Local and Global Asymptotic Stability of Nonlinear Cascade Interconnected Systems

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Abstract—In this paper, we derive some sufficient conditions for local and global asymptotic stability of both continuous-time and discrete-time nonlinear cascade interconnected systems. We prove our results using converse Lyapunov stability theorems and LaSalle's invariance principle for continuous-time and discrete-time nonlinear systems. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we derive some sufficient conditions for local and global asymptotic stability of both continuous-time and discrete-time nonlinear cascade interconnected systems. These results have important applications in the stability analysis of physical and engineering system models, which can be put in a cascade form [1], and also in the feedback stabilization of nonlinear control systems [2]. We establish our stability results using converse Lyapunov stability theorems and LaSalle's invariance principle for nonlinear systems.

This paper is organized as follows. In Section 2, we derive our main stability results for continuous-time nonlinear cascade systems. In Section 3, we derive the corresponding stability results for discrete-time nonlinear cascade systems.

2. MAIN RESULTS FOR CONTINUOUS-TIME NONLINEAR CASCADE SYSTEMS

In this section, we consider continuous-time nonlinear cascade systems of the form

\[
\dot{x} = \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_p
\end{bmatrix} = f(x) = \begin{bmatrix}
f_1(x_1) \\
f_2(x_1, x_2) \\
\vdots \\
f_p(x_1, x_2, \ldots, x_p)
\end{bmatrix},
\]

(1)
where $x = (x_1, x_2, \ldots, x_p)^T$ is the state of cascade system (1), $x_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, p$ and $n_1 + n_2 + \cdots + n_p = n$. We assume that $f$ is a $C^1$ vector field and $f(0) = 0$.

First, we state and prove a theorem giving sufficient conditions for global asymptotic stability of the equilibrium $x = 0$ of the dynamics (1).

**THEOREM 1.** Suppose that for each $i = 1, 2, \ldots, p$, $x_i = 0$ is a globally asymptotically stable equilibrium of the subsystem

$$\dot{x}_i = f_i(0,0,\ldots,0,x_i).$$

Suppose also that all the trajectories $x(t)$ of the system (1) are bounded for $t \geq 0$. Then, $x = 0$ is a globally asymptotically stable equilibrium of cascade system (1).

**PROOF.** Our proof is by induction on $p$. Theorem 1 is trivially true if cascade system (1) has the simple form

$$\dot{z}_1 = f_1(z_1).$$

Next, we suppose that Theorem 1 is true when cascade system (1) has the form

$$\dot{y} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_j \end{bmatrix} = g(y) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_j(x_1, x_2, \ldots, x_j) \end{bmatrix},$$

where $x_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, j$, $y = (x_1, x_2, \ldots, x_j)^T$, and $g$ is a $C^1$ vector field with $g(0) = 0$. We claim that Theorem 1 is also true when cascade system (1) has the form

$$\dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_j \\ \dot{x}_{j+1} \end{bmatrix} = h(z) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_j(x_1, x_2, \ldots, x_j) \\ f_{j+1}(x_1, x_2, \ldots, x_j, x_{j+1}) \end{bmatrix},$$

where $x_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, j + 1$, $z = (x_1, x_2, \ldots, x_j, x_{j+1})^T$, and $h$ is a $C^1$ vector field with $h(0) = 0$.

To prove Theorem 1 for cascade system (4), we assume that $x_i = 0$ is a globally asymptotically stable equilibrium of the subsystem

$$\dot{x}_i = f_i(0,0,\ldots,0,x_i),$$

for each $i = 1, 2, \ldots, j + 1$, and also that all the trajectories $z(t)$ of system (4) are bounded for $t \geq 0$.

We claim that the equilibrium $z = 0$ of the dynamics (4) is globally asymptotically stable. Our proof uses the induction hypothesis, Massera's converse Lyapunov theorem [3] for asymptotic stability of nonlinear systems and LaSalle's invariance principle [4] for nonlinear autonomous systems.

Clearly, we can rewrite $z$ as

$$z = \begin{bmatrix} y \\ x_{j+1} \end{bmatrix}, \quad \text{where} \quad y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix}.$$  

Hence, dynamics (4) reduces to

$$\dot{z} = \begin{bmatrix} \dot{y} \\ \dot{x}_{j+1} \end{bmatrix} = h(z) = \begin{bmatrix} g(y) \\ f_{j+1}(y, x_{j+1}) \end{bmatrix},$$

where $g(y)$ is as defined in (3).
By our induction hypothesis, it is immediate that \( y = 0 \) is a globally asymptotically stable equilibrium of the nonlinear system
\[
\dot{y} = g(y).
\] (6)

Hence, by Massera’s converse Lyapunov theorem [3], we know that there exists a \( C^1 \) Lyapunov function \( U : \mathbb{R}^m \rightarrow \mathbb{R} \) for system (6), where \( m \) is defined as \( m = n_1 + n_2 + \cdots + n_j \). Thus, \( U \) is a \( C^1 \) positive definite function on \( \mathbb{R}^m \) and
\[
\dot{U}(y) = \frac{\partial U}{\partial y}(y) \cdot g(y)
\]
is a negative definite function on \( \mathbb{R}^m \). To show that \((y, x_{j+1}) = (0, 0)\) is a globally asymptotically stable equilibrium of cascade system (5), we consider the candidate Lyapunov function
\[
V(z) = V(y, x_{j+1}) = U(y).
\]

Then, for all \( z = (y, x_{j+1}) \in \mathbb{R}^m \times \mathbb{R}^{n_{j+1}} \), we have
\[
\dot{V}(z) = \dot{V}(y, x_{j+1}) = \dot{U}(y) = \frac{\partial U}{\partial y}(y) \cdot g(y) \leq 0.
\]

Hence, by LaSalle’s invariance principle [4], it follows that as \( t \rightarrow \infty \), all trajectories \((y(t), x_{j+1}(t))\) of cascade system (5) (which are globally bounded for \( t \geq 0 \)) tend to the largest invariant subset of the locus of points defined by
\[
\dot{V}(y, x_{j+1}) = \dot{U}(y) = 0.
\]

Since \( \dot{U}(y) \) is a negative definite function, it is immediate that
\[
\dot{U}(y) = 0 \iff y = 0.
\]

Also, when \( y = 0 \), the differential equation for \( x_{j+1} \) reduces to the subsystem
\[
\dot{x}_{j+1} = f_{j+1}(0, 0, \ldots, 0, x_{j+1}),
\]
which has \( x_{j+1} = 0 \) as a globally asymptotically stable equilibrium.

Hence, we conclude that \((y, x_{j+1}) = (0, 0)\) is a globally asymptotically stable equilibrium of cascade system (5). This completes the induction, and hence, the proof of Theorem 1.

REMARK 1. For the particular case of \( p = 2 \), Theorem 1 is the same as the result derived by Seibert and Suarez ([2, Theorem 4.2]). We also note that our proof is new and direct.

Next, we state the local version of Theorem 1, which was first derived by Vidyasagar [1]. Our proof of the local stability result is new and direct.

THEOREM 2. (See [1].) Suppose that for each \( i = 1, 2, \ldots, p \), \( x_i = 0 \) is a locally asymptotically stable equilibrium of the subsystem
\[
\dot{x}_i = f_i(0, 0, \ldots, 0, x_i).
\]

Then, \( x = 0 \) is a locally asymptotically stable equilibrium of cascade system (1).

REMARK 2. Theorem 2 can easily be proved using Massera’s converse Lyapunov stability theorem [3] and LaSalle’s invariance principle [4] for autonomous nonlinear systems, very similar to the proof of Theorem 1. We just remark that it is redundant to assume that all the trajectories \( x(t) \) of system (1) with small initial conditions \( x(0) \) are bounded for \( t \geq 0 \). Indeed, if for each \( i = 1, 2, \ldots, p \), \( x_i = 0 \) is a locally asymptotically stable equilibrium of the subsystem \( \dot{x}_i = f_i(0, 0, \ldots, 0, x_i) \), then a total stability result [5, Corollary, p. 515] guarantees that \( x = 0 \) is a Lyapunov stable equilibrium of (1). In particular, it follows that all trajectories \( x(t) \) of (1) with small initial conditions \( x(0) \) are bounded for \( t \geq 0 \).
3. MAIN RESULTS FOR DISCRETE-TIME NONLINEAR CASCADE SYSTEMS

In this section, we consider discrete-time nonlinear cascade systems of the form

\[
x(k+1) = \begin{bmatrix}x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_p(k+1) \end{bmatrix} = f(x(k)) = \begin{bmatrix}f_1(x_1(k)) \\ f_2(x_1(k), x_2(k)) \\ \vdots \\ f_p(x_1(k), x_2(k), \ldots, x_p(k)) \end{bmatrix},
\]

where \(x = (x_1, x_2, \ldots, x_p)^T\) is the state of cascade system (7), \(x_i \in \mathbb{R}^{n_i}\) for \(i = 1, 2, \ldots, p\) and \(n_1 + n_2 + \cdots + n_p = n\). We assume that \(f\) is a \(C^1\) map and \(f(0) = 0\).

First, we state and prove a theorem giving sufficient conditions for global asymptotic stability of the equilibrium \(x = 0\) of cascade system (7).

**Theorem 3.** Suppose that for each \(i = 1, 2, \ldots, p\), \(x_i = 0\) is a globally asymptotically stable equilibrium of the subsystem

\[
x_i(k+1) = f_i(0, 0, \ldots, 0, x_i(k)).
\]

Suppose also that all the trajectories \(x(k)\) of system (7) are bounded for \(k \geq 1\). Then, \(x = 0\) is a globally asymptotically stable equilibrium of cascade system (7).

**Proof.** Our proof is by induction on \(p\). Theorem 3 is trivially true if cascade system (7) has the simple form

\[
y(k+1) = \begin{bmatrix}x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_j(k+1) \end{bmatrix} = g(y(k)) = \begin{bmatrix}f_1(x_1(k)) \\ f_2(x_1(k), x_2(k)) \\ \vdots \\ f_j(x_1(k), x_2(k), \ldots, x_j(k)) \end{bmatrix},
\]

where \(x_i \in \mathbb{R}^{n_i}\) for \(i = 1, 2, \ldots, j\), \(y = (x_1, x_2, \ldots, x_j)^T\), and \(g\) is a \(C^1\) map with \(g(0) = 0\). We claim that Theorem 3 is also true when cascade system (7) has the form

\[
z(k+1) = \begin{bmatrix}x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_j(k+1) \\ x_{j+1}(k+1) \end{bmatrix} = h(z(k)) = \begin{bmatrix}f_1(x_1(k)) \\ f_2(x_1(k), x_2(k)) \\ \vdots \\ f_j(x_1(k), \ldots, x_j(k)) \\ f_{j+1}(x_1(k), \ldots, x_j(k), x_{j+1}(k+1)) \end{bmatrix},
\]

where \(x_i \in \mathbb{R}^{n_i}\) for \(i = 1, 2, \ldots, j+1\), \(z = (x_1, x_2, \ldots, x_j, x_{j+1})^T\), and \(h\) is a \(C^1\) map with \(h(0) = 0\).

To prove Theorem 3 for cascade system (10), we assume that \(x_i = 0\) is a globally asymptotically stable equilibrium of the subsystem

\[
x_i(k+1) = f_i(0, 0, \ldots, 0, x_i(k)),
\]

for each \(i = 1, 2, \ldots, j+1\), and also that all the trajectories \(z(k)\) of system (10) are bounded for \(k \geq 1\).

We claim that the equilibrium \(z = 0\) of the dynamics (10) is globally asymptotically stable. Our proof uses the induction hypothesis, converse Lyapunov theorem for asymptotic stability of
discrete-time nonlinear systems [6] and LaSalle’s invariance principle for discrete-time nonlinear autonomous systems [6].

Clearly, we can rewrite \( z \) as

\[
Hence, the dynamics (10) reduces to
\]

\[
z(k + 1) = \begin{bmatrix} y(k + 1) \\ x_{j+1}(k + 1) \end{bmatrix} = h(z(k)) = \begin{bmatrix} g(y(k)) \\ f_{j+1}(y(k), x_{j+1}(k)) \end{bmatrix},
\]

where \( g(y(k)) \) is as defined in (9).

By our induction hypothesis, it is immediate that \( y = 0 \) is a globally asymptotically stable equilibrium of the nonlinear system

\[
y(k + 1) = g(y(k)).
\]

Hence, by the converse Lyapunov theorem for global asymptotic stability of discrete-time systems [6], we know that there exists a \( C^1 \) Lyapunov function \( U : \mathbb{R}^m \to \mathbb{R} \) for system (12), where \( m \) is defined as \( m = n_1 + n_2 + \cdots + n_j \). Thus, \( U \) is a \( C^1 \) positive definite function on \( \mathbb{R}^m \) and

\[
\Delta U(y) = U(g(y)) - g(y)
\]

is a negative definite function on \( \mathbb{R}^m \). To show that \( (y, x_{j+1}) = (0, 0) \) is a globally asymptotically stable equilibrium of cascade system (11), we consider the candidate Lyapunov function

\[
V(z) = V(y, x_{j+1}) = U(y).
\]

Then, for all \( z = (y, x_{j+1}) \in \mathbb{R}^m \times \mathbb{R}^{n_j+1} \), we have

\[
\Delta V(z) = \Delta V(y, x_{j+1}) = \Delta U(y) = U(g(y)) - g(y) \leq 0.
\]

Hence, by LaSalle’s invariance principle for discrete-time nonlinear systems [6], it follows that as \( k \to \infty \), all trajectories \( (y(k), x_{j+1}(k)) \) of cascade system (11) (which are globally bounded for \( k \geq 1 \)) tend to the largest invariant subset of the locus of points defined by

\[
\Delta V(y, x_{j+1}) = \Delta U(y) = 0.
\]

Since \( \Delta U(y) \) is a negative definite function, it is immediate that

\[
\Delta U(y) = 0 \iff y = 0.
\]

Also, when \( y = 0 \), the difference equation for \( x_{j+1} \) reduces to the subsystem

\[
x_{j+1}(k + 1) = f_{j+1}(0, 0, \ldots, 0, x_{j+1}(k)),
\]

which has \( x_{j+1} = 0 \) as a globally asymptotically stable equilibrium.

Hence, we conclude that \( (y, x_{j+1}) = (0, 0) \) is a globally asymptotically stable equilibrium of cascade system (11). This completes the induction, and hence, the proof of Theorem 3.

Next, we state the local version of Theorem 3.
THEOREM 4. Suppose that for each \( i = 1, 2, \ldots, p \), \( x_i = 0 \) is a locally asymptotically stable equilibrium of the subsystem

\[
x_i(k+1) = f_i(0, 0, \ldots, 0, x_i(k))
\]

Then, \( x = 0 \) is a locally asymptotically stable equilibrium of cascade system (7).

REMARK 3. Theorem 4 can be easily proved using the converse Lyapunov stability theorem for local asymptotic stability of discrete-time nonlinear systems and LaSalle's invariance principle for autonomous nonlinear systems [6], very similar to the proof of Theorem 3. We note that it is redundant to assume that all the trajectories \( x(k) \) of system (7) with small initial conditions \( x(0) \) are bounded for \( k \geq 1 \). Indeed, if for each \( i = 1, 2, \ldots, p \), \( x_i = 0 \) is a locally asymptotically stable equilibrium of the subsystem \( x_i(k+1) = f_i(0, 0, \ldots, 0, x_i(k)) \), then, a total stability result in Lyapunov stability theory guarantees that \( x = 0 \) is a Lyapunov stable equilibrium of (7). In particular, it follows that all trajectories \( x(k) \) of (7) with small initial conditions \( x(0) \) are bounded for \( k \geq 1 \).

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