flow of the composite system (6). This means that

\[ e(0) = 0 \Rightarrow e(t) \equiv 0, \quad \forall t \geq 0. \]

Since, by definition, \( e = z - q(x) \), it is immediate that

\[ z(0) = x(0) \Rightarrow z(t) = q(x(t)), \quad \forall t \geq 0. \]

Thus, we have also shown that (c) \( \Rightarrow \) (a).

This completes the proof.
3. NECESSARY AND SUFFICIENT CONDITIONS FOR GENERAL EXPONENTIAL OBSERVERS

In this section, we establish a basic theorem that completely characterizes the existence of general exponential observers of form (4) for Lyapunov stable nonlinear plants of form (2). For this purpose, we define

\[ C = Dh(0) \quad \text{and} \quad A = Df(0); \]  

i.e., \((C, A)\) is the system linearization pair for the given nonlinear plant (2). Also, define

\[ E = \frac{\partial g}{\partial z}(0, 0) \quad \text{and} \quad K = \frac{\partial g}{\partial y}(0, 0). \]  

Now, we state and prove the main result of this paper.

**THEOREM 1.** Suppose that the plant dynamics in (2) is Lyapunov stable at \(x = 0\). Then system (4) is a general exponential observer for plant (2) if, and only if,

(a) the submanifold defined via \(e = 0\) is invariant under the flow of the composite system (6);

(b) the dynamics

\[ \dot{e} = g(e, 0). \]  

is locally exponentially stable.

**PROOF. NECESSITY.** Suppose that system (4) is a local exponential observer for plant (2). Then, it satisfies Condition (O1) in Definition 1, which implies Condition (a) by Lemma 1. Next, we prove that Condition (b) also holds. Take \(x(0) = 0\). Then \(x(t) \equiv 0\), \(w(t) = q(x(t)) \equiv 0\), and \(y(t) = h(x(t)) \equiv 0\), for all \(t \geq 0\). Now, the dynamics for \(z\) becomes

\[ \dot{z}(t) = g(z(t), y(t)) = g(z(t), 0). \]  

Hence, by Condition (O2) in Definition 1, it is immediate that the dynamics (13) is locally exponentially stable. Thus, we have established the necessity of Conditions (a) and (b).

**SUFFICIENCY.** Suppose that Conditions (a) and (b) hold for plants (2) and (4). Since Condition (a) implies Condition (O1) in Definition 1 by Lemma 1, it suffices to show that Condition (O2) in Definition 1 also holds.

By hypotheses, the equilibrium \(e = 0\) of the dynamics (12) is locally exponentially stable, and the equilibrium \(x = 0\) of the plant dynamics in (2) is Lyapunov stable. Hence, the matrix \(E = \frac{\partial g}{\partial z}(0, 0)\) must be Hurwitz, and the matrix \(A = Df(0)\) must have all eigenvalues in the closed left-half plane. We have two cases to consider.

**CASE I: \(A\) IS HURWITZ.**

By the Hartman-Grobman Theorem [11, p. 69], it follows that the composite system (6) is locally topologically conjugate to the system

\[ \begin{align*}  
\dot{x} &= Ax, \\
\dot{e} &= Ee. 
\end{align*} \]  

Hence, it follows trivially that for the composite system (6), \(\|e(t)\|\) decays to zero as \(t \to \infty\) for all small initial conditions.

**CASE II: \(A\) IS NOT HURWITZ.**

Without loss of generality, we can assume that the plant dynamics in (2) has the form

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} A_1 x_1 + \phi_1(x_1, x_2) \\ A_2 x_2 + \phi_2(x_1, x_2) \end{bmatrix}, \]  

where \(A_1, A_2\) are matrices and \(\phi_1, \phi_2\) are functions.
where \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} (n_1 + n_2 = n) \), \( A_1 \) is an \( n_1 \times n_1 \) matrix having all eigenvalues with zero real part, \( A_2 \) is an \( n_2 \times n_2 \) Hurwitz matrix, and \( \phi_1 \) and \( \phi_2 \) are \( C^1 \) functions vanishing at \((x_1, x_2) = (0, 0)\) together with all their first-order partial derivatives.

By Lemma 1, we know that Condition (a) implies

\[
g(q(x), h(x)) = Dq(x) f(x), \quad \forall x \in X.
\]  

Now, by the center manifold theorem for flows [9], we know that the composite system (6) has a local center manifold at \((x, e) = (0, 0)\), the graph of a \( C^1 \) map,

\[
\begin{bmatrix} x_2 \\ e \end{bmatrix} = \pi(x_1) = \begin{bmatrix} \pi_1(x_1) \\ \pi_2(x_1) \end{bmatrix},
\]

with \( \pi(x_1) \) satisfying

\[
\begin{align*}
\pi(0) &= 0 \quad \text{and} \quad D\pi(0) = 0, \\
\end{align*}
\]

and the partial differential equations

\[
\begin{align*}
\left( \frac{\partial \pi_1}{\partial x_1} \right) f_1(x_1, \pi_1(x_1)) &= f_2(x_1, \pi_1(x_1)), \\
\left( \frac{\partial \pi_2}{\partial x_1} \right) f_1(x_1, \pi_1(x_1)) &= g(q(x_1, \pi_1(x_1)) + \pi_2(x_1), h(x_1, \pi_1(x_1))) \\
&- Dq(x_1, \pi_1(x_1)) f_1(x_1, \pi_1(x_1)).
\end{align*}
\]  

Using (16), we see that \( \pi_2 = 0 \) satisfies both (18) and the second equation in (19). Let \( \pi_1(x_1) \) be any solution of the first equation in (19) satisfying \( \pi_1(0) = 0 \) and \( D\pi_1(0) = 0 \).

Now, \( x = 0 \) is a Lyapunov stable equilibrium of \( \dot{x} = f(x) \). Also, \( e = 0 \) is a locally exponential stability equilibrium of the dynamics (12). Hence, by a total stability result [12, p. 515], it follows that \((x, e) = (0, 0)\) is a Lyapunov stable equilibrium of the composite system (6) (by its triangular structure).

This observation, together with the principle of asymptotic phase in the center manifold theory, implies the existence of a neighbourhood \( V \) of \((x, e) = (0, 0)\) such that, for all \((x(0), e(0)) \in V\), we have

\[
\left\| \begin{bmatrix} x_2(t) - \pi_1(x_1(t)) \\ e(t) \end{bmatrix} \right\| \leq M \exp(-at) \left\| \begin{bmatrix} x_2(0) - \pi_1(x_1(0)) \\ e(0) \end{bmatrix} \right\|, \quad \forall t \geq 0,
\]

for some positive constants \( M \) and \( a \).

Since

\[
\|e(t)\| \leq \left\| \begin{bmatrix} x_2(t) - \pi_1(x_1(t)) \\ e(t) \end{bmatrix} \right\|,
\]

it follows from (20) that \( e(t) \to 0 \) exponentially as \( t \to \infty \) for all small initial conditions. This completes the proof.

**REMARK 1.** Inspired by Theorem 1, one may try to give the following characterization for general asymptotic observers of form (4) for Lyapunov stable plants of form (2).

(a) The submanifold defined via \( e = 0 \) is invariant under the flow of the composite system (6).

(b) The dynamics \( \dot{e} = g(e, 0) \) is locally asymptotically stable.

While (a) and (b) are indeed necessary conditions for general asymptotic observers, they are not sufficient. This can be easily seen by considering the following counterexample.

Consider the problem of finding (identity) exponential observers for the following plant with its state \( x \) defined on \( \mathbb{R} \):

\[
\begin{align*}
\dot{x} &= 0 \triangleq f(x), \\
y &= x^2 \triangleq h(x).
\end{align*}
\]
Consider also the candidate observer
\[ \dot{z} = -z^3 + yz = g(z, y). \tag{22} \]

The plant dynamics in (21) is clearly Lyapunov stable as the state trajectories are all constant solutions.

Since
\[ g(x, h(x)) = -x^3 + x^3 = 0 = f(x), \]
it follows that the candidate observer (22) satisfies Condition (a) stated above. Also, since
\[ \dot{e} = g(e, 0) = -e^3 \]
is locally asymptotically stable, it follows that the candidate observer (22) also satisfies Condition (b) stated above.

To show that Conditions (a) and (b), stated above, are not sufficient for (22) to be an identity asymptotic observer for (21), we take a nonzero initial condition \( x(0) = A > 0 \), arbitrarily small.

Since \( x(t, \lambda) \equiv \lambda \), \( \forall t \), the candidate observer dynamics (22) assumes the simple form
\[ \dot{z} = \lambda^2 z - z^3 = z(\lambda^2 - z^2). \tag{23} \]

Now, (23) is a scalar, autonomous equation, and it has three equilibria, \( z = 0, \pm \lambda \). Clearly, for all \( z \) in the open interval \((-\lambda, 0)\), \( \dot{z} < 0 \), and, for all \( z \) in the open interval \((0, \lambda)\), \( \dot{z} > 0 \).

Therefore, if \( z(0) \in (-\lambda, 0) \), then \( z(t) \to -\lambda \) as \( t \to \infty \). Also, if \( z(0) \in (0, \lambda) \), then \( z(t) \to \lambda \) as \( t \to \infty \).

This calculation shows that for any initial state \( \lambda > 0 \) arbitrarily small, the candidate observer dynamics has initial conditions arbitrarily small such that the corresponding solutions converge to \(-\lambda\), and not to the state flow \( x(t, \lambda) \equiv \lambda \) asymptotically as \( t \to \infty \).

As an application of Theorem 1, we present a construction procedure for finding identity exponential observers for Lyapunov stable nonlinear systems of form (2).

**Theorem 2.** Suppose that plant (2) is Lyapunov stable at \( x = 0 \). If plant (2) has an identity exponential observer, then the system linearization pair \((C, A)\) is detectable; i.e., there exists a matrix \( K \) such that \( A - KC \) is Hurwitz. Conversely, if there exists a matrix \( K \) such that \( A - KC \) is Hurwitz, then the dynamical system defined by
\[ \dot{z} = f(z) + K(y - h(z)) \tag{24} \]
is an identity exponential observer for plant (2).

**Proof.** First, we suppose that plant (2) has an identity exponential observer of form (4). Then we wish to prove that \((C, A)\) is detectable. By Theorem 1, we know that the identity exponential observer (4) satisfies Conditions (a) and (b) in Theorem 1. Using Lemma 1, we know that Condition (a) is the same as
\[ g(q(x), h(x)) = g(x, h(x)) = Dq(x)f(x) = f(x), \quad \forall x \in X \tag{25} \]
(note that \( q \equiv \text{id}_X \) since (4) is an identity exponential observer).

Taking linearizations in (25) at \( x = 0 \), we get
\[ E + KC = A; \]
i.e., we get

\[ E = A - KC. \]

By Condition (b), it follows that the dynamics

\[ \dot{z} = g(z, 0) \tag{26} \]

is locally exponentially stable. Hence, from Lyapunov stability theory, we know that the linearization matrix

\[ E = \frac{\partial g}{\partial z}(0, 0) \]

must be Hurwitz, showing that \( A - KC \) is Hurwitz. Thus, the pair \((C, A)\) is detectable.

To prove the converse, suppose that \( K \) is some real matrix so that \( A - KC \) is Hurwitz, and consider the candidate identity observer

\[ \dot{z} = g(z, y) = f(z) + K(y - h(z)). \tag{27} \]

Since \( g(x, h(x)) = f(x) + K(h(x) - h(x)) = f(x) \), it follows from Lemma 1 that the candidate observer (27) satisfies Condition (a) in Theorem 1.

Moreover, the dynamics

\[ \dot{z} = g(z, 0) = f(z) - Kh(z) \tag{28} \]

has the linearization matrix \( A - KC \) at \( z = 0 \), and we know that \( A - KC \) is Hurwitz. Thus, from Lyapunov stability theory, it follows that the dynamics (28) is locally exponentially stable. Hence, the candidate observer (27) also satisfies Condition (b) in Theorem 1. Hence, by Theorem 1, we conclude that the candidate observer defined by (27) is an identity exponential observer for plant (2). This completes the proof.

### 4. EXAMPLES

First, we present an example illustrating the construction of identity exponential observers for nonlinear systems.

**Example 1.** Consider the nonlinear system described by

\[
\begin{align*}
\dot{x}_1 &= -x_2 + x_1 \left( x_1^2 + x_2^2 \right)^{1/2} \sin \left( \frac{\pi}{(x_1^2 + x_2^2)^{1/2}} \right), \\
\dot{x}_2 &= x_1 + x_2 \left( x_1^2 + x_2^2 \right)^{1/2} \sin \left( \frac{\pi}{(x_1^2 + x_2^2)^{1/2}} \right), \\
y &= x_1 + x_2^2.
\end{align*}
\tag{29}
\]

In polar coordinates,

\[ x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \]

the plant dynamics assumes a simple form

\[
\begin{align*}
\dot{r} &= r^2 \sin \left( \frac{\pi}{r} \right), \\
\dot{\theta} &= -1.
\end{align*}
\tag{30}
\]

Therefore, \( r = 1/n, \theta = t \) \((n \in \mathbb{Z}^+)\) describes a circular orbit with center at the origin and radius \(1/n\). Any orbit passing through a point in the \(x_1x_2\)-plane that is outside the unit circle diverges from the unit circle with increasing time since \( \dot{r} > 0 \) when \( r > 1 \). Also, the orbits spiral outward in the counterclockwise direction since \( \dot{\theta} = 1 \). It is also easy to see that the circular
orbits with radii $1/(2n + 1)$ ($n \in \mathbb{Z}^+$) are attractive, while the circular orbits with radii $1/(2n)$ ($n \in \mathbb{Z}^+$) are repulsive.

System (30) has a Lyapunov stable equilibrium at the origin since every neighborhood of the origin has a solution curve encircling the origin. This is a classical example of a Lyapunov stable system. For this plant, we consider the problem of finding identity local exponential observers.

Linearizing the plant (in its Cartesian coordinates), we obtain the system matrices,

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$  

Clearly, $(C, A)$ is an observable pair. In particular, the matrix $K = \text{col}(2, 0)$ is such that $A - KC$ is Hurwitz with eigenvalues $-1$ and $-1$.

Hence, by Theorem 2, the dynamical system defined by

$$\dot{z}_1 = -z_2 + z_1 \left( z_1^2 + z_2^2 \right) \sin \left( \frac{\pi}{\left( z_1^2 + z_2^2 \right)^{1/2}} \right)$$

$$\dot{z}_2 = z_1 + z_2 \left( z_1^2 + z_2^2 \right) \sin \left( \frac{\pi}{\left( z_1^2 + z_2^2 \right)^{1/2}} \right)$$

+ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \left[ y - z_1 - z_2 \right]$$

is an identity local exponential observer for plant (29).

Next, we present an example illustrating the construction of general exponential observers for nonlinear systems.

**EXAMPLE 2.** Consider the cascade nonlinear system described by

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_2), \\
y &= x_2,
\end{align*}$$

(31)

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ ($n_1 + n_2 = n$). For plant (31), we also suppose that the dynamics

$$\dot{x}_1 = f_1(x_1, 0)$$

is locally asymptotically stable, and the dynamics

$$\dot{x}_2 = f_2(x_2)$$

is Lyapunov stable. Then it follows by a total stability result [12, p. 515] that $(x_1, x_2) = (0, 0)$ is a Lyapunov stable equilibrium of the plant dynamics in (31).

Since the state $x_2$ is directly available by the output function $y = x_2$, we consider the problem of finding estimates for the state $x_1$, i.e., we set

$$w = q(x) = x_1,$$

and consider the problem of finding associated general exponential observers for plant (31).

In order that the candidate observer

$$\dot{z} = g(z, y), \quad z \in \mathbb{R}^{n_1},$$

(32)

forms a general exponential observer for the nonlinear plant (31), Theorem 1 gives the following necessary and sufficient conditions.

(a) $g(q(x), h(x)) = Dq(x) f(x), \forall x \in X$; i.e.,

$$g(x_1, x_2) = f_1(x_1, x_2), \quad \forall x_1, x_2.$$  

(b) The dynamics

$$\dot{z} = g(z, 0)$$

is locally exponentially stable.
Combining (a) and (b), we conclude that a necessary and sufficient condition for plant (31) to have a general exponential observer is that the dynamics

\[ \dot{x}_1 = f_1(x_1, 0) \]

is locally exponentially stable. Moreover, in this case, plant (31) has a unique general exponential observer given by

\[ \dot{z} = f_1(z, y). \]

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