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Some Topological and Geometric Properties of Sequence Spaces Involving Lacunary Sequence in $n$–normed Spaces

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Abstract. The aim of this paper is to study subsets of real linear $n$–normed spaces involving lacunary sequence and examine some topological and geometric properties of these sequence spaces.

Keywords: $n$-normed space, geometric properties, lacunary sequence, paranorm

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INTRODUCTION

The concept of a 2-normed space was improved by Gähler [4], [7], [14], [15], [16]. Let us recall the concept of an $n$-sequences) under the generalized by Misiak [10]. The standart concept of a norm has been extended to a new concept which is called $n$-norm. Many mathematicians have studied these concepts, see for instance [2], [4], [5], [6], [7], [14], [15], [16]. Let us first recall the concept of an $n$–norm; Let $n \in \mathbb{N}$ and $X$ be a real linear space of dimension $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\| : X^n \rightarrow \mathbb{R}$ satisfying the following four properties:

1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent vectors,
2. $\|x_1, \ldots, x_n\| = \|x_{j_1}, \ldots, x_{j_n}\|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$,
3. $\|\alpha x_1, \ldots, x_n\| = \|\alpha\|_2 \|x_1, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$,
4. $\|x+y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$ for all $x, y, x_2, \ldots, x_n \in X$, is called an $n$–norm on $X$ and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called a linear $n$–normed space.

A trivial example of an $n$–normed space is $X = \mathbb{R}^n$ equipped with the following Euclidean $n$-norm: $\|x_1, \ldots, x_n\| = \left(\det(x_{ij})\right)$ where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, \ldots, n$. The standart $n$-norm on $X$, where $X$ is a real inner product space of dimension $d \geq n$, is defined as;

$$\|x_1, \ldots, x_n\|_S := \left|\langle x_1, x_1 \rangle \cdots \langle x_1, x_n \rangle \right|^{\frac{1}{2}}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $X$. If $X = \mathbb{R}^n$ then this $n$-norm is exactly the same as the Euclidean $n$-norm $\|x_1, \ldots, x_n\|_E$ as mentioned earlier.

Notice that for $n = 1$ the above $n$-norm is the usual norm $\|x_1\|_S = \langle x_1, x_1 \rangle^{\frac{1}{2}}$ which gives the length of $x_1$, while for $n = 2$, which defines the standart 2–norm $\|x_1, x_2\|_S = \left(\|x_1\|_S^2 + \|x_2\|_S^2 - \langle x_1, x_1 \rangle \right)^{\frac{1}{2}}$ which represents the area of the parallelogram spanned by $x_1$ and $x_2$. Further if $X = \mathbb{R}^3$, then $\|x_1, x_2, x_3\|_E = \|x_1, x_2, x_3\|_E$ is represent the volume of the parallelograms spanned by $x_1, x_2$ and $x_3$. In general $\|x_1, \ldots, x_n\|_S$ represents the volume of the $n$–dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in $X$.

Recall that an Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. Note that if $M$ is an Orlicz function then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we will mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{h_r}{k_{r-1}}$ will be denoted by $q_r$. The space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman...)
[3] as,

\[ N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}. \]

There exists very close relation between the space of lacunary strongly convergent sequences and the space of strongly Cesaro summability sequences. This connection can be found in [1], [13]. Since this relation many geometric property of Cesaro sequence spaces can be generalized the lacunary sequence spaces.

In the section two, we introduce some sequence spaces and give some geometric properties related with these spaces. And the last section some geometric properties has been investigated by using a modular function and Luxemburg norm.

Now, we give some definitions, given in [7], which are necessary throughout the paper.

**Definition 1** A sequence \((x_k)\) in \(n\)-normed space \(X = \left\{ x = (x_k) : \lim_{r \to \infty} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}\)

**Definition 2** A sequence \((x_k)\) in \(n\)-normed space \(X = \left\{ x = (x_k) : \lim_{r \to \infty} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}\)

Recall that a paranorm \(g : X \to \mathbb{R}\) satisfies the conditions \(i. g(\theta) = 0\), \(ii. g(x) = g(-x)\), \(iii. g(x+y) \leq g(x) + g(y)\) and \(iv. |x - \lambda_0| \to 0\) and \(g(x-x_0) \to 0\) implies \(g(\lambda x - \lambda_0 x_0) \to 0\). Throughout the article \((X, \| \cdots \|)\) will be an \(n\)-normed space and \(w(X)\) will denote \(X\)-valued sequence space. Besides the following inequality will be used throughout the paper:

\[ |a_k + b_k|^p_k \leq K (|a_k|^{p_k} + |b_k|^{p_k}) \]

where \(K = \max \{1, 2^{H-1}\}\), \(0 < p_k \leq \sup p_k = H < \infty\).

### SOME TOPOLOGICAL PROPERTIES OF \(l(p, \theta, M, \| \cdot \|, \ldots, \|)\)

In this section we define some new sequence spaces involving lacunary sequence in \(n\)-normed spaces. Let \(\theta\) be a lacunary sequence and \(M\) be any Orlicz function. Then we denote by \(l(p, \theta, M, \| \cdot \|, \ldots, \|)\) the sequence space involving lacunary sequence defined by as the set of all \(x \in w(X)\) such that

\[
l(p, \theta, M, \| \cdot \|, \ldots, \|) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \leq \left\| \sum_{k \in I_r} \left| x_k \right| \right\| \leq \left\| M \left( \left\| \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \right\| \right) \right\| < \infty, \right. \)

for some \(p > 0\) and for every \(z_1, \ldots, z_{n-1} \in X\).

If \(M(x) = x\) then we get the sequence space as in the following,

\[
l(p, \theta, \| \cdot \|, \ldots, \|) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \leq \left\| \sum_{k \in I_r} \left| x_k \right| \right\| \leq \left\| M \left( \left\| \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \right\| \right) \right\| < \infty, \right. \)

for every \(z_1, \ldots, z_{n-1} \in X\).

If \(p_r = p\) for all \(r\), \(M(x) = x\) we write this sequence space as \(l(p, \theta, \| \cdot \|, \ldots, \|)\),

\[
l(p, \theta, \| \cdot \|, \ldots, \|) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \leq \left\| \sum_{k \in I_r} \left| x_k \right| \right\| \leq \left\| M \left( \left\| \frac{1}{M} \sum_{k \in I_r} \left| x_k \right| \right\| \right) \right\| < \infty, \right. \)

for every \(z_1, \ldots, z_{n-1} \in X\).

We denote \(l(p, \theta, M, \| \cdot \|, \ldots, \|)\) by \(l(\theta, \| \cdot \|, \ldots, \|)\) where \(M(x) = x\) and \(p_r = p = 1\) for all \(r\). In the special case where \(\theta = (2^*), \) we have \(ces[p, M, \| \cdot \|, \ldots, \|] = l(p, \theta, M, \| \cdot \|, \ldots, \|), \) \(ces[p, \| \cdot \|, \ldots, \|] = l(\theta, \| \cdot \|, \ldots, \|)\). We denote by \(ces[p, \| \cdot \|, \ldots, \|]\) the sequence space involving Cesaro sequence, defined as the set of all \(x \in w(X)\) such that
Theorem 2 Let \( p = (p_r) \) of strictly positive real numbers \( l(p, \theta, M, \|\cdot\|, \ldots) \) is a linear space over the complex numbers field \( \mathbb{C} \).

Proof The proof of this theorem is obvious, so we omit it.

Theorem 2 Let \( 1 \leq p_r < \infty \) and \( (X, \|\cdot\|, \ldots) \) be an \( n \)-Banach space. For any Orlicz function \( M \) and a bounded sequence \( p = (p_r) \) of strictly positive real numbers \( l(p, \theta, M, \|\cdot\|, \ldots) \) is a paranormed space by

\[
gr(x) = \inf \left\{ \rho^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{p_r} \sum_{k \in K_r} M\left( \left\| \frac{x_k}{p_r}, z_1, \ldots, z_{n-1} \right\| \right)^{p_r} \right)^{\frac{1}{p}} < \infty, \right\}
\]

for every \( z_1, \ldots, z_{n-1} \in X \), for some \( \rho > 0 \).

where \( H = \max(1, \sup p_r) \).

Proof The conditions (i. – iii.) are clearly hold.

iv. We prove the scalar multiplication is continuous. Let \( \lambda \) be any number and by using the definition of the paranorm,

\[
gr(\lambda x) = \inf \left\{ \rho^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{p_r} \sum_{k \in K_r} M\left( \left\| \frac{x_k}{p_r}, z_1, \ldots, z_{n-1} \right\| \right)^{p_r} \right)^{\frac{1}{p}} < \infty, \right\}
\]

for every \( z_1, \ldots, z_{n-1} \in X \), for some \( \rho > 0 \).

Then

\[
gr(\lambda x) = \inf \left\{ (\lambda p)^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{h_r} \sum_{k \in K_r} M\left( \left\| \frac{x_k}{h_r}, z_1, \ldots, z_{n-1} \right\| \right)^{h_r} \right)^{\frac{1}{h}} < \infty, \right\}
\]

where \( p = \frac{p}{\lambda} \). Since \( |\lambda|^{p_r} \leq \max \left( 1, |\lambda|^{H} \right) \) therefore \( |\lambda|^{p_r} \leq \left( \max \left( 1, |\lambda|^{H} \right) \right)^{\frac{1}{H}} \). Hence

\[
gr(\lambda x) \leq \left( \max \left( 1, |\lambda|^{H} \right) \right)^{\frac{1}{H}} \inf \left\{ \rho^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{h_r} \sum_{k \in K_r} M\left( \left\| \frac{x_k}{h_r}, z_1, \ldots, z_{n-1} \right\| \right)^{h_r} \right)^{\frac{1}{h}} < \infty, \right\}
\]

which converges to zero as \( g_r(x) \) converges to zero in \( l(p, \theta, M, \|\cdot\|, \ldots) \). Now suppose \( \lambda_r \rightarrow 0 \) and \( x \) is in \( l(p, \theta, M, \|\cdot\|, \ldots) \). Then there exists \( \rho > 0 \) such that

\[
gr(x) = \inf \left\{ \rho^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{h_r} \sum_{k \in K_r} M\left( \left\| \frac{x_k}{h_r}, z_1, \ldots, z_{n-1} \right\| \right)^{h_r} \right)^{\frac{1}{h}} < \infty, \right\}
\]

for every \( z_1, \ldots, z_{n-1} \in X \).

Now

\[
gr(\lambda x) = \inf \left\{ \rho^{\frac{p_r}{\rho}} : \left( \sum_{r=1}^{\infty} \frac{1}{h_r} \sum_{k \in K_r} M\left( \left\| \frac{\lambda x_k}{h_r}, z_1, \ldots, z_{n-1} \right\| \right)^{h_r} \right)^{\frac{1}{h}} < \infty, \right\} \rightarrow 0
\]
as \( \lambda \to 0 \), for every \( z_1, \ldots, z_{n-1} \in X \) and for some \( \rho > 0 \).

Let \((x')\) be any Cauchy sequence in \( l (p, \theta, M, \| \cdot \|, \ldots, \| \cdot \|) \). Let \( s \) and \( x_0 \) be fixed such that \( M (sx_0) \geq 1 \). Then for each \( \frac{\rho}{sx_0} > 0 \) there exists a positive integer \( N \) such that \( g_r (x' - x') < \frac{\rho}{sx_0} \), for all \( i, j \geq N \). Since \( g_r (x' - x') \) is positive so we can substitute \( \rho \) for \( g_r (x' - x') \). Using (6) for every \( z_1, \ldots, z_{n-1} \in X \) we get

\[
\left( \sum_{r=1}^{\infty} \frac{1}{H_r} \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right)^{\frac{1}{p_r}} \right) < \infty, \text{ for all } i, j \geq N.
\]

Thus

\[
\sum_{r=1}^{\infty} \frac{1}{H_r} \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right)^{\frac{1}{p_r}} < \infty, \text{ for all } i, j \geq N.
\]

Since \( 1 \leq p_r \leq \infty \) we have

\[
\frac{1}{H_r} \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right)^{\frac{1}{p_r}} \to 0, \text{ as } r \to \infty.
\]

Since \( \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right) \) is bounded then it follows that

\[
M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right) \leq 1
\]

for sufficiently large values of \( r \). Since \( M \left( \frac{sx_0}{2} \right) \geq 1 \), we obtain that

\[
M \left( \left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \right) \leq M \left( \frac{sx_0}{2} \right).
\]

Since \( M \) is non-decreasing and convex function, then

\[
\left\| \frac{x_k - x_k'}{g_r (x' - x')} \right\| \leq \frac{sx_0}{2}
\]

\[
\left\| x_k - x_k' \right\| \leq \frac{sx_0}{2} \cdot g_r (x' - x')
\]

\[
< \frac{sx_0}{2} \cdot \frac{\epsilon}{sx_0} = \frac{\epsilon}{2}
\]

for all \( i, j \geq N \). \((x')\) is convergent in \( X \) for all \( i \geq N \), since \( X \) is an \( n \)-Banach space. Using the continuity of \( M \) and \( \| \cdot \|, \ldots, \| \cdot \| \) functions and taking the limit as \( j \to \infty \) we have,

\[
\left( \sum_{r=1}^{\infty} \frac{1}{H_r} \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{\rho} \right\| \right)^{\frac{1}{p_r}} \right)^{\frac{1}{p'}} < \infty,
\]

for every \( z_1, \ldots, z_{n-1} \). Taking the infimum of such \( \rho' \)'s we get for every \( z_1, \ldots, z_{n-1} \in X \) and for all \( i \geq N \),

\[
\inf \left\{ \rho' : \left( \sum_{r=1}^{\infty} \frac{1}{H_r} \sum_{k \in I_r} M \left( \left\| \frac{x_k - x_k'}{\rho} \right\| \right)^{\frac{1}{p_r}} \right)^{\frac{1}{p'}} < \infty \right\} < \epsilon.
\]

The sequence space \( l (p, \theta, M, \| \cdot \|, \ldots, \| \cdot \|) \) is a linear space and \((x') \in l (p, \theta, M, \| \cdot \|, \ldots, \| \cdot \|) \) then we have \( x = x' - (x' - x) \in \ell (p, \theta, M, \| \cdot \|, \ldots, \| \cdot \|) \). This completes the proof.

**Theorem 3** \( l (p, \theta, M_1, \| \cdot \|, \ldots, \| \cdot \|) \cap l (p, \theta, M_2, \| \cdot \|, \ldots, \| \cdot \|) \subset l (p, \theta, M_1 + M_2, \| \cdot \|, \ldots, \| \cdot \|) \).
Let \( x \in l(p, \theta, \| \cdot \|, \ldots, \| \cdot \|) \cap l(p, \theta, \| \cdot \|, \ldots, \| \cdot \|) \) then,
\[
\sum_{r=1}^{\infty} \left[ \frac{1}{h_r} \sum_{k \in I_r} M_1 \left( \frac{x_k}{\rho \| x_k \|} \right) \right] < \infty,
\]
for some \( \rho_1 > 0 \).
\[
\sum_{r=1}^{\infty} \left[ \frac{1}{h_r} \sum_{k \in I_r} M_2 \left( \frac{x_k}{\rho_2 \| x_k \|} \right) \right] < \infty,
\]
for some \( \rho_2 > 0 \). Let \( \rho = \max \{ \rho_1, \rho_2 \} \)
\[
\left[ \frac{1}{h_r} \sum_{k \in I_r} (M_1 + M_2) \left( \frac{x_k}{\rho \| x_k \|} \right) \right] ^{p_r}
\]
\[
= \left[ \frac{1}{h_r} \sum_{k \in I_r} M_1 \left( \frac{x_k}{\rho \| x_k \|} \right) + \frac{1}{h_r} \sum_{k \in I_r} M_2 \left( \frac{x_k}{\rho_2 \| x_k \|} \right) \right] ^{p_r}
\]
\[
\leq K \left\{ \left[ \frac{1}{h_r} \sum_{k \in I_r} M_1 \left( \frac{x_k}{\rho_1 \| x_k \|} \right) \right] ^{p_r} + \left[ \frac{1}{h_r} \sum_{k \in I_r} M_2 \left( \frac{x_k}{\rho_2 \| x_k \|} \right) \right] ^{p_r} \right\}
\]
By taking sum from 1 to \( \infty \) we get,
\[
\sum_{r=1}^{\infty} \left[ \frac{1}{h_r} \sum_{k \in I_r} (M_1 + M_2) \left( \frac{x_k}{\rho \| x_k \|} \right) \right] ^{p_r}
\]
\[
\leq K \left\{ \sum_{r=1}^{\infty} \left[ \frac{1}{h_r} \sum_{k \in I_r} M_1 \left( \frac{x_k}{\rho_1 \| x_k \|} \right) \right] ^{p_r} + \sum_{r=1}^{\infty} \left[ \frac{1}{h_r} \sum_{k \in I_r} M_2 \left( \frac{x_k}{\rho_2 \| x_k \|} \right) \right] ^{p_r} \right\}
\]
\[
< \infty
\]
Hence \( x \in l(p, \theta, M_1 + M_2, \| \cdot \|, \ldots, \| \cdot \|) \).

**SOME GEOMETRIC PROPERTIES OF \( l(p, \theta, \| \cdot \|, \ldots, \| \cdot \|) \)**

Let \( X \) be a real vector space. A functional \( \sigma : X \to [0, \infty] \) is called a convex modular if \( i. \sigma(x) = 0 \) if and only if \( x = 0 \); 
\( ii. \sigma(\lambda x) = \sigma(x) \) for all scalar \( \lambda \) with \( |\lambda| = 1 \); 
\( iii. \sigma(\lambda x + \mu y) \leq \sigma(x) + \beta \sigma(y) \) for all \( x, y \in X \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \). Let \( X \) be a Banach space and \( S(X) \) and \( B(X) \) be the unit sphere and the unit ball of \( X \), respectively.

A Banach space \( X \) is said to have the Kadec-Klee property (or property \( H \)) if every weakly convergent sequence on the unit sphere with the weak limit in the sphere is convergent in norm.

Related articles can be seen in [11], [12], [17].

Let \( 1 \leq p < \infty \). A Banach space is said to have the Banach-Saks type \( p \), if every weakly null sequence has a subsequence \( (x_{k}) \) such that for some \( C > 0 \)
\[
\| \sum_{i=0}^{n} x_{k_{i}} \| \leq C(n+1)^{\frac{1}{p}}
\]
for all \( n \in \mathbb{N} \). Let \( \{ p_{i} \} \) be a bounded sequence of the positive real numbers. Recall that the sequence space \( l(p, \theta, \| \cdot \|, \ldots, \| \cdot \|) \) was defined in (2). Paranorm on \( l(p, \theta, \| \cdot \|, \ldots, \| \cdot \|) \) is given by
\[
g(x) = \left[ \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{\frac{p_r}{n}} \right]^{\frac{1}{p_r}}
\]
where $H = \sup_r p_r$. If $p_r = p$ for all $r$, we will use the notation $l_p (\theta, \ldots, \| \|)$ instead of $l (p, \theta, \ldots, \| \|)$. The norm on $l_p (\theta, \ldots, \| \|)$ is given by

$$
\| x \|_{l_p (\theta, \ldots, \| \|)} = \left[ \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^p \right]^{\frac{1}{p}}
$$

By using the properties of lacunary sequence in the space $l (p, \theta, \ldots, \| \|)$ we get the following sequences:

If $\theta = (2^r)$, then $l (p, \theta, \ldots, \| \|) = ces (p, \ldots, \| \|)$. If $\theta = (2^r)$ and $p_r = p$ for all $r$, then $l (p, \theta, \ldots, \| \|) = ces (p, \ldots, \| \|)$ For $x \in l (p, \theta, \ldots, \| \|)$, let

$$
\sigma (x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^p
$$

and define the generalized Luxemburg norm on $l (p, \theta, \ldots, \| \|)$ by

$$
\| x \| = \inf \left\{ \rho > 0 : \sigma (\frac{x}{\rho}) \leq 1 \right\}
$$

The generalized Luxemburg norm on $l_p (\theta, \ldots, \| \|)$ can be reduced to a usual norm $l_p (\theta, \ldots, \| \|)$, that is, $\| x \|_{l_p (\theta, \ldots, \| \|)} = \| x \|$. The proof is similar as in [8].

Firstly we give some theorems which show the relation between $l (p, \theta, \ldots, \| \|)$ and $ces (p, \ldots, \| \|)$ in this section.

**Theorem 4** If $\lim \inf q_r > 1$, then $ces (p, \ldots, \| \|) \subset l (p, \theta, \ldots, \| \|)$.

**Theorem 5** If $1 < \lim \sup q_r < \infty$, then $l (p, \theta, \ldots, \| \|) \subset ces (p, \ldots, \| \|)$.

**Proof** The proofs of above theorems can be obtained with the similar methods as in [8].

**Lemma 1** The functional $\sigma : l_\sigma (p, \theta, \ldots, \| \|) \to [0, \infty]$ is a convex modular on $l_\sigma (p, \theta, \ldots, \| \|)$.

**Proof** The proof is similar as in [8].

**Lemma 2** For $x \in l_\sigma (p, \theta, \ldots, \| \|)$, the modular $\sigma$ on $l_\sigma (p, \theta, \ldots, \| \|)$ satisfies the following properties:

(i) If $0 < \alpha < 1$, then $\alpha^H \sigma (\frac{x}{\alpha}) \leq \sigma (x)$ and $\sigma (\alpha x) \leq \alpha \sigma (x)$

(ii) If $\alpha > 1$, then $\sigma (x) \leq \alpha^H \sigma (\frac{x}{\alpha})$

(iii) If $\alpha \geq 1$, then $\sigma (\alpha x) \geq \alpha \sigma (x)$

**Proof** (i) Let $0 < \alpha < 1$. Then we have,

$$
\sigma (x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^p_r
$$

$$
= \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \frac{x_k}{\alpha}, z_1, \ldots, z_{n-1} \| \right)^p_r
$$

$$
= \sum_{r=1}^{\infty} \alpha^p_r \left( \frac{1}{h_r} \sum_{k \in I_r} \frac{x_k}{\alpha}, z_1, \ldots, z_{n-1} \| \right)^p_r
$$

$$
\geq \alpha^H \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \frac{x_k}{\alpha}, z_1, \ldots, z_{n-1} \| \right)^p_r
$$

$$
= \alpha^H \sigma (\frac{x}{\alpha})
$$
So that, $\alpha^H \sigma \left( \frac{x}{\alpha} \right) \leq \sigma(x)$ is obtained. For the proof of the $\sigma(\alpha x) \leq \alpha \sigma(x)$ we can write as in the following,

$$
\sigma(\alpha x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| \alpha x_k, \overline{z}_1, \ldots, \overline{z}_{n-1} \| \right)^{p_r}
$$

$$
= \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
< \alpha \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
= \alpha \sigma(x).
$$

(ii) Let $\alpha > 1$. Then we have,

$$
\sigma(x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
= \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \alpha \| \frac{x_k}{\alpha}, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
= \sum_{r=1}^{\infty} \alpha^{p_r} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
\leq \alpha^H \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
= \alpha^H \sigma \left( \frac{x}{\alpha} \right).
$$

(iii) Let $\alpha \geq 1$. Then we have,

$$
\sigma(\alpha x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| \alpha x_k, \overline{z}_1, \ldots, \overline{z}_{n-1} \| \right)^{p_r}
$$

$$
= \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \alpha \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
\geq \alpha \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \| x_k, z_1, \ldots, z_{n-1} \| \right)^{p_r}
$$

$$
= \alpha \sigma(x).
$$

The following known results give relationships between the modular $\sigma$ and the Luxemburg norm.

Lemma 3 For any $x \in I_0 (p, \theta, \| \cdot, \ldots, \| )$,

(i) If $\| x \|_L < 1$, then $\sigma(x) \leq \| x \|_L$.

(ii) If $\| x \|_L > 1$, then $\sigma(x) \geq \| x \|_L$.

(iii) If $\| x \|_L = 1$, if and only if $\sigma(x) = 1$.

(iv) If $\| x \|_L < 1$, if and only if $\sigma(x) < 1$.

(v) If $\| x \|_L > 1$, if and only if $\sigma(x) > 1$.

Proof The proof is similar as in [8].

Lemma 4 For any $x \in I_0 (p, \theta, \| \cdot, \ldots, \| )$,

(i) If $0 < \alpha < 1$ and $\| x \|_L > \alpha$, then $\sigma(x) > \alpha^H$
(ii) If $\alpha \geq 1$ and $\|x\|_L < \alpha$, then $\sigma(x) < \alpha^H$.

Proof (i) Suppose that $0 < \alpha < 1$ and $\|x\|_L > \alpha$. Then $\|\frac{x}{\alpha}\|_L > 1$. By Lemma 3 (ii) we have $\sigma(\frac{x}{\alpha}) > \|\frac{x}{\alpha}\|_L > 1$. Hence by Lemma 2 (i) we obtain that $\sigma(x) > \alpha^H \sigma(\frac{x}{\alpha}) > \alpha^H$.

(ii) Assume that $\alpha > 1$ and $\|x\|_L < \alpha$. Then $\|\frac{x}{\alpha}\|_L < 1$. By Lemma 3 (i) we have $\sigma(\frac{x}{\alpha}) < \|\frac{x}{\alpha}\|_L < 1$. If $\alpha = 1$, we have $\sigma(x) < 1$, by Lemma 2 (ii), we obtain that $\sigma(x) < \alpha^H \sigma(\frac{x}{\alpha}) < \alpha^H$.

Lemma 5 Let $(x_n)$ be a sequence in $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$.

(i) If $\lim_{n \to \infty} \|x_n\| = 1$, then $\lim_{n \to \infty} \sigma(x_n) = 1$

(ii) If $\lim_{n \to \infty} \sigma(x_n) = 0$, then $\lim_{n \to \infty} \|x_n\| = 0$

Proof The proof is similar as in [8].

Theorem 6 The space $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$ is a Banach space with respect to Luxemburg norm defined by

$$
\|x\|_L = \inf \left\{ \lambda > 0 : \sigma(\frac{x}{\lambda}) \leq 1 \right\}
$$

Proof We need to show that every Cauchy sequence in $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$ is convergent according to the Luxemburg norm. Let $(x^j(k))$ be any Cauchy sequence in $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$ and $\varepsilon \in (0, 1)$. Thus, there exists $i_0$ such that $\|x^i - x^j\| < \varepsilon^H$ for all $i, j \geq i_0$. By the Lemma 3 (i), we have

$$
\sigma (x^i - x^j) < \|x^i - x^j\| < \varepsilon^H.
$$

(7)

for all $i, j \geq i_0$. This implies that

$$
\sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in L} \|x^i(k) - x^j(k), z_1, \ldots, z_{n-1}\|_X \right)^{p_r} < \varepsilon.
$$

for all $i, j \geq i_0$ and for every $z_1, \ldots, z_{n-1} \in X$. For fixed $k \in \mathbb{N}$,

$$
\|x^i(k) - x^j(k), z_1, \ldots, z_{n-1}\|_X < \varepsilon.
$$

for all $i, j \geq i_0$ and for every $z_1, \ldots, z_{n-1} \in X$. Hence the sequence $(x^i(k))$ is a Cauchy sequence in $X$. Since $(X, \|\cdot\|, \ldots, \|\cdot\|_X)$ is an $n-$Banach space, $x^i(k) \to x(k)$ as $j \to \infty$. Therefore we have

$$
\sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in L} \|x^i(k) - x(k), z_1, \ldots, z_{n-1}\|_X \right)^{p_r} < \varepsilon.
$$

for all $i \geq i_0$ and for every $z_1, \ldots, z_{n-1} \in X$. Now show that the sequence $(x(k))$ is an element of $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$. From the equation (7) we have

$$
\sigma (x^i - x^j) \to \sigma (x^i - x)
$$

as $j \to \infty$. Since for all $i \geq i_0$,

$$
\sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in L} \|x^i(k) - x^j(k), z_1, \ldots, z_{n-1}\|_X \right)^{p_r} \to \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in L} \|x^i(k) - x(k), z_1, \ldots, z_{n-1}\|_X \right)^{p_r}
$$

as $j \to \infty$ and for every $z_1, \ldots, z_{n-1} \in X$ then by equation (7) we have

$$
\sigma (x^i - x) < \|x^i - x\|_L < \varepsilon.
$$

for all $i \geq i_0$ and for every $z_1, \ldots, z_{n-1} \in X$. This implies that $x^i \to x$ as $i \to \infty$. So we have $x = x_{i_0} = (x_{i_0} - x) \in l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$. Therefore the sequence space $l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$ is a Banach space with respect to Luxemburg norm. This completes the proof.

Theorem 7 Let $x \in l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$ and $(x_m) \subset l_\sigma(p, \theta, \|\cdot\|, \ldots, \|\cdot\|)$. If $\sigma(x_m) \to \sigma(x)$ as $m \to \infty$ and $x_m(i) \to x(i)$ as $m \to \infty$ for all $i \in \mathbb{N}$, then $x_m \to x$ as $m \to \infty$. 


Proof Let $\varepsilon > 0$ be given. Since

$$
\sigma(x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \|x_k, z_1, \ldots, z_{n-1}\| \right)^{p_r} < \infty,
$$

for every nonzero $z_1, \ldots, z_{n-1} \in X$, there is $m_0 \in \mathbb{N}$ such that

$$
\sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \|x_k, z_1, \ldots, z_{n-1}\| \right)^{p_r} < \varepsilon \left( \frac{1}{2} \right)^{n+1}
$$

(8)

and since $\sigma(x_m) \to \sigma(x)$ as $m \to \infty$ and $x_m(i) \to x(i)$ as $m \to \infty$ for all $i \in \mathbb{N}$, there is $m_0 \in \mathbb{N}$ such that $m \geq m_0$,

$$
\sigma(x_m) - \sum_{r=1}^{m_0} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} < \varepsilon \left( \frac{1}{2} \right)^{n+1}
$$

(9)

and also $x_m(i) \to x(i)$ as $m \to \infty$ for all $i \in \mathbb{N}$, we have $\sigma(x_m) \to \sigma(x)$ as $m \to \infty$. Hence for all $m \geq m_0$, we have $\|x_m(i) - x(i), z_1, \ldots, z_{n-1}\| < \varepsilon$. As a result for $m \geq m_0$, we have

$$
\sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i) - x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} < \frac{\varepsilon}{3}
$$

(10)

for every nonzero $z_1, \ldots, z_{n-1} \in X$. It follows from (8), (9) and (10) that for $m \geq m_0$,

$$
\sigma(x_m - x) = \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i) - x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r}
$$

$$
= \sum_{r=1}^{m_0} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i) - x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r}
$$

$$
+ \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i) - x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r}
$$

$$
\leq \frac{\varepsilon}{3} + K \left\{ \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} 
$$

$$
+ \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} \right\}
$$

$$
= \frac{\varepsilon}{3} + K \left\{ \sigma(x_m) - \sum_{r=1}^{m_0} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i), z_1, \ldots, z_{n-1}\| \right)^{p_r}
$$

$$
+ \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} \right\}
$$

$$
\leq \frac{\varepsilon}{3} + K \left\{ \sigma(x) - \sum_{r=1}^{m_0} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x_m(i), z_1, \ldots, z_{n-1}\| \right)^{p_r}
$$

$$
+ \frac{\varepsilon}{3K} + \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} \right\}
$$

$$
= \frac{\varepsilon}{3} + K \left\{ 2 \sum_{r=m_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{i \in I_r} \|x(i), z_1, \ldots, z_{n-1}\| \right)^{p_r} + \frac{\varepsilon}{3K} \right\}
$$

$$
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
$$
for every nonzero \( z_1, \ldots, z_{n-1} \in X \). This shows that \( \sigma(x_m - x) \to 0 \) as \( m \to \infty \). Hence, by Lemma 5 (ii), we obtain that 
\[
\|x_m - x\|_L \to 0 \quad \text{as} \quad m \to \infty.
\]

**Theorem 8** The space \( L_p(\theta, \|\cdot\|, \ldots, \|\cdot\|) \) has the property \((H)\).

**Proof** Let \( x \in S(L_p(\theta, \|\cdot\|, \ldots, \|\cdot\|)) \) and \( (x_m) \subset L_p(\theta, \|\cdot\|, \ldots, \|\cdot\|) \) such that \( \|x_m\|_L \to 1 \) and \( x_m \overset{w}{\to} x \) as \( m \to \infty \). From Lemma 3 (iii), it follows that \( \sigma(x_m) \to 1 \) as \( m \to \infty \). Here \( \sigma(x_m) \to \sigma(x) \) (\( m \to \infty \)) where \( \sigma(x) = 1 \). Since \( x_m \overset{w}{\to} x \) we have \( \sigma(x_m - x) \to 0 \) as \( m \to \infty \). So that by Theorem 7, \( \|x_m - x\|_L \to 0 \) as \( m \to \infty \).

**Theorem 9** The space \( L_p(\theta, \|\cdot\|, \ldots, \|\cdot\|) \) has the Banach-Saks type \( p \).

**Proof** Let \( (\varepsilon_m) \) be a sequence of positive numbers for which \( \sum \varepsilon_m \leq \frac{1}{2} \) and also let \( (x_m) \) be a weakly null sequence in \( B(L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)) \). Set \( b_0 = x_0 = 0 \) and \( b_1 = x_{m_1} - x_1 \). Then there exists \( s_1 \in \mathbb{N} \) such that
\[
\left\| \sum_{i=s_1+1}^{\infty} b_1(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_1
\]
Since \( x_m \overset{w}{\to} 0 \) implies \( x \to 0 \) coordinatewise, there is an \( m_2 \in \mathbb{N} \) such that
\[
\left\| \sum_{i=0}^{m_1} x_m(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_1
\]
when \( m \geq m_2 \). Set \( b_2 = x_{m_2} \). Then there exists a \( s_2 > s_1 \) such that
\[
\left\| \sum_{i=s_2+1}^{\infty} b_2(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_2
\]
Again using the fact \( x_m \to 0 \) coordinatewise, there exists an \( n_3 > n_2 \) such that
\[
\left\| \sum_{i=0}^{n_1} x_m(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_2
\]
when \( m \geq m_3 \). Continuing this process, we can find two increasing sequences \((s_j)\) and \((n_j)\) such that
\[
\left\| \sum_{i=0}^{n_j} x_m(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_j
\]
for each \( m \geq m_{j+1}, \) and
\[
\left\| \sum_{i=s_{j+1}+1}^{\infty} b_j(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} < \varepsilon_j
\]
where \( b_j = x_{m_j} \). Hence,
\[
\left\| \sum_{j=0}^{m} b_j \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} \leq \left\| \sum_{j=0}^{m} \sum_{i=s_{j+1}+1}^{n_j} b_j(i) e^{(i)} \right\|_{L_p(\theta,\|\cdot\|,\ldots,\|\cdot\|)} + 2 \sum_{j=0}^{m} \varepsilon_j.
\]

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and

\[
\left\| \sum_{j=0}^{m} \sum_{i=s_{j}-1+1}^{s_{j}} b_{j} (i) e^{(i)} \right\|_{p, \|\|, \ldots, \|}^{p} = \sum_{j=0}^{m} \sum_{i=s_{j}-1+1}^{s_{j}} \left( \frac{1}{h_{j}} \sum_{k\leq l_{i}} \|b_{j} (k), \tilde{z}_{1}, \ldots, \tilde{z}_{n-1}\| \right)^{p} \leq \sum_{j=1}^{m} \sum_{i=0}^{\infty} \left( \frac{1}{h_{j}} \sum_{k\leq l_{i}} \|b_{j} (k), \tilde{z}_{1}, \ldots, \tilde{z}_{n-1}\| \right)^{p} \leq (m+1)^{p}
\]

Hence, we obtain

\[
\left\| \sum_{j=0}^{m} \sum_{i=s_{j}-1+1}^{s_{j}} b_{j} (i) e^{(i)} \right\|_{p, \|\|, \ldots, \|}^{p} \leq (m+1)^{\frac{1}{p}} + 2 (m+1)^{\frac{1}{p}}.
\]

By using the fact \(1 \leq (m+1)^{\frac{1}{p}}\) for all \(m \in \mathbb{N}\), we have

\[
\left\| \sum_{j=0}^{m} b_{j} \right\|_{p, \|\|, \ldots, \|} \leq (m+1)^{\frac{1}{p}} + 1 \leq (m+1)^{\frac{1}{p}}.
\]

Hence \(l_{p}(\theta, \|\|, \ldots, \|)\) has the Banach-Saks type \(p\). This completes the proof.

REFERENCES