A worthy family of semisymmetric graphs

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Abstract

In this paper, we construct semisymmetric graphs in which no two vertices have exactly the same neighbors. We show how to do this by first considering bi-transitive graphs, and then we show how to choose two such graphs so that their product is regular. We display a family of bi-transitive graphs $D_N(a,b)$ which can be used for this purpose and we show that their products are semisymmetric by applying vectors due to Ivanov.

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1. Introduction

This paper investigates graphs which are both semisymmetric and worthy. A semisymmetric graph is regular (all vertices have the same degree) and its group of automorphisms acts transitively on its edges but not on its vertices. A graph is worthy provided that no two of its vertices have exactly the same set of neighbors.

In an investigation into symmetry of graphs, unworthy graphs are unwelcome intrusions. This is because, in such a graph, one can interchange two vertices which have the same neighbors without moving any of the rest of the graph. In a sense, unworthy graphs have symmetries which are “local”; unworthiness allows a graph to have symmetries which have nothing to do with its global structure.

In the field of semisymmetric graphs, in particular, we have a good reason to exclude unworthy graphs. Many constructions which create a graph whose group is transitive on edges but not on vertices yield a graph in which vertices have two different degrees;

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when we try to adjust the construction to make the degrees equal, the graph often becomes vertex-transitive. The challenge to avoid that trap is one of the topic’s compelling features. If unworthy graphs are permitted then the challenge can be avoided in a fairly trivial way: we can turn any non-regular edge-but-not-vertex-transitive graph into a semisymmetric graph merely by duplicating vertices to achieve equal degrees. In the body of the paper, we show that nearly every unworthy semisymmetric graph is constructed in just such a way. So, we admit that all such graphs are semisymmetric and restrict our attention to worthy graphs.

Particular cases have already been of interest. A graph is biprimitive provided that it is semisymmetric and the symmetry group acts primitively on the vertices of each color. Biprimitive graphs are considered in [4,5]. An unworthy graph cannot be biprimitive: sets of vertices having the same neighbors would be blocks in the action. There has also been considerable interest in semisymmetric graphs of prime degree, in particular of degree 3. Marusic constructs a family of such graphs in [14]. These must also be worthy, as the only possibility for an unworthy edge-transitive graph of prime degree \( p \) is \( K_{p,p'} \) and this is not semisymmetric. The important paper [10] brings these topics together.

2. Definitions

In this paper, all graphs are simple and connected. Almost all graphs will be bipartite. We will, in fact, assume that a bipartite graph is bi-colored; i.e., that colors black and white have been assigned to vertices so that each edge has one white and one black endpoint.

A symmetry or automorphism of a graph \( \Gamma \) is a permutation of its vertices which preserves adjacency. The symmetries of \( \Gamma \) form a group under composition, called \( \text{Aut}(\Gamma) \). In a bipartite graph, \( \text{Aut}^+(\Gamma) \) is the subgroup consisting of symmetries which preserve color. A bipartite graph is bi-transitive provided that \( \text{Aut}^+(\Gamma) \) acts transitively on edges. In a bi-transitive graph, \( \text{Aut}^+(\Gamma) \) also acts transitively on the vertices of each color. If \( \Gamma \) is bi-transitive and \( \text{Aut}^+(\Gamma) = \text{Aut}(\Gamma) \), we say that \( \Gamma \) is strictly bi-transitive. To say that another way, a bi-transitive graph is strictly bi-transitive if there is no symmetry which reverses color.

Suppose that \( \Gamma \) is bi-transitive and has \( B \) black vertices, which all must be of the same degree \( k \), and \( W \) white vertices, all of degree \( e \). Notice that \( Bk = We \); both expressions count the number of edges in the graph. If \( k \neq e \), then \( \Gamma \) is strictly bi-transitive. If \( k = e \) (and, hence, \( B = W \)), the graph is regular; if it is nevertheless strictly bi-transitive, we call it semisymmetric.

Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are two graphs. One possible product of the two graphs (called the categorical product and other names; see [9,12,17,19]) of these two is the graph whose vertices are ordered pairs of vertices from \( \Gamma_1 \) and \( \Gamma_2 \), two vertices \( (w,x) \) and \( (y,z) \) being joined by an edge when \( \{w,y\} \) is an edge of \( \Gamma_1 \) and \( \{x,z\} \) is an edge of \( \Gamma_2 \). When both graphs are bicolored, the graph so constructed is not connected and we modify the definition: we let \( \Gamma_1 \land \Gamma_2 \) be the component of the product containing all vertices \( (w,x) \) where \( w \) and \( x \) are either both white or both black. This is the “wedge product” of \( \Gamma_1 \) and \( \Gamma_2 \).
As an example, consider $\Gamma_1 = K_{1,3}$ and $\Gamma_2 = K_{1,2}$ (Fig. 1):

The un-modified product graph has the two components shown above. If the graphs are colored so that $B$ and 2 are both given the color white, then $\Gamma_1 \wedge \Gamma_2$ is the component in the left part of Fig. 2.

With these assumptions, it is not hard to prove the following:

**Theorem 1.** If $\Gamma_1$ and $\Gamma_2$ are both bi-transitive graphs, where, for $i = 1, 2$, $\Gamma_i$ has $B_i$ black vertices of degree $k_i$ and $W_i$ white vertices of degree $e_i$, then their wedge product is also bi-transitive; it has $B_1B_2$ black vertices of degree $k_1k_2$ and $W_1W_2$ white vertices of degree $e_1e_2$.

We will use the word *worthy* to describe a graph in which no two vertices have exactly the same set of neighbors.

Now suppose that $\Gamma$ is an *unworthy* bi-transitive graph. By symmetry, there must be numbers $r$ and $s$ so that every white vertex belongs to a class of exactly $r$ white vertices so that any two members of the class have exactly the same set of black neighbors, and similarly, the blacks come in classes of $s$ vertices all sharing the same set of white neighbors. We call the numbers $r$ and $s$ the *repeatednesses* of $\Gamma$. Collapsing each class to a single vertex gives us a graph $\Gamma^*$ having $B/s$ black vertices of degree $k/r$ and $W/r$ white vertices of degree $e/s$. $\Gamma^*$ must be worthy. As the symmetries of $\Gamma$ act on $\Gamma^*$, we see that $\Gamma^*$ must also be bi-transitive. Moreover, if we assign colors to $K_{r,s}$ so that the $s$ vertices are black and the remaining $r$ vertices are white, it is easy to see that $\Gamma$ must be isomorphic to $\Gamma^* \wedge K_{r,s}$.

On the other hand, it should be clear that the product of two worthy graphs is worthy. We can summarize our results with respect to semisymmetric graphs in this way:
Theorem 2. If $\Gamma$ is any worthy bi-transitive graph in which $k \neq e$ (i.e., $B \neq W$), then the graph $\Gamma \land K_{r,s}$ where $r$ and $s$ are natural numbers satisfying $r/s = e/k$, is semisymmetric. Moreover, any semisymmetric graph having distinct repeatednesses arises in this way.

Proof. $\Gamma$ has $B$ black vertices of degree $k$, $K_{r,s}$ has $s$ black vertices of degree $r$, so the product has $Bs$ black vertices of degree $kr$. Likewise, it has $Wr$ white vertices of degree $es$. By hypothesis, $kr = es$, so the product is regular and bitransitive. Because its repeatednesses are unequal, it must be strictly bi-transitive and so semisymmetric.

The paragraph preceding this theorem proves its last sentence. □

3. The graphs $D_N(a, b)$

If $a$, $b$, and $N$ are positive integers satisfying $a + b < N$, and $X$ is the set $[N] = \{1, 2, 3, \ldots, N\}$, we define the graph $D_N(a, b)$ to be the bipartite graph having black vertices corresponding to the subsets of $X$ of size $b$, white vertices corresponding to subsets of $X$ of size $a$, with vertices $A$ and $B$ connected when sets $A$ and $B$ are disjoint.

For example, the graph $D_4(1, 2)$ is shown in Fig. 3.

In $D_N(a, b)$,

$$B = \binom{N}{b}, \quad W = \binom{N}{a}, \quad k = \binom{N-b}{a}, \quad e = \binom{N-a}{b}.$$ 

Any permutation in $S_N$ acts as a color-preserving symmetry of this graph, and the graph is therefore bi-transitive. If $a \neq b$, then $B \neq W$ so the graph is strictly bi-transitive but not regular, while if $a = b$, switching black and white vertices having the same label is a symmetry and so the graph is regular but not strictly bi-transitive. $D_N(a, b)$ is worthy and connected for all $a, b, N$ with $a + b < N$.

4. Regular products of $D_N$’s

From Theorem 1, we see that if we wish to use the graphs $D_N(a, b)$ to form a regular bi-transitive graph by a product construction, we must choose the two graphs
so that \( k_1k_2 = e_1e_2 \), or equivalently, so that \( B_1B_2 = W_1W_2 \). We could, of course, consider \( D_N(a,b) \land D_N(b,a) \). This graph is clearly regular, but is, unfortunately, dart-transitive, as switching coordinates is a color-reversing symmetry.

A more interesting choice is to consider \( D_N(a,b) \land D_N(N - b,a) \), which we will abbreviate by \( E_N(a,b) \). This graph is clearly regular, but is, unfortunately, dart-transitive, as switching coordinates is a color-reversing symmetry.

A more interesting choice is to consider \( D_N(a;b) \land D_N(N - b;a) \), which we will abbreviate by \( E_N(a;b) \). This is a connected graph whenever \( 0 < a < N/2, a < b \) and \( a + b < N \). But the case \( b = N/2 \) gives the dart-transitive graph of the previous paragraph, and \( E_N(a,b) \) is isomorphic to \( E_N(a,N - b) \), so we will restrict \( a, b, N \) to satisfy \( 0 < a < b < N/2 \). \( E_N(a,b) \) is always worthy. Let us summarize the data for these graphs:

In \( D_N(a,b) \):

\[
B_1 = \binom{N}{b}, \quad k_1 = \binom{N - b}{a}, \quad W_1 = \binom{N}{a}, \quad e_1 = \binom{N - a}{b}.
\]

In \( D_N(N - b,a) \):

\[
B_2 = \binom{N}{a}, \quad k_2 = \binom{N - a}{N - b}, \quad W_2 = \binom{N}{N - b} = \binom{N}{b}, \quad e_2 = \binom{b}{a}.
\]

And so, in \( E_N(a,b) \):

\[
B = \binom{N}{b} \binom{N}{a}, \quad k = \binom{N - b}{a} \binom{N - a}{N - b}, \quad W = \binom{N}{a} \binom{N}{b},
\]

\[
e = \binom{N - a}{b} \binom{b}{a}.
\]

The smallest example of such a graph is \( E_5(1,2) = D_5(1,2) \land D_5(3,1) \), which has 50 vertices of each color, each of degree 12. Is it semisymmetric? We repeat here the argument from [20]: In \( D_5(1,2) \), any two black vertices have a common white neighbor. We say the blacks are “neighborly”. The whites are also neighborly in this graph. In \( D_5(3,1) \), the blacks are neighborly, but the whites are not. White vertices \( \{1,2,3\} \) and \( \{3,4,5\} \), for instance, have no common black neighbor. In the product graph, then, the blacks will be neighborly, but the whites will not. Therefore there can be no color-reversing symmetry.

This argument will generalize a little. First we need this:

**Lemma** (Powell [18]). In \( D_N(a,b) \), the blacks are neighborly iff

\[ a + 2b \leq N \]  \hspace{1cm} (1)

while the whites are neighborly iff

\[ 2a + b \leq N. \]  \hspace{1cm} (2)

**Corollary.** In \( D_N(N - b,a) \), the blacks are neighborly iff

\[ (N - b) + 2a \leq N \]  \hspace{1cm} (3)
while the whites are neighborly iff
\[ 2(N - b) + a \leq N. \] (4)

Notice that \((w, x)\) is a common neighbor of \((u, v)\) and \((y, z)\) iff \(w\) is a common neighbor of \(u\) and \(y\), \(x\) is a common neighbor of \(v\) and \(z\). It follows that the blacks of \(\Gamma_1 \land \Gamma_2\) are neighborly iff blacks are neighborly in \(\Gamma_1\) and in \(\Gamma_2\).

**Corollary.** In \(E_N(a, b)\), the whites are never neighborly and the blacks are neighborly iff
\[ a \leq N/5 \quad \text{and} \quad 2a \leq b \leq (N - a)/2. \] (5)

**Proof.** Inequality (4) implies \(b \geq (N + a)/2\), which contradicts the condition \(b < N/2\), so some white vertices in \(D_N(N - b, a)\) have no common neighbors, and so white vertices in \(E_N(a, b)\) having those as second coordinates have no common neighbors either. Inequalities (1) and (3) are equivalent to (5). \(\square\)

Thus we know that \(E_6(1, 2)\), \(E_7(1, 2)\) and \(E_7(1, 3)\) are semisymmetric because in each case, the blacks are neighborly but the whites are not. However, in \(E_7(2, 3)\) neither the whites nor the blacks are neighborly. In order to show that these and other cases are semisymmetric, we need a finer distinction about common neighbors.

5. The Ivanov vectors

The mechanism for this distinction is a pair of vectors constructed by Ivanov [11]. In this paper, we modify his vectors very slightly. For a given bi-transitive graph \(\Gamma\), there are two \(a\)-vectors: \(ab\) for the blacks, \(aw\) for the whites. (Ivanov also introduced \(b\)-vectors which we will not find as useful here.) The vector \(ab\) has \(k + 1\) entries indexed by the numbers \(0\) through \(k\); similarly, \(aw\) has \(e + 1\) entries indexed \(0\) through \(e\). Fix a given black vertex \(u\). The entry in position \(i\) of \(ab\) is the number \(abi\) of black vertices having exactly \(i\) common neighbors with \(u\). The white vector \(aw\) is defined similarly. Notice that, unlike Ivanov, we count \(u\) itself and so \(ab_k\) will be at least \(1\), more if the graph is unworthy. The sum of the entries in \(ab\) is \(B\), in \(aw\), the sum is \(W\). Because of symmetry, the vector will not depend on the choice of \(u\).

For example, consider \(D_4(1, 2)\) shown in Fig. 3 above. Let \(u\) be the black vertex \(\{1, 2\}\), of degree 2. The four black vertices \(\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\) each have one white neighbor in common with \(u\), so entry 1 in the vector is 4. Vertex \(\{3, 4\}\) has no common neighbor with \(u\), so entry 0 is 1. And \(u\) has 2 common neighbors with itself, so entry 2 is 1. The vector \(ab\), then, is \((1, 4, 1)\).

If we let \(u\) be the white vertex \(\{1\}\), of degree 3, then each of the three other whites has exactly 1 neighbor in common with \(u\), and so \(aw\) is \((0, 3, 0, 1)\).

Because the Ivanov vectors will have many 0 entries, we introduce an abbreviation: let \(\{i, c\}\) stand for the vector of the appropriate length whose \(i\)th entry is \(c\) and which has 0 in every other entry. Then in this example, \(ab = \{0, 1\} + \{1, 4\} + \{2, 1\}\), while \(aw = \{1, 3\} + \{3, 1\}\).
For the graph $D_N(a,b)$ the number of common neighbors of two vertices depends entirely on the size of the intersection of the corresponding sets. We make that precise in this:

**Lemma** (Powell [18]). For $0 \leq r \leq b$, the number of sets $B_2$ which intersect a given set $B_1$, both of size $b$, in a set of size $r$ is

\[
\binom{b}{r} \binom{N-b}{b-r}.
\]

The number of common neighbors of a pair of such black vertices is

\[
\binom{N-2b+r}{a}.
\]

**Theorem 3** (Powell [18]). In the graph $D_N(a,b)$,

\[
ab = \sum_{r=0}^{b} \left\{ \binom{N-2b+r}{a}, \binom{b}{r} \binom{N-b}{b-r} \right\}
\]

and

\[
aw = \sum_{r=0}^{a} \left\{ \binom{N-2a+r}{b}, \binom{a}{r} \binom{N-a}{a-r} \right\}.
\]

**Example.** In the graph $D_7(2,4)$, the black vector $ab$ is

\[
\left\{ \begin{array}{c}
-1 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
4 \\
0
\end{array} \right\} + \left\{ \begin{array}{c}
0 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
4 \\
1
\end{array} \right\} + \left\{ \begin{array}{c}
1 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
4 \\
2
\end{array} \right\} \\
+ \left\{ \begin{array}{c}
2 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
4 \\
3
\end{array} \right\} + \left\{ \begin{array}{c}
3 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
4 \\
0
\end{array} \right\} \\
= \{0,0\} + \{0,4\} + \{0,18\} + \{1,12\} + \{3,1\} = \{0,22\} + \{1,12\} + \{3,1\}
\]

= (22,12,0,1)

and the white vector $aw$ is

\[
\left\{ \begin{array}{c}
3 \\
4
\end{array} \right\} \left\{ \begin{array}{c}
5 \\
0
\end{array} \right\} + \left\{ \begin{array}{c}
4 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
5 \\
1
\end{array} \right\} + \left\{ \begin{array}{c}
5 \\
2
\end{array} \right\} \left\{ \begin{array}{c}
5 \\
0
\end{array} \right\} \\
= \{0,10\} + \{1,10\} + \{5,1\} = (10,10,0,0,0,1).
\]

The idea of neighborliness is expressed in the Ivanov vectors: the black vertices are neighborly exactly when there are no other blacks that share no common neighbors with $u$; i.e., when $ab_0 = 0$. 
Now suppose that \( I_1 \) and \( I_2 \) are both bi-transitive graphs having black vectors \( a_1b, a_2b \) and white vectors \( a_1w, a_2w \). Then we want to claim that the Ivanov vectors for \( I_1 \cap I_2 \) are formed from those for \( I_1 \) and \( I_2 \) by multiplication of both indices and entries. More precisely:

**Theorem 4** (Powell [18]). With notation as in the paragraph above, the black vector for \( I_1 \cap I_2 \) is \( \sum_j \sum_i \{ij, a_1b, a_2b\} \), and a similar result holds for the white vector.

**Proof.** This is because for a given vertex \((u, v)\), the \( a_1b, a_2b \) vertices \((y, z)\) such that \( u \) and \( y \) have \( i \) common neighbors while \( v \) and \( z \) have \( j \) such each has \( ij \) common neighbors with \((u, v)\). This is true even when \( u = y \) and/or \( v = z \). □

**Example.** \( D_4(1, 2) \), as we have seen, has vectors \( ab = (1, 4, 1) \) and \( aw = (0, 3, 0, 1) \). \( D_5(3, 1) \) has \( ab = (0, 4, 0, 0, 1) \) and \( aw = (3, 6, 1, 1) \). Regarding the black vectors as \( \{0, 1\} + \{1, 4\} + \{2, 1\} \) and \( \{1, 4\} + \{5, 1\} \), we compute that the product is \( \{0, 1, 1, 4\} + \{0, 5, 1, 1\} + \{1, 1, 4, 4\} + \{1, 5, 1, 4\} + \{2, 1, 4\} + \{2, 5, 1, 1\} + \{3\}^2 = (0, 4) + (0, 1) + (1, 16) + (5, 4) + (2, 4) + (10, 1) = (5, 16, 4, 0, 0, 0, 0, 0, 1) \).

We can easily write code to compute these vectors for the \( D_N(a, b) \)'s and for their regular products, \( E_N(a, b) \). We list here the output vectors for all integer values of \( a, b, N \) satisfying \( 0 < a < b < N/2 \) for \( 5 \leq N \leq 7 \):

**\( E_5(1, 2) \):**
\[
ab = \{1, 12\} + \{2, 24\} + \{3, 4\} + \{4, 3\} + \{8, 6\} + \{12, 1\},
\]
\[
aw = \{0, 15\} + \{3, 24\} + \{6, 10\} + \{12, 1\}.
\]

**\( E_6(1, 2) \):**
\[
ab = \{2, 30\} + \{3, 40\} + \{4, 5\} + \{10, 6\} + \{15, 8\} + \{20, 1\},
\]
\[
aw = \{0, 36\} + \{6, 40\} + \{10, 8\} + \{12, 5\} + \{20, 1\}.
\]

**\( E_7(1, 2) \):**
\[
ab = \{3, 60\} + \{4, 60\} + \{5, 6\} + \{18, 10\} + \{24, 10\} + \{30, 1\},
\]
\[
aw = \{0, 70\} + \{10, 60\} + \{15, 10\} + \{20, 6\} + \{30, 1\}.
\]

**\( E_7(1, 3) \):**
\[
ab = \{5, 24\} + \{10, 108\} + \{15, 76\} + \{20, 6\} + \{30, 18\} + \{45, 12\} + \{60, 1\},
\]
\[
aw = \{0, 28\} + \{10, 108\} + \{20, 90\} + \{30, 6\} + \{40, 12\} + \{60, 1\}.
\]

**\( E_7(2, 3) \):**
\[
ab = \{0, 394\} + \{1, 180\} + \{3, 120\} + \{5, 18\} + \{6, 10\} + \{15, 12\} + \{30, 1\},
\]
\[
aw = \{0, 462\} + \{1, 120\} + \{3, 10\} + \{4, 120\} + \{10, 12\} + \{12, 10\} + \{30, 1\}.
\]
Every graph in this list is regular and bi-transitive. In every case, the vectors \( ab \) and \( aw \) are different, showing that the bi-transitivity is strict, and this implies that the graph is semisymmetric.

We make an observation: In each pair of vectors, while the final entries are the same, a one in the place corresponding to the common degree, the second-to-last non-zero entry in each pair occurs in different places. In fact, it is always later in the black vector than in the white. For example, in the last graph listed above, \( E_7(2,3) \), \( ab \) has a 12 in position 15, while \( aw \) has a 10 in place 12. Neither has any more non-zero entries until the 1 in place 30. We will then use the phrase **critical index** to indicate the index of the second-to-last non-zero entry in a vector.

**Theorem 5.** For all integers \( a, b, N \) with \( 0 < a < b < N/2 \), the critical index of the black vector is greater than that of the white vector in the graph \( E_N(a,b) \). Thus, every \( E_N(a,b) \) is semisymmetric.

**Proof.** Let \( D_N(a,b) \) have vectors \( a1b, a1w \) and let \( D_N(N-b,a) \) have vectors \( a2b, a2w \). Then the last few terms of these vectors are:

\[
\begin{align*}
a1b &= \ldots + \left\{ \begin{pmatrix} N - b - 1 \\ a \end{pmatrix}, b(N - b) \right\} + \left\{ \begin{pmatrix} N - b \\ a \end{pmatrix}, 1 \right\}, \\
a2b &= \ldots + \left\{ \begin{pmatrix} N - a - 1 \\ N - b \end{pmatrix}, a(N - a) \right\} + \left\{ \begin{pmatrix} N - a \\ N - b \end{pmatrix}, 1 \right\}, \\
a1w &= \ldots + \left\{ \begin{pmatrix} N - a - 1 \\ b \end{pmatrix}, a(N - a) \right\} + \left\{ \begin{pmatrix} N - a \\ b \end{pmatrix}, 1 \right\}, \\
a2w &= \ldots + \left\{ \begin{pmatrix} b - 1 \\ a \end{pmatrix}, b(N - b) \right\} + \left\{ \begin{pmatrix} b \\ a \end{pmatrix}, 1 \right\}.
\end{align*}
\]

Then, by Theorem 4, the critical index for the blacks in \( E_N(a,b) \) is

\[
b_1 = \begin{pmatrix} N - a \\ N - b \end{pmatrix} \begin{pmatrix} N - b - 1 \\ a \end{pmatrix}
\]
or

\[
b_2 = \begin{pmatrix} N - b \\ a \end{pmatrix} \begin{pmatrix} N - a - 1 \\ N - b \end{pmatrix},
\]

whichever is larger. Similarly, the critical index for the whites in the product is

\[
w_1 = \begin{pmatrix} N - a - 1 \\ b \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}
\]
or

\[
w_2 = \begin{pmatrix} b - 1 \\ a \end{pmatrix} \begin{pmatrix} N - a \\ b \end{pmatrix}.
\]
whichever is larger. If we expand these and let $D = (N - a - 1)! / (a!(N - a - b)!(b - a)!)$, then after some algebra we see that

$$b_1 - b_2 = \frac{D}{N - b}((N - a)(N - a - b) - (N - b)(b - a))$$

$$= \frac{D}{N - b}((N - b)^2 - a(N - a))$$

and this is positive because $N - b$ is larger than $N/2$, so its square is larger that $N^2/4$, while $a(N - a)$ is the product of two numbers whose average is $N/2$, and such a product cannot be larger than $N^2/4$. Thus $b_1$ is the critical index for black.

Now $w_1$ or $w_2$ may be the critical index for white, and we claim that both are less than $b_1$. First note that

$$b_1 = D \frac{N - a}{N - b}(N - a - b),$$

$$w_1 = D(N - a - b),$$

and

$$w_2 = (D/b)(N - a)(b - a).$$

Because $N - a > N - b$, $b_1$ is clearly larger than $w_1$. And

$$b_1 - w_2 = D(N - a) \left[ \frac{N - a - b}{N - b} - \frac{b - a}{b} \right]$$

$$= \frac{Da(N - a)}{b(N - b)}(N - 2b);$$

because $b < N/2$, this is positive and so $w_2 < b_1$. Thus the black critical index is greater than the white critical index, proving the theorem and showing that all graphs $E_N(a,b)$ are semisymmetric.  

6. History

Harary and Dauber’s unpublished paper [8] seems to have been the beginning of the idea of “line-symmetric but not point-symmetric” graphs. Folkman [7] refers to it in 1967 as the motivation for his constructions. The graph $\tilde{G}$ he constructs in Theorem 3 of [7] is $G \wedge K_{r,r}$. The graph in Theorem 4 can be expressed as Cay$(A,S) \wedge K_{1,r}$, where $S = \{1,a,a^2,a^3,\ldots\}$. The graph in Fig. 1 of [7] can be expressed as $D_5(1,3) \wedge K_{1,2}$. In other parts of the paper, he establishes conditions for semisymmetric (“admissible”) graphs to occur.

Bouwer’s 1968 paper [1] describes a worthy semisymmetric graph of degree 3 attributed to Marion Gray in 1932. This graph is discussed in detail in [15]. His 1972 paper [2] relates semisymmetric graphs to configurations. In Section 1, he uses a configuration of cubes to construct a worthy semisymmetric graph having $n^m$ vertices of each color, all vertices having degree $n$; for $n = 3$, this construction gives the Gray
graph. In Section 2, he constructs the unworthy graph \( L(n, h, s, t) \), which we would describe here as \( D_n(1, n-h) \land K_s \).

In 1978, Klin [13] introduced the term “semisymmetric” and constructed a worthy family of semisymmetric graphs in which \( B \) and \( k \) are relatively prime.

Ivanov’s 1987 paper [11] we have partly discussed. He uses the vectors to classify possibilities for parameters for semisymmetric graphs. This was the first of several papers [3–6,14–16] on this topic.

The 2000 paper [16] of Marušić and Potocnik generalizes Folkman’s Theorem 4 construction to a graph still of repeatednesses \( 1, r \) and begins to relate semisymmetric graphs with \( \frac{1}{x} \)-transitive ones.

7. And finally

There are two modifications of the constructions of this paper that are as yet unexplored. First, we may consider as base graphs the bi-transitive graphs \( T_N(a, b, c) \). In such a graph, \( X = \{1, 2, 3, \ldots, N\} \), the white vertices correspond to subsets of \( X \) of size \( a \), the blacks to those of size \( b \). An edge is formed between sets \( A \) of size \( a \) and \( B \) of size \( b \) if their intersection has size \( c \). Thus \( D_N(a, b) = T_N(a, b, 0) \). The formulas for degrees and common neighbors are a little more complex than for \( D_N(a, b) \), and no general results have been obtained concerning them.

The second is the mix-and-match option. Given, say, \( D_5(1, 2) \), what \( D_N(a, b) \) can we pair it with so that the product is regular? Observe that \( \begin{pmatrix} N \\ 2 \end{pmatrix} / \begin{pmatrix} N \\ 1 \end{pmatrix} = \frac{10}{5} = 2 \). If we can find \( N, a, b \) so that \( \begin{pmatrix} N \\ a \end{pmatrix} / \begin{pmatrix} N \\ b \end{pmatrix} = \frac{1}{2} \), then the product will be regular. Because \( \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 28 \) and \( \begin{pmatrix} 8 \\ 3 \end{pmatrix} = 56 \), \( D_8(3, 2) \) will do. So \( D_5(1, 2) \land D_8(3, 2) \) is bi-transitive and regular. Is it strictly bi-transitive? Yes, the Ivanov vectors for this product are unequal. Is this true in general? The answer is unknown. And a preliminary question is this:

Given a bi-partite graph with \( B/W = p/q \), how can we find all \( a, b, N \) so that \( \begin{pmatrix} N \\ a \end{pmatrix} / \begin{pmatrix} N \\ b \end{pmatrix} = q/p \)?

These questions are as yet uninvestigated.

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References

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